The core model induction

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This is a set of notes on the proceedings of the joint Muenster-Irvine-Berlin-Gainesville-Oxford seminar in core model theory, held in cyberspace April–June 2006.

The plan now is to eventually turn this set of notes into a reasonable paper.

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Chapter 1

The successor case

The core model K is a canonical inner model that is close to V in some sense. The basic problem of core model theory is to construct and study such a model K. Because K is close to V, it will contain all the canonical inner models for large cardinal hypotheses which there are. However, "all there are" can only be made precise, so far as we know, by assuming some limitation on the large cardinal hypotheses admitting canonical inner models, and therefore core model theory is always developed under some such *anti-large-cardinal hypothesis*.

Core model theory began in the mid-1970's with the work of Dodd and Jensen, who developed a good theory of K under the assumption that there is no inner model with a measurable cardinal ([3], [4],[5]). The theory was further developed under progressively weaker anti-large-cardinal hypotheses by Mitchell ([17],[18]) and Steel ([37]) in the early 80s and early 90s. Certain defects in Steel's work were remedied by Jensen and Steel in 2007. The upshot is that there are Σ_2 formulae $\psi_K(v)$ and $\psi_\Sigma(v)$) such that one can prove:

Theorem 1.0.1 (Jensen, Steel 2007) *Suppose there is no transitive proper class model satisfying* ZFC *plus "there is a Woodin cardinal"; then*

(1) $K = \{v \mid \psi_K(v)\}$ *is a transitive proper class extender model satisfying* ZFC*,*

(2) $\{v \mid \psi_\Sigma(v)\}$ *is an iteration strategy for* K *for set-sized iteration trees, and moreover the unique such strategy,*

(3) *(Generic absoluteness)* $\psi_K^V = \psi_K^{V[g]}$, *and* $\psi_\Sigma^V = \psi_\Sigma^{V[g]} \cap V$, *whenever* g *is* V*-generic over a poset of set size,*

(4) *(Inductive definition)* $K|(\omega_1^V)$ *is* Σ_1 *definable over* $(J_{\omega_1}(\mathbb{R}), \in)$,

(5) *(Weak covering) For any* K*-cardinal* $\kappa \geq \omega_2^V$, $\mathrm{cof}(\kappa^{+K}) \geq |\kappa|$*; thus* $\kappa^{+K} = \kappa^+$, *whenever* κ *is a singular cardinal of* V *(Mitchell, Schimmerling [20]).*

It is easy to formulate this theorem without referring to proper classes, and so formulated, it can be proved in ZFC. The theorem as stated can be proved in GB.

Items (1)-(4) say that K is absolutely definable and, through (1), that its internal properties can be determined in fine-structural detail. Notice that by combining (3) and (4) we get that for any uncountable cardinal μ, $K|\mu$ is Σ_1 definable over $L(H_\mu)$, uniformly in μ. This is the best one can do if $\mu = \omega_1$ (see [37, §6]), but for $\mu \geq \omega_2$ there is a much simpler definition of $K|\mu$ due to Schindler (see Lemma 2.3.4 below).

Item (5) says that K is close to V in a certain sense. There are other senses in which K can be shown close to V; for example, every extender which coheres with its sequence is on its sequence ([28]), and if there is a measurable cardinal, then K is Σ_3^1-correct ([37, §7]).

It is probably safe to say that Theorem 1.0.1 has never been used by anyone who believed its hypothesis. The theorem is most often used in its contrapositive form: one has some hypothesis H which implies there can be no core model as in the conclusion of 1.0.1, and from this one gets that H implies that there is a proper class inner model with a Woodin cardinal. This establishes a consistency strength lower bound for H, among other things. Establishing such lower bounds is one of the main applications of core model theory.[1]

For example, the Proper Forcing Axiom, or PFA , implies that Jensen's square principle fails at all cardinals κ. But $\Box_\kappa$ holds in iterable extender models below the minimal model of a superstrong cardinal as a consequence of their fine structure, and if $\Box_\kappa$ holds in a transitive model which computes κ^+ correctly, then it holds in V. Thus PFA implies that there is an inner

[1] Perhaps one can prove something like the conclusions of 1.0.1 under true hypotheses, but even making a precise conjecture concerning the existence of such an *ultimate* K is a grand project indeed, for one would need to extend the meaning of "extender model" so as to accomodate all large cardinals.

model with a Woodin cardinal.[2]

What about better lower bounds? The known consistency strength upper bound for PFA, obtained by a very natural forcing argument, is one supercompact cardinal. It seems unlikely that one Woodin cardinal is enough, and indeed it seems likely that PFA implies there are inner models with supercompact cardinals. At the very least, PFA should yield inner models with two Woodin cardinals. The proof must use some version of core model theory, as that is our most all-purpose method for constructing inner models with large cardinals, and the strength in PFA is far enough from the surface that less powerful methods seem doomed. Or take the failure of $\Box$ at a singular cardinal itself: this should imply there are inner models with superstrongs, but it seems hopeless to obtain anything from the non-structure asserted by not-$\Box$ without making use of a covering theorem. For these and many other reasons, one wants a variant of Theorem 1.0.1 which can be used to produce inner models of large cardinal hypotheses much stronger than "there is a Woodin cardinal".

There is a subtlety here, in that the anti-large-cardinal hypothesis of Theorem 1.0.1 cannot be weakened, unless one simultaneously strengthens the remainder of the hypothesis, i.e., ZFC. For suppose δ is Woodin, that is, V is our proper class model with a Woodin. Suppose toward contradiction we had a formula $\psi_K(v)$ defining a class K, and that (3), (4), and (5) held. Let g be V-generic for the full stationary tower below δ.[3] Let

$$j\colon V \to M \subseteq V[g],$$

where $M^{<\delta} \subseteq M$ holds in $V[g]$. We can choose g so that $\text{crit}(j) = \aleph_{\omega+1}^V$. Let $\mu = \aleph_\omega^V$. Then

$$(\mu^+)^K = (\mu^+)^V < (\mu^+)^M = (\mu^+)^{j(K)} = (\mu^+)^K,$$

a contradiction. The first relation holds by (5), the second by the choice of j, the third by (5) applied in M, and the last by (3) and (4), and the agreement between M and $V[g]$.[4]

So the anti-large-cardinal hypothesis of Theorem 1.0.1 cannot be weakened, unless one simultaneously strengthens the remainder of the hypothesis,

[2] This last result is due to Schimmerling. Credits and references for the various steps in the argument are given in Chapter 2.

[3] See [14]. The reader who is not familiar with stationary tower forcing needn't worry, as we shall not use it in any essential way in this book.

[4] The same proof shows there is no formula ψ such that (3) holds, and (5) holds in all set generic extensions of V.

ZFC. Of course, whatever hypothesis we are trying to prove is strong does go beyond ZFC, but is there anything we can use to erect a theory which is not hypothesis-specific? Well, there is Theorem 1.0.1 itself! Using it, we will have obtained inner models with one Woodin cardinal, and in the simplest case, we will have obtained such models *over arbitrary sets x*. It turns out that in the theory ZFC + "Over every set, there is a proper class model with one Woodin cardinal" + "There is no proper class model with two Woodin cardinals", we can prove the existence of a core model K satisfying the conclusions of 1.0.1. Our hypothesis will be inconsistent with the existence of such a K, and thus from it we get inner models with two Woodin cardinals. In the simplest case, we will in fact have such models over arbitrary sets x. But again, it turns out that in the theory ZFC + "Over every set, there is a proper class model with two Woodin cardinals" + "There is no proper class model with three Woodin cardinals", we can prove the existence of a core model K satisfying the conclusions of 1.0.1. So now we have inner models with three Woodin cardinals, and so on.

This leads to general method for obtaining consistency strength lower bounds beyond one Woodin cardinal. One shows by induction that the universe is closed under mouse operators (functions like $x \mapsto M_n^{\#}(x)$) of ever greater complexity. The successor step of such a *core model induction* uses core model theory, in the form of a generalization of Theorem 1.0.1. The hypothesis whose strength is being mined will imply that there can be no generalized K over any set in the appropriate domain, and then via our generalization of 1.0.1, we have that the next mouse operator is defined on that domain. The next operator produces, roughly, mice with one Woodin cardinal closed under the previous operator.[5]

There are some white lies in the last paragraph. We shall correct them, and give a more precise account of the generalization of 1.0.1 one needs at successor steps in a core model induction. We shall also say a bit there about what happens at limit steps. In the intervening sections, we shall look more closely at the construction of K that goes into the proof of Theorem 1.0.1, and at what blocks it in the case that there are proper class models with a Woodin cardinal. To our intended reader, the answer will not be a complete surprise: the problem is the existence of iteration strategies for the premice we construct. The special problem occurs when our premouse has a Woodin cardinal.

[5] W.H. Woodin first discovered the core model induction method, in about 1991.

1.1 Iteration strategies for V

Much of inner model theory centers on the construction of iteration strategies. The basic techniques and problems can be illustrated with coarse-structural iterations of V, so before diving into the complexities of fine structure theory, let us look at this simpler situation. In order to keep the technicalities to a minimum, we work in Kelley-Morse set theory throughout this section.

The basic existence theorem for branches of iteration trees is

Theorem 1.1.1 (Martin, Steel [16]) *Let $\pi\colon M \to V_\eta$ be elementary, where M is countable and transitive, and let $\mathcal{T}$ be a countable putative iteration tree on M; then either*

(a) $\mathcal{T}$ has a last model N, and letting $i\colon M \to N$ be the iteration map, there is an elementary $\sigma\colon N \to V_\eta$ such that $\pi = \sigma \circ i$, or

(b) $\mathcal{T}$ has a maximal branch b such that letting $i_b\colon M \to M_b$ be the iteration map, there is an elementary $\sigma\colon M_b \to V_\eta$ such that $\pi = \sigma \circ i_b$.

Putative iteration trees are just like iteration trees, except that they may have a last, illfounded model. The map σ described in (a) or (b) is called a *π-realization* of N or of b. The theorem says that choosing π-realizable branches is an ω_1-iteration strategy for M, provided we never encounter a countable tree with distinct, cofinal, π-realizable branches. We turn now to the uniqueness question.

Let $\mathcal{T}$ be an iteration tree on some elementary submodel of V. We let $\nu(E_\alpha)$ be the strength of the extender $E_\alpha = E_\alpha^{\mathcal{T}}$ in the model $M_\alpha^{\mathcal{T}}$ where it appears, and say $\mathcal{T}$ is *increasing* if $\alpha < \beta \Rightarrow \nu(E_\alpha) < \nu(E_\beta)$, for all $\alpha < \beta$. For increasing $\mathcal{T}$,

$$\delta(\mathcal{T}) = \sup\{\nu(E_\alpha) \mid \alpha < \mathrm{lh}(\mathcal{T})\},$$

and

$$M(\mathcal{T})^6 = \bigcup_{\alpha<\mathrm{lh}(\mathcal{T})} V_{\nu(E_\alpha)}^{M_\alpha}.$$

Thus for any cofinal branch b of $\mathcal{T}$ such that $\delta(\mathcal{T}) \in M_b$,

$$V_{\delta(\mathcal{T})}^{M_b} = M(\mathcal{T}).$$

[6] $M(\mathcal{T})$ is also called the *common part model*.

We note that $(M(\mathcal{T}), \in)$ satisfies Zermelo Set Theory plus Choice, together with "$\forall\alpha(V_\alpha$ exists)" and "every set belongs to some V_α. We call this theory ZCP. If P is any transitive model of ZCP, we say that *A kills the Woodinness of P* iff $A \subseteq o(P)$,[7] and either

(1) $\exists\gamma < o(P)(A \cap \gamma \notin P)$, or

(2) there is no $\kappa < o(P)$ such that $(P, \in, A) \models \kappa$ is A-reflecting in OR.

