ZFC without parameters (A note on a question of Kai Wehmeier)

Ralf Schindler
Institut für Mathematische Logik und Grundlagenforschung, Universität Münster
Einsteinstr. 62, 48149 Münster, Germany

Abstract

It is shown that the parameter free formulation of ZFC is as strong as ZFC itself.

It is well–known that if the induction schema of Peano arithmetic, PA, is formulated without parameters, then the resulting theory is as strong as PA itself in that both theories prove the same theorems. The reason is basically that every natural number is definable in the language of PA, so that parameters may be replaced by their definition.

As it is certainly not true that every set is definable (cf. [1], though), we may ask if the "parameter free version" of ZFC is weaker than ZFC itself. Zermelo–Fraenkel set theory has the follwing axioms: Extensionality (Ext), Foundation (Fund), Pairing (Par), Union (Union), Power set (Pow), Infinity (Inf), choice (AC), as well as the schemas Aus and Repl of Aussonderung (separation) and replacement. Aus says that for every formula $\varphi(x, v_1, \ldots, v_n)$ in which the variable b does not occur,

$$\forall v_1 \ldots \forall v_n \forall a \exists b \forall x (x \in b \longleftrightarrow x \in a \wedge \varphi(x, v_1, \ldots, v_n)), \tag{1}$$

and Repl says that for every formula $\varphi(x, y, v_1, \ldots, v_n)$,

$$\forall v_1 \ldots \forall v_n (\forall x \exists y \forall y' (\varphi(x, y', v_1, \ldots, v_n) \leftrightarrow y' = y) \longrightarrow \tag{2}$$
$$\forall a \exists b \forall y (y \in b \leftrightarrow \exists x \in a \varphi(x, y, v_1, \ldots, v_n))).$$

Repl plus $\exists x(x = \emptyset)$ proves Aus. We may also phrase Repl by saying that if $F \colon x \mapsto y$ is a class function definable by the formula $\varphi(x, y, v_1, \ldots, v_n)$, then for all sets a, $F''a = \{y \colon \exists x \in a F(x) = y\}$ exists.

In (1) and (2), $v_1, \ldots, v_n$ play the role of parameters. We may let Aus° be the statement that for every formula $\varphi(x)$ (with $x \neq b$ as variables),

$$\forall a \exists b \forall x (x \in b \longleftrightarrow x \in a \wedge \varphi(x)), \tag{1°}$$

and we may let Repl° be the statement that for every formula $\varphi(x, y)$,

$$\forall x \exists y \forall y' (\varphi(x, y') \leftrightarrow y' = y) \longrightarrow \tag{2°}$$
$$\forall a \exists b \forall y (y \in b \leftrightarrow \exists x \in a \varphi(x, y)).$$

We let ZFC° be the theory with the axioms Ext, Fund, Par, Union, Pow, Inf, AC, Aus°, and Repl°. We now prove the following.[1]

[1] We thank Kai Wehmeier for asking us the question as to whether ZFC° is weaker than ZFC. We also thank Rene Schipperus for discussions about this question. Theorem 0.1 may be part of the set theoretic folklore. If so, then the autor of this note just displays his ignorance.

Theorem 0.1 ZFC° *proves both* Aus *and* Repl*.*

Naive attempts of proving Aus and Repl in ZFC° all go through trying to first show that the cross product $a \times b$ exists, which seems to require Aus. We basically follow this route and cook up a version of such a cross product which can be shown to exist in ZFC° with not much effort.

Lemma 0.2 (ZFC°) *For all sets a, both $a \times \{0\}$ and $a \times \{1\}$ exist.*

PROOF. $a \times \{0\} = F''a$, where $F(x) = (x, 0)$, and $a \times \{1\} = G''a$, where $G(x) = (x, 1)$. □

Lemma 0.2 immediately yields

Lemma 0.3 (ZFC°) *For all sets a and b, $(a \times \{0\}) \cup \{(b, 1)\}$ exists.* □

Lemma 0.4 (ZFC°) *For all sets a and b, $\{((u, 0), (b, 1)) \colon u \in a\}$ exists.*

PROOF. By Lemma 0.3 and Pow, $d = \mathcal{P}(\mathcal{P}((a \times \{0\}) \cup \{(b, 1)\}))$ exists. We then have that

$$\{((u, 0), (b, 1)) \colon u \in a\} = \{x \in d \colon \exists u \exists v x = ((u, 0), (v, 1))\},$$

so that the desired set exists by Aus°. □

Lemma 0.5 *Let $\varphi(x, v)$ be a formula. For all sets a and b, $\{x \in a \colon \varphi(x, b)\}$ exists.*

PROOF. Let $F(x) = 0$ unless there are u, c with $x = ((u, 0), (c, 1))$ in which case $F(((u, 0), (c, 1))) = u$. Then

$$\{x \in a \colon \varphi(x, b)\} \cup \{0\} = F''\{((u, 0), (b, 1)) \colon u \in a\}$$

exists by Lemma 0.4 and Repl°. We may then easily use Aus° to get the desired set $\{x \in a \colon \varphi(x, b_1, \ldots, b_n)\}$. □

As finitely many parameters may be easily amalgamated into one, using Par, this shows that Aus follows from ZFC°. To prove Repl from ZFC°, it also suffices to consider just one parameter and prove the following by a slight generalization of the argument for Lemma 0.5.

Lemma 0.6 *Let $\varphi(x, y, v)$ be a formula, Let b be a set such that for every x there is exactly one y with $\varphi(x, y, b)$. Then for all sets a, $\{y \colon \exists x \in a \varphi(x, y, b)\}$ exists.*

PROOF. We may let $F(z) = 0$ unless there are x, c with $z = ((x, 0), (c, 1))$ and there is a unique y with $\varphi(x, y, c)$ in which case $F(z) = y$. Then

$$\{y \colon \exists x \in a \varphi(x, y, b)\} \cup \{0\} = F''\{((x, 0), (b, 1)) \colon x \in a\}$$

exists by Lemma 0.4 and Repl°. We may then easily use Aus° to get the desired set $\{y \colon \exists x \in a \varphi(x, y, b)\}$. □

It was of course redundant to prove Lemma 0.5 before Lemma 0.6. In any event, we have verified Theorem 0.1.

References

[1] Hamkins, J., *Pointwise definable models of set theory*, talk at Oberwolfach, 2011, cf. http://www.logic.univie.ac.at/~holy/ow2011/joelhamkins.pdf