The basic result on uniqueness of branches in an iteration tree is:

Theorem 1.1.2 (Martin, Steel [16]) *Let M be a transitive model of ZFC, and $\mathcal{T}$ an increasing iteration tree on M. Suppose b and c are cofinal branches of $\mathcal{T}$ such that $\delta(\mathcal{T}) \in M_b \cap M_c$, and some $A \in M_b \cap M_c$ kills the Woodinness of $\delta(\mathcal{T})$; then $b = c$.*

In the contrapositive: if b and c are distinct cofinal branches of $\mathcal{T}$, then $\delta(\mathcal{T})$ is Woodin over $M(\mathcal{T})$ with respect to all $A \in M_b \cap M_c$.

The following is typical of how these results combine to produce an iteration strategy.

Theorem 1.1.3 *Suppose that there is no proper class model with a Woodin cardinal; then V has a unique iteration strategy defined on all set-sized trees.*

Proof. Suppose P is a transitive model of ZCP. We cannot have $L(P) \models P = V_{o(P)} \wedge o(P)$ is Woodin. (This is not quite so obvious as it sounds, as $L(P)$ may have no wellorder of P. See Exercise 1.1.4.) So some $A \in L(P)$ kills the Woodinness of P. We let

$$Q(P) = J_{\alpha_P}(P),$$

where α_P is the least α such that some $A \in J_\alpha(P)$ kills the Woodinness of P.

Now let $\pi \colon M \to V$ be elementary, where M is countable and transitive. We claim M is ω_1-iterable, for trees based on some V_η^M, by the strategy Σ of choosing the unique π-realizable branch. For suppose $\mathcal{T}$ is a tree of countable limit length which has been played according to Σ. By 1.1.1, $\mathcal{T}$ has a cofinal π-realizable branch. Suppose b and c are cofinal π-realizable branches. Since $\mathcal{T}$ is based on some V_η^M, $M(\mathcal{T}) \in M_b \cap M_c$. Moreover,

[7] $o(P)$, sometimes written $P \cap \mathrm{OR}$, is the ordinal height of P

$M(\mathcal{T})$ is in the domain of the Q-functions of M_b and M_c. Since M_b and M_c are wellfounded, they compute $Q(M(\mathcal{T}))$ correctly:

$$Q(M(\mathcal{T}))^{M_b} = Q(M(\mathcal{T})) = Q(M(\mathcal{T}))^{M_c}.$$

Since some set in $Q(M(\mathcal{T}))$ kills the Woodinness of $\delta(\mathcal{T})$, 1.1.2 implies $b = c$.

Note that Σ can also described as choosing the unique cofinal branch such that M_b is wellfounded, or as choosing the unique branch b such that $Q(M(\mathcal{T}))^{M_b} = Q(M(\mathcal{T}))$. (Cf. Exercise 1.1.5.) This final description is the one which generalizes best.

We now define an iteration strategy Γ for V by

$$\Gamma(\mathcal{T}) = \text{ unique cofinal } b \text{ such that } Q(M(\mathcal{T}))^{M_b} = Q(M(\mathcal{T})),$$

whenever $\mathcal{T}$ has limit length $<$ OR and is played by Γ. By 1.1.2, there is at most one such branch, so our problem is existence. Suppose $\mathcal{T}$ has limit length $<$ OR, and is played according to Γ. Let

$$\pi \colon M \to V$$

be elementary, with M countable transitive, and $\pi(\bar{\mathcal{T}}) = \mathcal{T}$. Since $\mathcal{T}$ is based on some V_η, $\bar{\mathcal{T}}$ is based on some V_η^M. Then $\bar{\mathcal{T}}$ is played according to the ω_1-iteration strategy Σ described above, because the Q-structure function collapses to itself, that is

$$\pi^{-1}(Q(M(\mathcal{T} \restriction \lambda))) = Q(\pi^{-1}(M(\mathcal{T} \restriction \lambda))),$$

for all $\lambda \in \mathrm{ran}(\pi)$. So there is a unique cofinal branch $\bar{b}$ of $\bar{\mathcal{T}}$ such that

$$Q(M(\bar{\mathcal{T}}))^{M_{\bar{b}}^{\bar{\mathcal{T}}}} = Q(M(\bar{\mathcal{T}})).$$

But $Q(M(\bar{\mathcal{T}})) = \pi^{-1}(Q(M(\mathcal{T})))$ is in M, and $\bar{b}$ is Σ^1_1 in every real coding it, so $\bar{b} \in M[g]$ whenever g is M-generic for $\mathrm{Col}(\omega, |Q(M(\bar{\mathcal{T}}))|)$, so $\bar{b} \in M$. But then

$$\Gamma(\mathcal{T}) = \pi(\bar{b})$$

works. □

Exercise 1.1.4 *Let P be a transitive model of* ZCP*, and suppose that $L(P) \models P = V_{o(P)} \wedge o(P)$ is Woodin (but not necessarily $L(P) \models$ the axiom of choice). Show that there is a transitive proper class model of* ZFC *with a Woodin cardinal.*

Exercise 1.1.5 *Show that in the proof of Theorem 1.1.3, Σ can also described as choosing the unique cofinal branch such that M_b is wellfounded, or as choosing the unique branch b such that $Q(M(\mathcal{T}))^{M_b} = Q(M(\mathcal{T}))$.*

All iteration strategies we know how to construct, in any circumstance, trace back to strategies which pick the unique branch correctly moving some canonical failure of Woodinness. Those strategies are *guided by Q-structures*, in the standard, uninspired terminology. The crucial thing is that the Q-structure function condenses to itself under Skolem hulls. Here is another example.

Theorem 1.1.6 *Suppose that every set has a sharp, and δ is small enough that for all $\eta < \delta$, η is not Woodin in $L(V_\eta)$; then V has a unique iteration strategy for set sized trees based on V_δ.*

Proof. For any transitive P, put

$$J(P) = \text{ rudimentary closure of } P^\# \cup \{P^\#\}.$$

We show V has a J-guided strategy for trees based on V_δ; that is, given $\mathcal{T}$ on V_δ played according to Γ, and of limit length $< \mathrm{OR}$, we put

$$\Gamma(\mathcal{T}) = \text{ unique } b \text{ such that } J(M(\mathcal{T}))^{M_b} = J(M(\mathcal{T})).$$

To see that there can be at most one such b, notice that for all $\eta \leq \delta$, there is an $A \in J(V_\eta)$ which kills the Woodinness of V_η. For $\eta < \delta$, this is clear, as $V_{\eta+1} \cap L(V_\eta) = V_{\eta+1} \cap (V_\eta)^\# \subset J(V_\eta)$. δ itself may be Woodin in $L(V_\delta)$, but if so, it is the least such cardinal, and hence δ is singular in $J(V_\delta)$ (cf. Exercise 1.1.7). Since $\mathcal{T}$ is based on V_δ, if b is cofinal in $\mathcal{T}$, then $J(M(\mathcal{T})^{M_b})$ exists. But then $J(M(\mathcal{T}))^{M_b} = J(M(\mathcal{T}))^{M_c}$ implies $b = c$, by theorem 1.1.2.

The proof that there is a cofinal branch b of $\mathcal{T}$ such that $J(M(\mathcal{T}))^{M_b} = J(M(\mathcal{T}))$ is a reflection argument like that in the proof of 1.1.3, using the fact that J condenses to itself, that is, that $\pi^{-1}(J(P)) = J(\pi^{-1}(P))$, for elementary $\pi\colon M \to V$, with M transitive. □

Exercise 1.1.7 *Show that in the context of the proof of Theorem 1.1.6, if δ is Woodin in $L(V_\delta)$, then δ is singular in $J(V_\delta)$*

Exercise 1.1.8 *It is consistent with* ZFC *that there is a γ such that no $\eta \leq \gamma$ is Woodin in $L(V_\eta)$, and yet $L(V_\gamma)$ is not fully iterable. It is not true in M_1.*

It is true in $M_1[G]$ for almost all γ below the Woodin cardinal of M_1, with G generic for the full stationary tower. Thus in such a $L(V_\gamma)$, there are no proper class models with a Woodin of the form $L(V_\eta)$, and yet V is not fully iterable.

Let us state an abstract version of Theorem 1.1.6.

Definition 1.1.9 *Let F be a function on V. We say F has the* coarse condensation property *iff whenever $\pi\colon M \to V$ is elementary, where M is transitive, and $(P, F(P)) \in \mathrm{ran}(\pi)$, then*

$$\pi^{-1}(F(P)) = F(\pi^{-1}(P)).$$

We generally use this terminology when $F(P) = 0$ whenever P is not transitive, and for P transitive, $F(P)$ is transitive and $P \in F(P)$.

Definition 1.1.10 *Let F have the coarse condensation property. We say that P* is F-Woodin *iff P is a transitive model of* ZCP, *and no $A \in F(P)$ kills the Woodinness of P.*

The proof of Theorem 1.1.6 gives at once:

Theorem 1.1.11 *Let F be a function on V with the coarse condensation property, and suppose that for all $\eta \leq \delta$, V_η is not F-Woodin; then there is a unique iteration strategy for V with respect to set-sized trees based on V_δ.*

For example, if V is closed under the $M_n^\sharp$ function, then the function $F_n(P) =$ rudimentary closure of $M_n^\#(P) \cup \{M_n^\sharp(P)\}$ has the coarse condensation property. So V is fully iterable unless there is a δ which is Woodin in $M_n^\sharp(V_\delta)$. If there is such a δ, then V is iterable for trees based on V_{δ_0}, where δ_0 is the least such δ.

Exercise 1.1.12 *Show that the iteration strategy we have obtained for such trees yields an iteration strategy for $M_n^\sharp(V_{\delta_0})$, so that we have an iterable $M_{n+1}^\sharp$.*

Thus if V is closed under the $M_n^\sharp$ function, then either V is fully iterable for set-sized trees, or $M_{n+1}^\sharp$ exists. The fine-structural counterpart of this argument is a basic engine in the core model induction.

Where does one get functions with coarse condensation? Probably only from mouse operators of some kind. Precise conjectures along these lines will come later, as forms of the fundamental *Mouse Set Conjecture*. In a more abstract vein:

Exercise 1.1.13 *Given a function F on $H(\omega_1)$ whose codeset $F^* \subseteq \mathbb{R}$ is universally Baire, there is a function F^+ on V such that $F \subseteq F^+$, and F^+ has the coarse condensation property. Conversely, if F has the coarse condensation property and determines itself on generic extensions in a certain sense, then the codeset $(F \restriction H(\omega_1))^*$ is universally Baire.*

1.2 Counterexamples to uncountable iterability

Before we look at some applications of 1.6.15 and other similar arguments, let us collect some examples which show that we cannot expect that the full iterability of K^c will be provable in ZFC. The theme here is: Woodin cardinals are the enemy of full iterability.

(1) (H. Woodin) $M_1 \models$ "I am not $\delta^+ + 1$ iterable, where δ is my Woodin cardinal".

(2) (I. Neeman) $M_1 \models$ "I am not $\delta + 1$ iterable, where δ is my Woodin cardinal".

(3) (J. Steel) Let M be an iterable extender model satisfying $\mathsf{ZFC}^- +$ "δ is Woodin". Then $M \models$ "I am not $\delta^+ + 1$ iterable, where δ is my least Woodin cardinal".

The original argument here is Woodin's (1), of which the other two are variants. In each case it is a genericity iteration which cannot be exectuted in the model in question. Genericity iterations are a very important source of logically complicated iterations.

In contrast to (1)-(3), we have that if ξ is strictly less than the Woodin cardinal of M_1, then M_1 knows its own iteration strategy restricted to set length iteration trees based on $M_1|\xi$. This is because the $\mathcal{Q}$ structures for such a tree $\mathcal{T}$ is an initial segments of $L[\mathcal{M}(\mathcal{T})]$.

(4) (Woodin) Suppose there is a set-length iterable proper class model with a Woodin cardinal; then every set has a sharp. More generally, if there is a set-length iterable proper class model with $n+1$ Woodin cardinals, then for all sets x, $M_n^\sharp(x)$ exists and is set-length iterable.

This shows pretty clearly why we need $M_n^\sharp$-closure in order to build a fully iterable K^c reaching M_{n+1}. If we find a fully iterable M_{n+1}, then V was $M_n^\sharp$-closed.

(5) (J. Steel) Assume boldface $\mathbf{\Delta^1_2}$ determinacy; then for a Turing cone of x, $L[x] \models K^c$ is not Ω_x-iterable, where Ω_x is the least x-indiscernible, and K^c is built up to that point.

This comes out the proof in CMIP that $\mathbf{\Delta^1_2}$ determinacy implies for all reals x, $M_1^\sharp(x)$ exists and is ω_1-iterable. (The result itself is due to Woodin.)

(6) (J. Steel) Let M be a fully iterable, tame extender model, and suppose that $M \models \delta$ is Woodin. Then for no $\kappa \geq \delta$ do we have $(\kappa^+)^M = \kappa^+$.

So within the tame mice, an iterable K with weak covering and a Woodin cardinal is impossible. Unpublished results of Woodin show that it *is* possible just a bit past tame (i.e. below the $\mathsf{AD}_\mathbb{R}$ hypothesis).

1.3 F-mice and $K^{c,F}$.

In this section, we shall generalize the concept of a premouse by replacing the operator $x \mapsto \text{rud}(x \cup \{x\})$ (or, more generally, $x \mapsto \text{rud}_{\vec{B}}(x \cup \{x\})$ for some finite set $\vec{B}$) by an arbitrary *model operator*.

Definition 1.3.1 . *Let $\nu \geq \aleph_1$ be a cardinal or $\nu = \infty$, and let $A \in H_\nu$ be swo'd. A* model operator over A on H_ν *is a function F which sends every transitive rud-closed amenable model $\mathcal{M} = (|\mathcal{M}|; \in, A, E, B, S)$ with $A \in |\mathcal{M}| \in H_\nu$ and E, B, $S \subset |\mathcal{M}|$ to a transitive rud-closed amenable model $F(\mathcal{M}) = \mathcal{N} = (|\mathcal{N}|; \in, A, E, B')$ with $\mathcal{M} \in |\mathcal{N}| \in H_\nu$, $B' \subset |\mathcal{N}|$, and*

$$F(\mathcal{M}) = \text{Hull}_{\Sigma_1}^{F(\mathcal{M})}(|\mathcal{M}| \cup \{\mathcal{M}\}),$$

i.e., $F(\mathcal{M})$ is the Σ_1-hull generated from elements of $|\mathcal{M}|$ inside $F(\mathcal{M})$.

For technical reasons we demand $F(\mathcal{M})$ to have a different type than $\mathcal{M}$ has. All the models in the domain (and in the range) of F have A as an element as well as a distigiuished predicate available. In what follows, we shall refer to models in the domain (or in the range) of F as *models over A*. Examples for F which we have in mind are (beyond $x \mapsto \text{rud}(x \cup \{x\})$ or $x \mapsto \text{rud}_{\vec{B}}(x \cup \{x\})$) the following ones.

(1) *Mouse operators.* For $\mathcal{M}$ an ordinary sound premouse, $F(\mathcal{M})$ = the n^{th} reduct of some ordinary premouse $\mathcal{N} \rhd \mathcal{M}$ with $\rho_{n+1}(\mathcal{N}) \leq \mathcal{M} \cap \text{OR} < \rho_n(\mathcal{N})$ such that $\mathcal{N}$ is $(n+1)$-sound above $\mathcal{M} \cap \text{OR}$ (and where we assume

the first standard parameter of this reduct to be coded into its master code). In this case, we shall actually think of $F(\mathcal{M})$ as being $\mathcal{N}$ itself rather that its n^{th} reduct.

(2) *Hybrid mouse operators.* For $\mathcal{M}$ being a "hybrid" premouse having some fixed ordinary sufficiently iterable premouse $\mathcal{N}$ as an element, $F(\mathcal{M}) =$ some code for a structure which feeds in information about the iteration strategy of $\mathcal{N}$.

We shall only be interested in model operators with condensation in the following sense.

Definition 1.3.2 *Let ν, A, F be as in Definition 1.3.1. Set $\kappa = \mathrm{Card}(A) \cdot \aleph_0$. We then say that F* condenses well *iff the following holds true.*

Let $\bar{\mathcal{M}}$, $\mathcal{M} \in H_\nu$ be models over A such that $\mathrm{Card}^V(|\bar{\mathcal{M}}|) = \kappa$. Let G be $\mathrm{Col}(\omega, \kappa)$-generic over V, and let $\bar{\mathcal{M}}^+ \in V[G]$ be a model over A with $\bar{\mathcal{M}} \in |\bar{\mathcal{M}}^+|$ and $\bar{\mathcal{M}}^+ = \mathrm{Hull}_{\Sigma_1}^{\bar{\mathcal{M}}^+}(|\bar{\mathcal{M}}|)$. Suppose that either

(1) there is a map $\pi\colon \bar{\mathcal{M}}^+ \to F(\mathcal{M})$, $\pi \in V[G]$, with $\pi(\bar{\mathcal{M}}) = \mathcal{M}$ and $\pi \restriction (A \cup \{A\}) = \mathrm{id}$ which is Σ_0-cofinal or Σ_2-elementary, or else

(2) there is some $F(\mathcal{P}) \in H_\nu$, where $\mathcal{P}$ is a model over A, and there are maps $i\colon F(\mathcal{P}) \to \bar{\mathcal{M}}^+$ and $\pi\colon \bar{\mathcal{M}}^+ \to F(\mathcal{M})$, i, $\pi \in V[G]$, $\pi \circ i \in V$, with $i(\mathcal{P}) = \bar{\mathcal{M}}$, $\pi(\bar{\mathcal{M}}) = \mathcal{M}$, $i \restriction (A \cup \{A\}) = \pi \restriction (A \cup \{A\}) = \mathrm{id}$, i is Σ_0-cofinal or Σ_2-elementary, and π is a weak Σ_1-embedding.

Then $\bar{\mathcal{M}}^+ = F(\bar{\mathcal{M}}) \in V$.

A straightforward absoluteness argument shows that if $\kappa \leq \aleph_0$, then in Definition 1.3.2 we might have equivalently restricted ourselves to $\bar{\mathcal{M}}^+$, π, i existing in V (cf. Exercise 1.3.3).

In the general case (i.e., with κ being arbitrary) there is a reformulation of the concept of "condenses well" in terms of an embedding game which avoids having to step into $V[G]$. Given $\mathcal{M}$ and $\bar{\mathcal{M}}$ as in Definition 1.3.2, let $\mathcal{G}(\mathcal{M}, \bar{\mathcal{M}})$ denote the following game, in which two players I and II alternate playing $k_0 \in \omega$ and sets a_n and b_n as follows.

I	k_0, a_0, b_0		a_2, b_1, b_2		a_4, b_3, b_4	$\ldots$	
II		a_1		a_3			$\ldots$

Letting $(\varphi_k\colon k < \omega)$ be a redundant enumeration of all Σ_2-formulae in which every Σ_2-formula is mentioned ω times, the rules of the game are:

(a) $a_k \in |\bar{\mathcal{M}}|$ and $b_k \in |\mathcal{M}|$.

(b) If there is some $x \in |\mathcal{M}|$ such that $F(\mathcal{M}) \models \varphi_k(b_0, \ldots, b_{2k-1}, x)$, then $F(\mathcal{M}) \models \varphi_k(b_0, \ldots, b_{2k-1}, b_{2k})$.

(c) $\bar{\mathcal{M}} \models \varphi_k(a_0, \ldots, a_k)$ iff $\mathcal{M} \models \varphi_k(b_0, \ldots, b_k)$.

(d) $F(\bar{\mathcal{M}}) \models \varphi_{k_0}(a_0)$ holds true iff $F(\mathcal{M}) \models \varphi_{k_0}(b_0)$ does not hold true.

The first player to break one of these rules loses; if no player breaks a rule, then I wins. Notice that $\mathcal{G}(\mathcal{M}, \bar{\mathcal{M}})$ is a closed game.

Playing in $V[G]$, where G is $\mathrm{Col}(\omega, \kappa)$-generic over V, player I can easily arrange that $|\bar{\mathcal{M}}| = \{a_k | k \in \omega\}$, so that if I wins, then $a_k \mapsto b_k$ defines a Σ_2-elementary embedding from $\bar{\mathcal{M}}$ to $\mathcal{M}$ such that $\{b_k | k \in \omega\} = X \cap |\mathcal{M}|$, where $X = \mathrm{Hull}_{\Sigma_2}^{F(\mathcal{M})}(\{b_k | k \in \omega\})$. By clause (d), X does then not collapse to $F(\bar{\mathcal{M}})$. It is now not hard to show that there is a counterexample to F condensing well, with (1) of Definition 1.3.2 holding true and π being Σ_2-elementary, if and only if player I wins $\mathcal{G}(\mathcal{M}, \bar{\mathcal{M}})$ (cf. Exercise 1.3.4). In much the same way all of Definition 1.3.2 may be reformulated in terms of having I lose games similar to $\mathcal{G}(\mathcal{M}, \bar{\mathcal{M}})$.

Let ν, A, F, and κ be as in Definition 1.3.2. If $\nu > \kappa^+$, then F is completely determined by $F \restriction H_{\kappa^+}$. For the same reason, if $\lambda > \nu$, then there is at most one model operator over A on H_λ which extends F and condenses well. (Cf. exercise 1.3.5.)

Exercise 1.3.3 *Show that if A is at most countable, then the model operator F over A condenses well iff the criterion from Definition 1.3.2 is satisfied with $V[G]$ being replaced by V.*

Exercise 1.3.4 *Show that for an arbitrary A, if F is a model operator over A, then there is a reformulation of "condenses well" in terms of games as in the discussion after Definition 1.3.2.*

Exercise 1.3.5 *If $\lambda > \nu$, then there is at most one model operator over A on H_λ which extends F and condenses well.*

Let us now turn towards defining the concept of an "F-premouse." If ν, A, F are as in Definition 1.3.1, then an F-premouse will be a model of the form

$$\mathcal{M} = (|\mathcal{M}|; \in, A, \vec{E}, B, \vec{\mathcal{M}}) \tag{1.1}$$

over A, where

(a) $\vec{\mathcal{M}} = (\mathcal{M}_i : i < \theta)$ is a sequence of models over A, which will be informally referred to as the "history" of $\mathcal{M}$, and

(b) $\vec{E}$ codes a fine extender sequence $(E_\alpha \mid \alpha \in \text{dom}(\vec{E}))$ in the spirit of [41, Definition 2.4], where $\text{dom}(\vec{E}) \subset \theta + 1$.

Definition 1.3.6 *Let $\nu \geq \aleph_1$ be a cardinal, let $A \in H_\nu$ be swo'd, and let F be a model operator over A on H_ν. A model $\mathcal{M}$ as in (1.1) is called a* potential F-premouse *provided the following hold true.*

1. *$\vec{\mathcal{M}} = (\mathcal{M}_i : i < \theta)$, a sequence of models over A. In order to simplify the statement of the following clauses, let us also write $\mathcal{M}_\theta$ for $\mathcal{M}$ itself.*
2. *$\vec{E}$ codes a sequence $(E_\alpha \mid \alpha \in \text{dom}(\vec{E}))$ of extenders, where $\text{dom}(\vec{E}))$ is a subset of the limit ordinals in $\theta + 1$.*
3. *Let $i \leq \theta$. Then $\mathcal{M}_i = (|\mathcal{M}_i|; \in, A, \vec{F}_i, B_i, (\mathcal{M}_j | j < i))$, some $\vec{F}_i$,B_i, and in fact $\vec{F}_i$ codes $(E_\alpha \mid \alpha \in \text{dom}(\vec{E}) \cap (i+1))$.*
4. *Let $i + 1 \leq \theta$. Then $F(\mathcal{M}_i) = (|\mathcal{M}_{i+1}|; \in, A, \vec{F}_{i+1}, B_{i+1})$. (I.e., $\mathcal{M}_{i+1}$ is the expansion obtained from $F(\mathcal{M}_i)$ by adding its "history.")*
5. *Let $\lambda \leq \theta$ be a limit ordinal. Then $B_\lambda = \emptyset$.*
6. *The sequence $(E_\alpha \mid \alpha \in \text{dom}(\vec{E}))$ codes a fine extender sequence in the sense of [41, Definition 2.4] (cf. also [22, Definition 1.0.4] and [33, Definition 2.4]).[8]*

In this situation, $\mathcal{M}_0$ is called the base model *of $\mathcal{M}$.*

In the case of F being the appropriate rud-closure operator, an F-premouse $\mathcal{M}$ is basically an ordinary potential premouse in the sense of [22, Definition 1.0.5]. In this simple case, the "history" of $\mathcal{M}$ is Σ_1-definable over the reduct of $\mathcal{M}$ obtained by removing the "history," so that the relation $\vec{\mathcal{M}}$ of $\mathcal{M}$ can be dispensed with.[9] Notice that by amenability, if $i < \theta$, then $(\mathcal{M}_j : j < i) \in |\mathcal{M}|$, i.e., the initial segments of the "history" are elements of $\mathcal{M}$.

In much the same way as in [22, 3.5.1], F-premice are supposed to be potential F-premice all of whose proper initial segments are sound. In order to see that there is a reasonable fine structure theory for potential F-premice, we need the following straightforward consequence of Definition 1.3.2.

[8] Of course, this definition has to be adjusted to the present context in that if $\lambda \in \text{dom}(\vec{E})$, then the relevant initial segment of $\mathcal{M}$ to consider in the coherence and closure under initial segment conditions is $(\mathcal{M}_\lambda; (\mathcal{M}_i \mid i < \lambda))$.

[9] In fact, this will be true for all the examples considered in this book.

Lemma 1.3.7 (Condensation for F-premice) *Let ν, B, F be as in Definition 1.3.6, and let $\mathcal{M} \in H_\nu$ be a potential F-premouse. Let G be $\mathrm{Col}(\omega, \kappa)$-generic over V, where $\kappa = \mathrm{Card}(A) \cdot \aleph_0$. Let $\bar{\mathcal{M}} \in H_\nu$ be a model over A such that either*

(1) there is a map $\pi\colon \bar{\mathcal{M}} \to \mathcal{M}$, $\pi \in V[G]$, with $\pi \restriction (A \cup \{A\}) = \mathrm{id}$ which is Σ_0-cofinal or Σ_2-elementary, or else

(2) there is some $\mathcal{P} \in H_\nu$, where $\mathcal{P}$ is a potential F-premouse, and maps $i\colon \mathcal{P} \to \bar{\mathcal{M}}$ and $\pi\colon \bar{\mathcal{M}} \to \mathcal{M}$, i, $\pi \in V[G]$, $\pi \circ i \in V$, with $i \restriction (A \cup \{A\}) = \pi \restriction (A \cup \{A\}) = \mathrm{id}$, i is Σ_0-cofinal or Σ_2-elementary, and π is a weak Σ_1-embedding.

Then $\bar{\mathcal{M}}$ is a potential F-premouse.

A trivial consequence of the Condensation Lemma 1.3.7 is that if $\mathcal{M}$ is a potential F-premouse with base model $\mathcal{M}_0$, then

$$\mathcal{M} = \mathrm{Hull}^{\mathcal{M}}_{\Sigma_1}(|\mathcal{M}_0| \cup (|\mathcal{M}| \cap \mathrm{OR})),$$

i.e., $\mathcal{M}$ is the Σ_1-hull generated inside $\mathcal{M}$ from the ordinals in $\mathcal{M}$ together with elements of $\mathcal{M}_0$. In fact, on the basis of Lemma 1.3.7, a full fine structure theory for potential F-premice may be developped in much the same way as for J-structures.

All our potential F-premice which we'll consider in the future will actually be J-structures, for which a fine structure theory is presented in [34]. We therefore here refrain from making the development of a fine structure theory for potential F-premice explicit.

Definition 1.3.8 *Let ν, A, F be as in Definition 1.3.6, where F condenses well. Let $\mathcal{M}$ be a potential F-premouse as in (1.1) and Definition 1.3.6. Then $\mathcal{M}$ is called an F–*premouse *iff for every $i < \theta$, $\mathcal{M}_i$ is sound above $\mathcal{M}_0$.*

The easiest example of an F-premouse is L^F, the least transitive model closed under F. More precisely:

Definition 1.3.9 *Let $\nu \geq \aleph_1$ be a cardinal or $\nu = \infty$, and let $A \in H_\nu$ be swo'd. Suppose that F is a model operator over A on H_ν which condenses well. Let $\mathcal{P} = (|\mathcal{P}|; \in, B)$ be a transitive and rud-closed amenable model such that $A \in |\mathcal{P}|$.*

We then denote by $L^F_\nu(\mathcal{P})$ the unique F-premouse of height ν with base model $\mathcal{P}$ which does not have any extender on its $\vec{E}$-sequence. We also write $L^F(\mathcal{P})$ for $L^F_\infty(\mathcal{P})$.

The Condensation Lemma 1.3.7 for F-premice will also be the key tool for showing that the $K^{c,F}$-construction succeeds, unless M_1^F, the least "F-mouse" with a Woodin cardinal, exists. Namely, we are now going to inductively define F-premice $\mathcal{N}_\xi$ in much the same way as in [41, Definition 6.3], except that we start with some transitive and rud-closed amenable model $\mathcal{P} = (|\mathcal{P}|; \in, B)$ such that $A \in |\mathcal{P}|$ rather than with $(V_\omega; \in)$ and that we use F rather than $x \mapsto \mathrm{rud}_{\vec{B}}(x \cup \{x\})$, some $\vec{B}$, at "successor" stages of the construction. More precisely:

Definition 1.3.10 *Let $\nu \geq \aleph_1$ be a cardinal or $\nu = \infty$, and let $A \in H_\nu$ be swo'd. Suppose that F is a model operator over A on H_ν which condenses well. Let $\mathcal{P} = (|\mathcal{P}|; \in, B) \in H_\nu$ be a transitive and rud-closed amenable model such that $A \in |\mathcal{P}|$.*

A $K^{c,F}(\mathcal{P})$–construction is a sequence $(\mathcal{N}_\xi \; | \xi \leq \theta)$ of F-premice in H_ν such that the following hold true.

1. $\mathcal{N}_0 = (|\mathcal{P}|; \in, A, \emptyset, B, \emptyset)$.

2. *Let $\xi + 1 < \theta$. Then $\mathcal{N}_\xi$ is ω-solid, and letting*[10]

$$\mathcal{M} = \mathfrak{C}_\omega(\mathcal{N}_\xi) = (|\mathcal{M}|; \in, A, \vec{E}, B', \vec{\mathcal{M}})$$

and $\gamma = |\mathcal{M}| \cap \mathrm{OR}$, either

   (a) $\mathcal{M}$ is passive, i.e., $\gamma \notin \mathrm{dom}(\vec{E})$, and

$$\mathcal{N}_{\xi+1} = (|\mathcal{M}|; \in, A, \vec{E}^\frown E_\gamma, B', \vec{\mathcal{M}}),$$

   where E_γ is countably certified in the sense of [41, Definition 6.2], i.e., $\mathcal{N}_{\xi+1}$ is obtained from $\mathcal{M}$ by adding a countably certified top extender, or else

   (b) if $F(\mathcal{M}) = (|F(\mathcal{M})|; \in, A, \vec{E}, B')$, some B', then

$$\mathcal{N}_{\xi+1} = (|F(\mathcal{M})|; \in, A, \vec{E}, B', \vec{\mathcal{M}}^\frown \mathcal{M}),$$

   i.e., $\mathcal{N}_{\xi+1}$ is the "reorganization of $F((\mathcal{M})$ as an F-premouse.

[10] When forming $\mathfrak{C}_\omega(\mathcal{N}_\xi)$, we throw in $|\mathcal{P}|$ into the hull. By Lemma 1.3.7, $\mathfrak{C}_\omega(\mathcal{N}_\xi)$ is thus an F-premouse with base model $\mathcal{N}_0$.

3. *Let $\lambda < \theta$ be a limit cardinal. Let, for $\xi \leq \lambda$,*

$$\mathcal{N}_\xi = (|\mathcal{N}_\xi|; \in, A, \vec{E}_\xi, B_\xi, \vec{\mathcal{M}}_\xi),$$

where $\vec{E}_\xi$ is of the form $(E^\xi_\alpha | \alpha \in \mathrm{dom}(\vec{E}_\xi))$ and $\vec{\mathcal{M}}_\xi)$ is of the form $(\vec{\mathcal{M}}^\xi_\alpha | \alpha < \theta^\xi)$. Then $B_\lambda = \emptyset$, and for all ξ, $\xi < \theta^\lambda$ and $E^\lambda_\xi = E$ iff for all but boundedly many $i < \lambda$, $E^i_\xi = E$, and similarily $\mathcal{M}^\lambda_\xi = \mathcal{N}$ iff for all but boundedly many $i < \lambda$, $\mathcal{M}^i_\xi = \mathcal{N}$.

The simplest case of a $K^{c,F}(\mathcal{P})$-construction is one which produces $L^F_\nu(\mathcal{P})$, but of course in general there are more complicated ones.

The following is parallel to Theorem 1.1.1.

Theorem 1.3.11 *Let $\nu \geq \aleph_1$ be a cardinal or $\nu = \infty$, and let $A \in H_\nu$ be swo'd. Suppose that F is a model operator over A on H_ν which condenses well. Set $\kappa = \mathrm{Card}(A) \cdot \aleph_0$. Let $\mathcal{P} = (|\mathcal{P}|; \in, B) \in H_\nu$ be a transitive and rud-closed amenable model with $A \in |\mathcal{P}|$. Let $\gamma \geq \kappa$ be a cardinal, $\gamma < \nu$, and let $(\mathcal{N}_\xi \mid \xi \leq \theta)$ be a $K^{c,F}(\mathcal{P})$-construction with additivity $\geq \gamma^+$,[11] where $\theta \leq \nu$. Let $\mathcal{N}_\xi$ be k-sound, let $\bar{\mathcal{N}}$ be a k-sound F-premouse of size at most γ, and let $\pi \colon \bar{\mathcal{N}} \to \mathcal{N}_\xi$ be a weak k-embedding with $\pi \restriction (A \cup \{A\}) = \mathrm{id}$. Let $\mathcal{T}$ be a normal iteration tree on $\bar{\mathcal{N}}$ of length $< \gamma^+$. Let G be $\mathrm{Col}(\omega, \gamma)$-generic over V.*

Then either

1. *$\bar{\mathcal{T}}$ has successor length $\gamma+1$, and if $\mathcal{M}^{\bar{\mathcal{T}}}_\gamma$ is ℓ-sound, then in $V[G]$ there is a weak ℓ-embedding $\sigma \colon \mathcal{M}^{\bar{\mathcal{T}}}_\gamma \to \mathcal{N}_{\bar{\xi}}$ for some $\bar{\xi} \leq \xi$, or else*

2. *$\bar{\mathcal{T}}$ has limit length and in $V[G]$ there is a* maximal *branch b through $\bar{\mathcal{T}}$ such that if $\mathcal{M}^{\bar{\mathcal{T}}}_b$ is ℓ-sound, in $V[G]$ there is a weak ℓ-embedding $\sigma \colon \mathcal{M}^{\bar{\mathcal{T}}}_b \to \mathcal{N}_{\bar{\xi}}$, where again $\bar{\xi} \leq \xi$, such that $\sigma \circ i_b = \pi$.*

This theorem is shown by the method of [37, §9].

We aim to use Theorem 1.3.11 together with an adaptation of Theorem 1.1.2, namely Theorem 1.3.13, to the current context to show that all models from a $K^{c,F}(\mathcal{P})$-construction are "sufficiently iterable" under favorable circumstances.

[11] I.e., if $\bar{\xi} \leq \xi$ and $\mathcal{N}_{\bar{\xi}}$ has a top extender, E_μ say, then $\mathrm{crit}(E_\mu) \geq \gamma^+$.

Definition 1.3.12 *Let $\mathcal{M}$ be an F-premouse as in (1.1). Then $\mathcal{M}$ is called* F–small *iff for all $\alpha \in \mathrm{dom}(\vec{E})$,*

$$\mathcal{M}_\alpha \models \text{ “there is no Woodin cardinal.” }[12]$$

Theorem 1.3.13 *Let ν, A, F, κ, γ, $\mathcal{N}$, $(\mathcal{N}_\xi \mid \xi \leq \theta)$ be as in Theorem 1.3.11. Suppose that every $\mathcal{N}_\xi$, $\xi \leq \theta$, is F-small, and that $\mathcal{N}_\xi$ does not have a definable Woodin cardinal.*[13] *Let $\mathcal{N}_\xi$ be k-sound, let $\bar{\mathcal{N}}$ be a k-sound F-premouse of size at most γ, and let $\pi\colon \bar{\mathcal{N}} \to \mathcal{N}_\xi$ be a weak k-embedding with $\pi \restriction (A \cup \{A\}) = \mathrm{id}$. Let $\mathcal{T}$ be a normal iteration tree on $\bar{\mathcal{N}}$ of* limit *length $< \gamma^+$. Let G be $\mathrm{Col}(\omega, \gamma)$-generic over V.*

In $V[G]$ there is then at most one cofinal *branch b through $\bar{\mathcal{T}}$ such that if $\mathcal{M}_b^{\bar{\mathcal{T}}}$ is ℓ-sound, then in $V[G]$ there is a weak ℓ-embedding $\sigma\colon \mathcal{M}_b^{\bar{\mathcal{T}}} \to \mathcal{N}_{\bar{\xi}}$, where $\bar{\xi} \leq \xi$, such that $\sigma \circ i_b = \pi$.*

Proof. Assume b_0, b_1 to be two such cofinal branches. For $h = 0, 1$, let $\mathcal{Q}_h$ be the least initial segment $\mathcal{Q}$ of $\mathcal{M}_{b_h}^{\bar{\mathcal{T}}}$ such that $\delta(\bar{\mathcal{T}})$ is not definably Woodin in $\mathcal{Q}$. By the Condensation Lemma 1.3.7, both $\mathcal{Q}_0$ and $\mathcal{Q}_1$ are F-premice (here we might use (2) of Lemma 1.3.7). By our hypotheses, both $\mathcal{Q}_0$ and $\mathcal{Q}_1$ are F-small, so that neither $\mathcal{Q}_0$ nor $\mathcal{Q}_1$ has any extenders above $\delta(\bar{\mathcal{T}})$. This implies that $\mathcal{Q}_0$ and $\mathcal{Q}_1$ are both initial segments of $L_\nu^F(\mathcal{M}(\bar{\mathcal{T}}))$, so that in fact $\mathcal{Q}_0 = \mathcal{Q}_1$.

This implies that $b_0 = b_1$ by standard arguments (cf. Theorem 1.1.2; for a proof cf. [36]). □

We refer the reader to [41, Definition 4.4] on the concept of (k, λ, θ)-iteration strategies.

Definition 1.3.14 *Let $\mathcal{M}$ be an F-premouse. Then $\mathcal{M}$ is (k, λ, θ)–F–*iterable *(or, $(\mathcal{M}, F)$ is (k, λ, θ)-*iterable*) iff there is a (k, λ, θ)-iteration strategy Σ for $\mathcal{M}$ such that if $\mathcal{M}^*$ is an iterate of $\mathcal{M}^*$ according to Σ, then $\mathcal{M}^*$ is an F-premouse.*

The following is a corollary to Theorems 1.3.11 and 1.3.13.

Corollary 1.3.15 *Let ν, A, F, κ, γ, $\mathcal{N}$, $(\mathcal{N}_\xi \mid \xi \leq \theta)$ be as in Theorem 1.3.13. Suppose that $\mathcal{N}_\xi$ does not have a definable Woodin cardinal. Let*

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[12] This is equivalent to saying that $\mathcal{M}_\alpha | \mathrm{crit}(E_\alpha^{\mathcal{M}_\alpha}) \models$ there is no Woodin cardinal.

[13] I.e., if $\delta \leq \mathcal{N}_\xi$, then either $\rho_\omega(\mathcal{N}_\xi) < \delta$ or else there is some $n < \omega$ and some $r\Sigma_n^{\mathcal{N}_\xi}$-definable counterexample to the Woodinness of δ.

$\mathcal{N}_\xi$ be k-sound, let $\bar{\mathcal{N}}$ be a k-sound F-premouse of size at most γ, and let $\pi : \bar{\mathcal{N}} \to \mathcal{N}_\xi$ be a weak k-embedding with $\pi \restriction (A \cup \{A\}) = \mathrm{id}$.

Then $\bar{\mathcal{N}}$ is (k, γ^+, γ^+)-iterable.

Proof. Let Σ be the following strategy for iterating $\bar{\mathcal{N}}$. Let $\mathcal{T}$ be a tree of limit length on $\bar{\mathcal{N}}$ as in the definition of (k, γ^+, γ^+)-iterability. We then set $b = \Sigma(\mathcal{T})$ iff b is the unique cofinal branch c through $\mathcal{T}$ such that if $\mathcal{M}_c^\mathcal{T}$ is ℓ-sound, then there is a weak ℓ-embedding $\sigma : \mathcal{M}_c^\mathcal{T} \to \mathcal{N}_{\bar{\xi}}$, where $\bar{\xi} \leq \xi$, such that $\sigma \circ i_b = \pi$. (Σ is often referred to as the "realizing strategy.") We need to see that Σ is total on trees of limit length on $\bar{\mathcal{N}}$ as in the definition of (k, γ^+, γ^+)-iterability, and that if $\mathcal{T}$ is a putative tree on $\bar{\mathcal{N}}$ according to Σ and of successor length, then the last model of $\mathcal{T}$ is well-founded.

Suppose this were false, and let $\mathcal{T}$ be a counterexample of minimal length. By Theorem 1.3.11, $\mathcal{T}$ must then have limit length, and in $V[G]$, where G is $\mathrm{Col}(\omega, \gamma)$-generic over V, there is a maximal branch b through $\mathcal{T}$ such that if $\mathcal{M}_b^\mathcal{T}$ is ℓ-sound, there is a weak ℓ-embedding $\sigma : \mathcal{M}_b^\mathcal{T} \to \mathcal{N}_{\bar{\xi}}$, where $\bar{\xi} \leq \xi$, such that $\sigma \circ i_b = \pi$. By the minimality of $\mathcal{T}$, b must in fact be cofinal (rather than maximal and non-cofinal). But then by Theorem 1.3.13, b is unique with the properties as stated. Hence by the homogeneity of $\mathrm{Col}(\omega, \kappa)$, $b \in V$, and of course b is still unique in V with the desired properties. This shows that $\Sigma(\mathcal{T})$ is well-defined after all. Contradiction! □

Definition 1.3.16 *Let ν, A, F, κ, $\mathcal{P}$, $(\mathcal{N}_\xi \mid \xi \leq \theta)$ be as in Theorem 1.3.13. Let Γ be a set or class of regular cardinals $\geq \kappa^+$. $(\mathcal{N}_\xi \mid \xi \leq \theta)$, with $\theta = \nu$, is then called a* maximal $K^{c,F}(\mathcal{P})$-construction with additivity in Γ of height ν for an inner model with an F-Woodin cardinal *iff for all $\xi \leq \theta$, if*

$$\mathcal{M} = \mathfrak{C}_\omega(\mathcal{N}_\xi) = (|\mathcal{M}|; \in, A, \vec{E}, B', \vec{\mathcal{M}})$$

is passive and there is a countably certified extender E with critical point μ with $\mathrm{cf}^V(\mu) \in \Gamma$ such that

$$(|\mathcal{M}|; \in, A, \vec{E}^\frown E, B', \vec{\mathcal{M}})$$

is an F-small F-premouse, then $E_{|\mathcal{M}| \cap \mathrm{OR}} \neq \emptyset$ is a countably certified extender whose critical point has cofinality in Γ, i.e., $\mathcal{N}_{\xi+1}$ is obtained from $\mathcal{M}$ by adding some such extender.

If $\Gamma \subset (\nu \setminus \kappa^{++})$, then by Corollary 1.3.15 we may inductively prove that each standard parameter of every $\mathfrak{C}_\omega(\mathcal{N}_\xi)$ is solid and universal, that there is in fact exactly one maximal $K^{c,F}(\mathcal{P})$-construction over $\mathcal{P}$ with additivity

$\geq \gamma^+$ of height ν for an inner model with an F-Woodin cardinal (cf. [41, §6.2]).

Definition 1.3.17 *Let ν, A, F, κ, $\Gamma \subset (\nu \setminus \kappa^{++})$, $\mathcal{P}$ be as in Definition 1.3.16. We shall then write (somewhat ambiguously) $K^{c,F}(\mathcal{P})|\nu$, for $\mathcal{N}_\nu$, where $(\mathcal{N}_\xi \mid \xi \leq \nu)$ is the unique maximal $K^{c,F}(\mathcal{P})$-construction with additivity in Γ of height ν for an inner model with an F-Woodin cardinal*

We shall now be interested in isolating $K^F(\mathcal{P})$, the "true" F-core model over $\mathcal{P}$. In order for this to work out we need to see that $K^{c,F}(\mathcal{P})|\nu$ is "fully iterable" in a sense to be made precise. In order to develop the theory of $K^F(\mathcal{P})$, it is useful (but not necessary, cf. [11]) to assume that ν, henceforth written Ω, be an ineffable cardinal. The weak compactness of Ω will easily yield that if $K^{c,F}(\mathcal{P})|\Omega$ is (ω, Ω, Ω)-iterable, then it is $(\omega, \Omega, \Omega+1)$-iterable also, and the subtlety of Ω will easily yield that $K^{c,F}(\mathcal{P})|\Omega$ satisfies "cheapo" covering (cf. Exercise 1.3.23).

Definition 1.3.18 *Let Ω be an ineffable cardinal. Let $A \in H_\Omega$ be swo'd, and suppose that F is a model operator over A on H_ν which condenses well. Set $\kappa = \mathrm{Card}(A)$. Let $\mathcal{P} = (|\mathcal{P}|; \in, B) \in H_\nu$ be a rud-closed transitive model with $A \in |\mathcal{N}|$. Let $K^{c,F}(\mathcal{P})$ be the maximal $K^{c,F}(\mathcal{P})$–construction with additivity Γ, where Γ is some non-empty set of regular uncountable cardinals between κ^+ and Ω. Let $\xi \leq \Omega$, and let $\mathcal{T} \in V_\Omega$[14] be an iteration tree on $\mathcal{N}_\xi$ as in the definition of (ω, Ω, Ω)-iterability.*

If b is a cofinal branch through $\mathcal{T}$, then the $\mathcal{Q}$-structure for b *(if it exists, in which case it is unique) is the least initial segment $\mathcal{Q} = \mathcal{Q}(b, \mathcal{T})$ of $\mathcal{M}_b^{\mathcal{T}}$ such that $\delta(\mathcal{T})$ is not definably Woodin in $\mathcal{Q}$. We say that $\mathcal{T}$ is* guided by L^F *iff for every limit ordinal $\lambda < \mathrm{lh}(\mathcal{T})$, $[0, \lambda)_{\mathcal{T}}$ is the unique cofinal branch b through $\mathcal{T}$ such that $\mathcal{Q}(b, \mathcal{T}) \lhd L^F(\mathcal{M}(\mathcal{T}))$.*

If Σ is a (possibly partial) iteration strategy for $\mathcal{N}_\xi$, then we say that Σ is guided by L^F *iff for every iteration tree $\mathcal{T} \in V_\Omega$ on $\mathcal{N}_\xi$ as in the definition of (ω, Ω, Ω)-iterability and which is guided by L^F, if $\Sigma(\mathcal{T})$ is defined, then the extension of $\mathcal{T}$ by $\Sigma(\mathcal{T})$ is guided by L^F also.*

Definition 1.3.19 *We say that an iteration strategy Σ for $\mathcal{N}_\xi$ which is guided by L^F* produces an F-closed model with a Woodin cardinal *iff there is some iteration tree $\mathcal{T} \in V_\Omega$ on $\mathcal{N}_\xi$ according to Σ and as in the definition*

[14]I.e., the tree order of $\mathcal{T}$ is in V_Ω

of (ω,Ω,Ω)-iterability such that

$$L^F(\mathcal{M}(\mathcal{T})) \models \text{“}\,\delta(\mathcal{T}) \text{ is a Woodin cardinal.”}$$

Finally, we say that the $K^{c,F}(\mathcal{P})$-construction reaches $\mathcal{P}^{F\#}$ *iff there is an extender on the sequence of $K^{c,F}(\mathcal{P})$; in this case we write $\mathcal{P}^{F\#}$ for $\mathfrak{C}_\omega(\mathcal{N}_\xi)$, where $\xi < \Omega$ is least such that $\mathcal{N}_\xi$ has an extender (which will then be the unique top extender). We also say that the $K^{c,F}(\mathcal{P})$-construction* reaches $M_1^F(\mathcal{P})$ *iff there is some $\xi < \Omega$ and some $\delta \in \mathcal{N}_\xi$ such that*

$$\mathcal{N}_\xi \models \text{“}\,\delta \text{ is a Woodin cardinal;”} \tag{1.2}$$

in this case we write $M_1^F(\mathcal{P})$ for $\mathfrak{C}_\omega(\mathcal{N}_\xi)$, where $\xi < \Omega$ is least such that (1.2) holds.

For sake of illustration, let us give some examples. If F is the usual rud-closure operator, then $L^F(\mathcal{M}) = L(\mathcal{M})$, $\mathcal{P}^{F\#} = \mathcal{P}^\#$, and $M_1^F(\mathcal{P}) = M_1(\mathcal{P})$ is the least iterable premouse of height $\delta+\omega$, some δ, in which δ is Woodin. If F is the $M_n^\#$-operator, $n < \omega$,[15] then $L^F(\mathcal{M})$ is the least model containing $\mathcal{M}$ which is closed under $M_n^\#$, $\mathcal{P}^{F\#}$ is the least active iterable $\mathcal{P}$-premouse which is closed under $M_n^\#$, and $M_1^F(\mathcal{P}) = M_{n+1}^\#$.

We may now state our two K^F-Existence Dichotomies. The first one is "global" and presupposes the existence of an ineffable cardinal. The second one is more "local" but also often more useful (in fact the second one subsumes the first one).

Of course there is a unique maximal iteration strategy for $K^{c,F}(\mathcal{P})$ which is guided by L^F. The reason is that if $\mathcal{T}$ is guided by L^F, then there is at most one cofinal branch b through $\mathcal{T}$ such that the $\mathcal{Q}$-structure $\mathcal{Q}$ for b is an initial segment of $L^F(\mathcal{M}(\mathcal{T}))$ (cf. the proof of Theorem 1.3.13). If this maximal strategy is not total with respect to trees on $K^{c,F}(\mathcal{P})$ which exist in V_Ω, are guided by L^F, and are as in the definition of (ω,Ω,Ω)-iterability, then this strategy will produce an F-closed model with a Woodin cardinal.

Theorem 1.3.20 (K^F-Existence Dichotomy) *Let Ω be an ineffable cardinal. Let $A \in H_\Omega$ be swo'd, and suppose that F is a model operator over A on H_ν which condenses well. Set $\kappa = \mathrm{Card}(A)$. Let $\mathcal{P} = (|\mathcal{P}|;\in,B) \in H_\nu$ be a rud-closed transitive model with $A \in |\mathcal{N}|$. Let us write $K^{c,F}(\mathcal{P})$ for $K_\Gamma^{c,F}(\mathcal{P})|\Omega$, where Γ is a non-empty set of regular uncountable cardinals between κ^+ and Ω. Let Σ be the unique maximal (albeit possibly partial)*

[15]If $n = 0$, then the $M_n^\#$-operator is just the $\#$-operator.

iteration strategy for $K^{c,F}(\mathcal{P})$ which is guided by L^F. Then the following hold true.

1. *If Σ produces an F-closed model with a Woodin cardinal, then the $K^{c,F}(\mathcal{P})$-construction reaches $M_1^F(\mathcal{P})$ and $M_1^F(\mathcal{P})$ ist $(\omega, \Omega, \Omega+1)$-iterable via the unique strategy which is guided by L^F.*

2. *If Σ does not produce an F-closed model with a Woodin cardinal, then $K^{c,F}(\mathcal{P})$ is $(\omega, \Omega, \Omega+1)$-iterable. This implies that $K^F(\mathcal{P})$ exists and is $(\omega, \Omega, \Omega+1)$-iterable as being witnessed by a strategy which is guided by L^F in the obvious sense.*

PROOF. Let $\xi \leq \Omega$ be such that $\mathcal{N}_\xi$ does not have a definable Woodin cardinal. Let us assume that there is a putative iteration tree $\mathcal{T} \in V_\Omega$ on $\mathcal{N}_\xi$ as in the definition of (k, Ω, Ω)-iterability and which is according to a strategy which is guided by L^F such that either

(a) $\mathcal{T}$ has successor length, and $\mathcal{T}$ has a last ill-founded model, or else

(b) $\mathcal{T}$ has limit length and $L^F(\mathcal{M}(\mathcal{T})) \models$ "$\delta(\mathcal{T})$ is a Woodin cardinal," but the $K^{c,F}$-construction doesn't reach $M_1^F(\mathcal{P})$, or else

(c) $\mathcal{T}$ has limit length and $L^F(\mathcal{M}(\mathcal{T})) \models$ "$\delta(\mathcal{T})$ is not a Woodin cardinal," but there is no cofinal branch b through $\mathcal{T}$ such that the $\mathcal{Q}$-structure for b is an initial segment of $L^F(\mathcal{M}(\mathcal{T}))$.

Let us fix $\mathcal{T}$ witnessing this. Let

$$\pi \colon H \to V_{\Omega+2}$$

be elementary such that H is transitive, $\text{Card}(H) = \kappa$, $\pi \upharpoonright A \cup \{A\} = \text{id}$, and $\mathcal{N}_\xi$, $\mathcal{T} \in \text{ran}(\pi)$. Write $\bar{\mathcal{N}} = \pi^{-1}(\mathcal{N}_\xi)$ and $\bar{\mathcal{T}} = \pi^{-1}(\mathcal{T})$.

Let G be $\text{Col}(\omega, \kappa)$-generic over V. By the proof of Corollary 1.3.15, $\bar{\mathcal{N}}$ is (k, κ^+, κ^+)-iterable via the "realization strategy." This means that either $\text{lh}(\mathcal{T}) = \theta + 1$, some θ, and there is $\bar{\xi} \leq \xi$ and a weak ℓ-embedding $\sigma \colon \mathcal{M}^{\bar{\mathcal{T}}}_{\pi^{-1}(\theta)} \to \mathcal{N}_{\bar{\xi}}$ such that $\sigma \in V[G]$, or else $\mathcal{T}$ has limit length and there is a cofinal branch $b \in V$ through $\bar{\mathcal{T}}$ there is $\bar{\xi} \leq \xi$ and a weak ℓ-embedding $\sigma \colon \mathcal{M}^{\bar{\mathcal{T}}}_b \to \mathcal{N}_{\bar{\xi}}$ such that $\sigma \in V[G]$. It is clear that if the former holds true, then (a) fails. Let us now suppose that the latter holds true.

We may let $\bar{\mathcal{Q}}$ be the least initial segment of $\mathcal{M}^{\bar{\mathcal{T}}}_b$ such that $\delta(\bar{\mathcal{T}})$ is not definably Woodin in $\bar{\mathcal{Q}}$.

If

$$L^F(\mathcal{M}(\mathcal{T})) \models \text{ "}\delta(\mathcal{T}) \text{ is not a Woodin cardinal,"} \tag{1.3}$$

then we may let $\mathcal{Q}$ be the least initial segment of $L^F(\mathcal{M}(\mathcal{T}))$ in which $\delta(\mathcal{T})$ is not definably Woodin. Of course then $\mathcal{Q} \in \text{ran}(\pi)$ and by the Condensation Lemma 1.3.7 $\pi^{-1}(\mathcal{Q}) = \bar{\mathcal{Q}}$. By the proof of Corollary 1.3.15, $b \in H$, so that in H we have that b is the (unique) cofinal branch c through $\bar{\mathcal{T}}$ such that $\bar{\mathcal{Q}} \trianglelefteq \mathcal{M}_c^{\bar{\mathcal{T}}}$. Therefore in V we have that $\pi(b)$ is the (unique) cofinal branch c through $\mathcal{T}$ such that $\mathcal{Q} \trianglelefteq \mathcal{M}_c^{\mathcal{T}}$. Hence (c) fails.

But it is now straightforward to see that if (1.3) fails, then $F(\mathcal{M}(\bar{\mathcal{T}})) \models$ "$\delta(\bar{\mathcal{T}})$ is a Woodin cardinal" and $F(\mathcal{M}(\bar{\mathcal{T}}))$ is (or may be reorganized as) an initial segment of $\mathcal{M}_b^{\bar{\mathcal{T}}}$. But then $\sigma(F(\mathcal{M}(\bar{\mathcal{T}}))) = F(\mathcal{M}(\mathcal{T}))$ is (or may be reorganized as) an initial segment of $\mathcal{N}_{\bar{\xi}}$ and $F(\mathcal{M}(\mathcal{T})) \models$ "$\sigma(\delta(\bar{\mathcal{T}}))$ is a Woodin cardinal." This implies that $K^{c,F}(\mathcal{P})$ reaches $M_1^F(\mathcal{P})$. Hence (b) fails.

In order to finish the proof of Theorem 1.3.20 it thus remains to show that if $K^{c,F}(\mathcal{P})$ reaches $M_1^F(\mathcal{P})$, then $M_1^F(\mathcal{P})$ is (ω, Ω, Ω)-iterable via the unique strategy which is guided by L^F. However, this is shown by what we did so far. Notice that if $\mathcal{N}_\xi = M_1^F(\mathcal{P})$, then (1.3) must always be true, because $\delta(\bar{\mathcal{T}})$ cannot be definably Woodin in $F(\mathcal{M}(\bar{\mathcal{T}}))$, so that $\pi(\delta(\bar{\mathcal{T}})) = \delta(\mathcal{T})$ is not definably Woodin in $\pi(F(\mathcal{M}(\bar{\mathcal{T}}))) = F(\mathcal{M}(\mathcal{T}))$. $\square$

In order to state our second K^F-Existence Dichotomy, we need to introduce another property beyond "condenses well" which the model operators arising in nature all share.

Definition 1.3.21 *Let $\nu \geq \aleph_1$ be a cardinal or $\nu = \infty$, let $A \in H_\nu$ be swo'd, and let F be a model operator over A on H_ν. We say that F* relativizes well *iff there is a formula $\theta(u, v, w, z)$ such that whenever $\mathcal{N}$ and $\mathcal{N}'$ are models over A with $\mathcal{N} \in |\mathcal{N}'|$, and $\mathcal{M}$ is an F-premouse with base model $\mathcal{N}'$ and with $\mathcal{M} \models \mathsf{ZFC}^-$ then $F(\mathcal{N}) \in |\mathcal{M}|$ and $F(\mathcal{N})$ is the unique $x \in |\mathcal{M}|$ such that $\mathcal{M} \models \theta(x, \mathcal{N}, \mathcal{N}', A)$.*

The following theorem is shown in [7, Theorem 3.11] in a special case. The proof given there routinely also produces this general statement, though.

Theorem 1.3.22 (Stable K^F-Existence Dichotomy) *Let $\nu > 2^{\aleph_0}$, let $A \in H_\nu$ be swo'd, and let F be a model operator over A on H_ν such that F condenses well and F relativizes well. Suppose that for every model $\mathcal{N} \in H_\nu$ over A,*

$$L_\nu^F(\mathcal{N}) \models \text{"There is an ineffable cardinal } < \nu, \Omega_{\mathcal{N}}, \text{ say."}$$

(For instance, suppose H_ν to be closed under $F^{\#}$, and let $\Omega_{\mathcal{N}}$ be the critical point of $F^{\#}(\mathcal{N})$.) Let $\mathcal{N}_0 \in H_\nu$ be a model over A.

Then either

1. *there is some model $\mathcal{N} \in H_\nu$ over A with $\mathcal{N}_0 \in \mathcal{N}$ such that in $L^F_{\Omega_{\mathcal{N}}}(\mathcal{N})$, L^F does not guide an $(\Omega_{\mathcal{N}}, \Omega_{\mathcal{N}}+1)$-iteration strategy for $K^{c,F}(\mathcal{N}_0)^{L^F_{\Omega_{\mathcal{N}}}(\mathcal{N})}$, in which case $M_1^F(\mathcal{N}_0)$ exists and is (ν,ν)-iterable, or else*

2. *there is an F-small stable $K^F(\mathcal{N}_0)$ of height ν such that L^F guides a (ν,ν)-iteration strategy for it, by which we mean that there is an F-small F-premouse $\mathcal{K}(\mathcal{N}_0)$ over $\mathcal{N}_0$ of height ν such that*

   (a) for all $\alpha < \nu$ there is a cone $\mathcal{C}$ of models $\mathcal{N}$ over A such that for all $\mathcal{N} \in \mathcal{C}$, $K^F(\mathcal{N}_0)^{L^F_{\Omega_{\mathcal{N}}}(\mathcal{N})}$ exists inside $L^F_{\Omega_{\mathcal{N}}}(\mathcal{N})$ and $\mathcal{K}(\mathcal{N}_0)|\alpha = K^F(\mathcal{N}_0)^{L^F_{\Omega_{\mathcal{N}}}(\mathcal{N})}|\alpha$, and

   (b) in V, L^F guides a (ν,ν)-iteration strategy for $\mathcal{K}(\mathcal{N}_0)$.

Exercise 1.3.23 *Let Ω be subtle. Show "cheapo" covering for $K^{c,F}(\mathcal{P})|\Omega$. Also show that if Ω is weakly compact and if $\mathcal{T}$ is an iteration tree on some F-premouse $\mathcal{M}$ of length Ω with $\mathcal{T} \subset V_\Omega$, then there is a cofinal well-founded branch through $\mathcal{T}$.*

1.4 Capturing, correctness, and genericity iterations

Our proofs that iteration strategies exist are constructive, in a loose sense, in that we always give a definition of the strategy in question. An ω_1-iteration strategy Σ for a countable premouse $\mathcal{M}$ can be coded by a set Σ^* of reals, and the definability of Σ^* can be carefully calibrated using concepts from descriptive set theory. It is crucial in the core model induction that we do this for the $(\mathcal{M},\Sigma)$ that we construct. At a given stage in the induction, we will have some next appropriate pointclass Γ, and we will be trying to construct those $(\mathcal{M},\Sigma)$ such that $\Sigma^* \in \Gamma$. It is important to have constructed those $(\mathcal{N},\Lambda)$ such that Λ^* lies in an appropriate pointclass strictly below Γ, because such $(\mathcal{N},\Lambda)$ *capture* all sets in their corresponding pointclasses, and thereby get us ready to try to construct $(\mathcal{M},\Sigma)$.

Exercise 1.4.1 *Let $\mathcal{M}$ and $\mathcal{N}$ be countable premice, and let Σ and Λ be* unique $\omega_1 + 1$*-strategies for $\mathcal{M}$ and $\mathcal{N}$ resp. Then Σ^* is projective in Λ^*, or vice-versa.*

We now define the relevant notion of capturing. First a preliminary definition.

Definition 1.4.2 *Let $(\mathcal{M}, F)$ be a premouse, and Σ an iteration strategy for $(\mathcal{M}, F)$. Let $\delta < o(\mathcal{M})$. We say that $(\mathcal{M}, \Sigma)$* absorbs reals at δ *iff whenever $\eta < \delta$ and $i\colon \mathcal{M} \to \mathcal{N}$ comes from an iteration tree by Σ based on $\mathcal{M}|\eta$, and $x \in \mathbb{R}$, then there is an iteration tree $\mathcal{U}$ on $\mathcal{N}|i(\delta)$ such that*

(1) all critical points of $\mathcal{U}$ are $> i(\eta)$,

(2) $\mathcal{U}$ gives rise to an iteration map $j\colon \mathcal{N} \to \mathcal{P}$, and

(3) $x \in \mathcal{P}[g]$, for some $\mathcal{P}$-generic g on $\mathrm{Col}(\omega, j(i(\delta)))$.

Exercise 1.4.3 *Modulo the other clauses of Definition 1.4.2 Clause (3) of 1.4.2 is equivalent to: if l is* $\mathrm{Col}(\omega, i(\eta))$*-generic over $\mathcal{N}$, and $x \in \mathbb{R}$, then there is a* $\mathrm{Col}(\omega, j(i(\delta)))$*-generic g over $\mathcal{P}[l]$ such that $x \in \mathcal{P}[l][g]$.*

One version of a basic result of Woodin is

Theorem 1.4.4 (Woodin) *Suppose Σ is an $(\omega_1 + 1)$-iteration strategy for the r-premouse $(\mathcal{M}, F)$, and $\mathcal{M} \models \delta$ is Woodin; then $(\mathcal{M}, \Sigma)$ absorbs reals at δ.*

See Appendix 1 for a discussion. Neeman [23] proves 1.4.4 under the weaker hypothesis that Σ is an $(\omega + 1)$-iteration strategy, but only for $\mathcal{M}$ satisfying a reasonable fragment of ZFC. Woodin cardinals are needed to absorb reals:

Exercise 1.4.5 *Suppose that $(\mathcal{M}, \Sigma)$ absorbs reals at δ and $\mathcal{M} \models$* ZFC$^-$ *plus δ^+ exists, then δ is either Woodin or a limit of Woodins in $\mathcal{M}$.*

Definition 1.4.6 *Let A be a set of reals. Let Σ be an iteration strategy for a countable r-premouse $(\mathcal{M}, F)$, and let $\tau, \delta \in \mathcal{M}$.*

(1) We say that τ is a $(\mathcal{M}, \Sigma)$ term for A at δ *iff whenever $i\colon \mathcal{M} \to \mathcal{N}$ is an iteration map arising from a tree on $\mathcal{M}|\delta$ by Σ, and g is* $\mathrm{Col}(\omega, i(\delta))$*-generic over $\mathcal{N}$, then*
$$i(\tau)_g = A \cap \mathcal{N}[g].$$

(2) We say that $(\mathcal{M}, \tau, \Sigma)$ captures A *iff* τ *is an* $(\mathcal{M}, \Sigma)$*-term for* A *at* δ*, and* $(\mathcal{M}, \Sigma)$ *absorbs reals at* δ*.*

(3) We say $(\mathcal{M}, \Sigma)$ understands A at δ *just in case there is an* $(\mathcal{M}, \Sigma)$*-term for* A *at* δ*. We say* $(\mathcal{M}, \Sigma)$ captures A at δ *just in case there is an* $(\mathcal{M}, \Sigma)$*-term which captures* A *at* δ*.*

It is easy to see that if $(\mathcal{M}, \Sigma)$ understands A at δ, then it understands A at all $\eta < \delta$. On the other hand, $(\mathcal{M}, \Sigma)$ may understand A at δ, but fail to understand it at some $\eta > \delta$. This is because collapsing at η may destroy some of the structure of $\mathcal{M}$ above δ that was used to understand A at δ.

For example, let $\mathcal{M}$ be a countable, active premouse, let Σ be an ω_1-iteration strategy for $\mathcal{M}$, and let η be a cardinal of $\mathcal{M}$. The Shoenfield Absoluteness Theorem implies that if A is Σ^1_2 or Π^1_2, then $(\mathcal{M}, \Sigma)$ understands A at η. In the special case that $\mathcal{M}$ is the minimal active premouse (that is, $0^\sharp$), then there are no sets of reals which are captured by $(\mathcal{M}, \Sigma)$. This is because $0^\sharp$ is not generic over L.

Understanding or capturing a set A is especially useful if it is done via Suslin representations.

Definition 1.4.7 *A* Suslin term *is a term of the form "*$p[\check{T}]$*", where* $T \in \mathcal{M}$ *is a tree on some* $\omega \times \kappa$*. We say* $(\mathcal{M}, \Sigma)$ *has a* Suslin understanding of A at η *(or a* tree for A at η*) iff* $(\mathcal{M}, \Sigma)$ *understands* A *at* η *via a Suslin term. We say* $(\mathcal{M}, \Sigma)$ Suslin-captures A at η *if it captures* A *at* η *via a Suslin term.*

Exercise 1.4.8 *If* $(\mathcal{M}, \Sigma)$ *has a tree for* A *at* η*, then we can take the tree to be on* $\omega \times (\eta^+)^{\mathcal{M}}$*. Moreover,* $(\mathcal{M}, \Sigma)$ *has trees for* A *at all* $\gamma < \eta$*.*

Much of the usefulness of r-premice lies in the fact that, in practice, if Σ is an iteration strategy for $(\mathcal{M}, F)$, then $(\mathcal{M}, \Sigma)$ has trees at all η for the set of reals coding $F \restriction \mathrm{HC}$. The following makes this more precise.

Let us code countable transitive structures by reals as follows: put $x \in C$ iff x is a real coding a wellfounded, extensional structure $(\omega, E_x, ...)$ with universe ω. If $x \in C$, then $\pi_x \colon (\omega, E_x, ...) \cong \mathcal{P}_x$ is the transitive collapse. Recalling that our model operators F are such that $F(\mathcal{P})$ is pointwise Σ_1 definable from parameters in $\mathcal{P}$, we can code $F \restriction \mathrm{HC}$ as follows.

Definition 1.4.9 *Let* F *be a model operator defined on the* H_ν *cone above* y*, for some* $y \in HC$ *and* $\nu \geq \omega_1$*. We put*

$$(x, \langle n_0, ..., n_k \rangle, \varphi) \in F^* \Leftrightarrow x \in C \wedge F(\mathcal{P}_x) \models \varphi[\pi_x(n_0), ..., \pi_x(n_k)].$$

Of course, we can regard F^* as a set of reals. Our coding has the property that $F(\mathcal{P}_x)$ can be easily recovered in any $\mathcal{M}$ containing x and $F^* \cap \mathcal{M}$. In a similar vein, any Suslin term for F^* can be easily converted to a Suslin term for $\mathbb{R} \setminus F^*$.

Definition 1.4.10 *Let F be a model operator defined on an HC-cone. We say that F* determines itself on generic extensions *if there is a formula $\theta(v_0, v_1)$ in the language of r-premice such that whenever $\mathcal{M}$ is an F-premouse, and*

$$\mathcal{M} \models \mathsf{KP} + \text{ “there are arbitrarily large cardinals”},$$

then putting

$$\tau_\eta^{\mathcal{M}} = \text{ unique } \tau \text{ such that } \mathcal{M} \models \theta[\eta, \tau],$$

we have that whenever η is a cardinal of $\mathcal{M}$, and g is $\mathrm{Col}(\omega, \eta)$-generic over $\mathcal{M}$, then $HC^{\mathcal{M}[g]}$ is closed under F, and

$$(\tau_\eta^{\mathcal{M}})_g = F \restriction HC^{\mathcal{M}[g]}.$$

Lemma 1.4.11 *Let $(\mathcal{M}, F)$ be a countable r-premouse, where F is defined on an HC-cone, and let Σ be an iteration strategy for $(\mathcal{M}, F)$. Let η be a cardinal of $\mathcal{M}$. Suppose that F condenses well and determines itself on generic extensions, that $\eta < o(\mathcal{M})$, and that $\mathcal{M} \models \mathsf{KP}+$ “there are arbitrarily large cardinals”. Then $(\mathcal{M}, \Sigma)$ has trees for F^* and $\mathbb{R} \setminus F^*$ at η.*

Proof. It follows at once from the definitions that $(\mathcal{M}, \Sigma)$ has terms for F^* and its complement at η. To see that it has Suslin terms, we use the condensation property of F. For instance, if $\alpha < o(\mathcal{M})$, then we may let $T \in \mathcal{M}$ search for x, $\bar{\mathcal{M}}$, π, and g such that

1. $\pi \colon \bar{\mathcal{M}} \to \mathcal{M}$ is fully elementary, where $\bar{\mathcal{M}}$ is countable and transitive,
2. g is $\mathrm{Col}(\omega, \pi^{-1}(\eta))$-generic over $\bar{\mathcal{M}}$ with $x \in \bar{\mathcal{M}}[g]$, and
3. $x \in \pi^{-1}(\tau)^g$.

$\square$

Exercise 1.4.12 *Let F be a model operator defined on the H_κ-cone above y, where $y \in HC$. Suppose F has condensation, and $F \restriction HC$ determines itself on generic extensions. Then F^* is $< \kappa$-UB.*

Exercise 1.4.13 *Let $\mathcal{M}$ be active and 1-small; then $\mathcal{M}$ is not generic over any iterate of $\mathcal{M}$, and hence no sets of reals are captured by $(\mathcal{M}, \Sigma)$.*

Exercise 1.4.14 *Every $\Game$-ωn-Π^1_1 set is understood by any active iterable mouse. The function $(x, n) \mapsto x^\sharp(n)$ is not understood by $M_0^\# = 0^\#$. This function is understood by $L[\mu]$. (A subset of ω understood by $\mathcal{M}$ is in $\mathcal{M}$.)*

Exercise 1.4.15 *Let $\mathcal{M}$ be $\omega_1 + 1$ iterable, sound, and project to ω. Let Σ be its unique strategy, and Σ^* be a set of reals coding $\Sigma \restriction HC$. Then $(\mathcal{M}, \Sigma)$ does not capture Σ^*.*

Exercise 1.4.16 *The class of sets captured (understood) by $(\mathcal{M}, \Sigma)$ at η is closed under Boolean combinations.*

In the case that $(\mathcal{M}, \tau, \Sigma)$ captures A, we have

$$A = \bigcup\{i(\tau)_g \mid i\colon \mathcal{M} \to \mathcal{P} \wedge g \text{ is } \mathcal{P}\text{-generic over } \mathrm{Col}(\omega, i(\eta))\}.$$

Thus $(\mathcal{M}, \tau, \Sigma)$ does determine A, and in fact, A is Σ^1_1 definable from $(\mathcal{M}, \tau)$ over the structure $(V_{\omega+1}, \in, \Sigma^*)$, where Σ^* is a set of reals naturally coding $\Sigma \cap \mathrm{HC}$.[16]

As an immediate corollary of Theorem 1.4.4, we have

Corollary 1.4.17 *Suppose Σ is an $(\omega_1 + 1)$-iteration strategy for the r-premouse $(\mathcal{M}, F)$, and $\mathcal{M} \models \delta$ is Woodin. Suppose also that $(\mathcal{M}, \Sigma)$ understands A at δ; then $(\mathcal{M}, \Sigma)$ captures A at δ.*

For $A \subseteq \mathbb{R} \times \mathbb{R}$, we let $\exists^{\mathbb{R}} A = \{x \mid \exists y (x, y) \in A\}$, and $\forall^{\mathbb{R}} A = \{x \mid \forall y (x, y) \in A\}$. Capturing A leads to understanding of $\exists^{\mathbb{R}} A$:

Lemma 1.4.18 *Let $\mathcal{M}, F)$ be a countable r-premouse, and Σ be an iteration strategy for $(\mathcal{M}, F)$, and let $\eta < \delta$ be cardinals of $\mathcal{M}$. Suppose $(\mathcal{M}, \Sigma)$ captures $A \subset \mathbb{R} \times \mathbb{R}$ at δ; then $(\mathcal{M}, \Sigma)$ understands $\exists^{\mathbb{R}} A$ and $\forall^{\mathbb{R}} A$ at η.*

Proof: Let τ be a $(\mathcal{M}, \Sigma)$-term for A at δ. Let τ^0 be the equivalent $\mathrm{Col}(\omega, \eta) \times \mathrm{Col}(\omega, \delta)$-term. Working in $\mathcal{M}$, we define a $\mathrm{Col}(\omega, \eta)$-term σ by

$$(p, \rho) \in \sigma \Leftrightarrow \rho \in H_{\eta^+} \text{ and } \exists q \in \mathrm{Col}(\omega, \delta)[(p, q) \Vdash \exists y (\rho_{\dot{g}}, y) \in \tau^0_{\dot{g} \times \dot{h}}].$$

[16]For the careful reader: the universal Π^1_1 set will be reducible to Σ^*, so we need not go all the way to Σ^1_2-in-Σ^*.

Here $\dot{g}$ and $\dot{h}$ are terms for the left and right factors in a $\mathrm{Col}(\omega,\eta)\times\mathrm{Col}(\omega,\delta)$-generic. We claim that σ is an $(\mathcal{M},\Sigma)$-term for $\exists^{\mathbb{R}}A$ at η. For let $i\colon\mathcal{M}\to\mathcal{N}$ be an iteration map by Σ coming from a tree on $\mathcal{M}|\eta$. Let g be $\mathcal{N}$-generic over $\mathrm{Col}(\omega,i(\eta))$, and let x be a real in $\mathcal{N}[g]$.

If $x\in i(\sigma)_g$, let $(p,\rho)\in i(\sigma)$ be such that $x=\rho_g$. Let q witness that $(p,\rho)\in i(\sigma)$, and pick h which is $\mathcal{N}[g]$-generic over $\mathrm{Col}(\omega,i(\delta))$ such that $q\in h$. Then we have y such that $(\rho_g,y)\in i(\tau^0)_{g\times h}$, so that $(x,y)\in A$ as τ is an $(\mathcal{M},\Sigma)$-term for A. So $x\in\exists^{\mathbb{R}}A$.

If $x\in\exists^{\mathbb{R}}A$, then let $(x,y)\in A$. Since $(\mathcal{M},\Sigma)$ absorbs reals at δ, there is a Σ-iteration map $j\colon\mathcal{N}\to\mathcal{P}$ with $\mathrm{crit}(j)>i(\eta)$, and k generic for $\mathrm{Col}(\omega,j(i(\delta)))$ over $\mathcal{P}$, such that $y,g\in\mathcal{P}[k]$. Basic forcing theory yields h which is $\mathcal{P}[g]$-generic over $\mathrm{Col}(\omega,j(i(\delta)))$ such that $y\in\mathcal{P}[g][h]$. Letting $x=\rho_g$, we then have $(p,q)\in g\times h$ such that $(p,q)\Vdash\exists y(\rho_{\dot{g}},y)\in j(i({\tau^0}_{\dot{g}\times\dot{h}}))$. But then $(p,\rho)\in j(i(\sigma))$, so $(p,\rho)\in i(\sigma)$ because $\mathrm{crit}(j)>i(\eta)$. But then $x=\rho_g\in i(\sigma)_g$, as desired.

The sets undertood by $(\mathcal{M},\Sigma)$ at γ form a Boolean algebra. So $(\mathcal{M},\Sigma)$ understands $\forall^{\mathbb{R}}A=\neg\exists^{\mathbb{R}}\neg A$ at η. □

We turn now to correctness. Clearly, if $(\mathcal{M},\Sigma)$ understands A at some η, then $A\cap\mathcal{M}\in\mathcal{M}$. However, even if $\mathcal{M}$ has many Woodin cardinals below η, it may not be correct for $\Sigma^1_1(A)$ assertions. Understanding the truth is not the same as believing it. For example, if $A=\{x\in\mathrm{WO}\mid |x|=\omega_1^{\mathcal{M}}$, then A is nonempty, but $A\cap\mathcal{M}$ is empty. For correctness, $\mathcal{M}$ must have some way of constructing reals witnessing true existential statements about A and $\neg A$. In order to do that, it must Suslin-capture A and $\neg A$.

Lemma 1.4.19 *Suppose $(\mathcal{M},\Sigma)$ Suslin captures A at δ, and $\mathcal{M}\models\mathsf{ZFC}^-+$ "δ^+ exists". Suppose also A is nonempty; then $A\cap\mathcal{M}$ is nonempty.*

Proof. Suppose $(\mathcal{M},\Sigma)$ Suslin-captures A at δ via T. Let $x\in A$. There is an $i\colon\mathcal{M}\to\mathcal{N}$ so that $x\in\mathcal{N}[g]$ for some g on $\mathrm{Col}(\omega,i(\delta))$. But then $x\in p[i(T)]$. So $\mathcal{N}\models p[i(T)]\neq\emptyset$, and thus $\mathcal{M}\models p[T]\neq\emptyset$ by absoluteness of wellfoundedness and elementarity. But $p[T]\subseteq A$. □

On propagating Suslin terms, we have

Lemma 1.4.20 *Let $(\mathcal{M},F)$ be a countable r-premouse, and Σ an (ω_1+1)-iteration strategy for $(\mathcal{M},F)$. Suppose δ is Woodin in $\mathcal{M}$, and $\mathcal{M}\models\mathsf{ZFC}^-+$ δ^+ exists. Let $A\subseteq\mathbb{R}\times\mathbb{R}$, and suppose $(\mathcal{M},\Sigma)$ Suslin-captures A at δ; then $(\mathcal{M},\Sigma)$ has a Suslin understanding of $\exists^{\mathbb{R}}A$ and $\forall^{\mathbb{R}}A$ at all $\eta<\delta$.*

Proof. Suppose $(\mathcal{M}, \Sigma)$ Suslin-captures A via T. Let γ be such that $T \in \mathcal{M}|\gamma$, and $\mathcal{M}|\gamma \models \mathsf{KP}$. Let $\eta < \delta$. Working in $\mathcal{M}$, we construct a tree U for $\forall^{\mathbb{R}} A$ at η.

U will be a tree on $\omega^3 \times \mathcal{M}|(\gamma+1)$. We describe informally what is being built on each of the 4 coordinates of a potential branch of U, and leave the formal details to the reader.

Let $\mathcal{L}$ be the language of r-premice, expanded by new Henkin constants $c_0, c_1, \ldots$. On coordinate 1, U builds a real x. On the remaining coordinates it will try to prove $x \in \forall^{\mathbb{R}} A$. On coordinate 2, U builds a real z coding a complete Henkinized theory S_z of a pointwise definable $\mathcal{L}$-structure $\mathcal{R}_z$. On coordinate 4, U builds an elementary embedding $\pi\colon \mathcal{R}_z \to \mathcal{M}|(\gamma+1)$, with $\pi(c_0^{\mathcal{R}_z}) = \eta, \pi(c_1^{\mathcal{R}_z}) = \delta$, and $\pi(c_2^{\mathcal{R}_z}) = T$. On coordinate 3, U builds a real t proving the Σ^1_1 fact about x and z that for some g generic over $\mathcal{R}_z$ for $\mathrm{Col}(\omega, c_0)^{\mathcal{R}_z}$, we have $x = (c_3^{\mathcal{R}_z})_g$ and $c_4^{\mathcal{R}_z} \in g$. Finally, U must arrange that S_z has the sentence φ in it, where

$$\varphi = c_4 \overset{\mathrm{Col}(\omega,c_0)}{\Vdash} (1 \overset{\mathrm{Col}(\omega,c_1)}{\Vdash} (\forall y(c_3, y) \in p[c_2])).$$

We check that U yields an $(\mathcal{M}, \Sigma)$-term for $\forall^{\mathbb{R}} A$ at η. So let $i\colon \mathcal{M} \to \mathcal{N}$ be an iteration map by Σ, and let $x = \tau_g$ where g is $\mathcal{N}$-generic over $\mathrm{Col}(\omega, i(\eta))$.

Suppose first $x \in \forall^{\mathbb{R}} A$. Because T Suslin-captured A at δ, we must have

$$\mathcal{N} \models \varphi[i(\eta), i(\delta), i(T), \tau, p],$$

for some $p \in g$. But as $\mathcal{N}|i(\gamma) \models \mathsf{KP}$, and admissible sets can define well-founded parts of relations belonging to them,

$$\mathcal{N}|i(\gamma+1) \models \varphi[i(\eta), i(\delta), i(T), \tau, p].$$

Working in $\mathcal{N}[g]$, we can find $\pi^*\colon \mathcal{R}[g] \to \mathcal{N}|i(\gamma+1)[g]$ with $\mathcal{R}$ countable, and everything relevant in the range of π^*. Let $\pi = \pi^* \restriction \mathcal{R}$. Let z be chosen so that $\mathcal{R}_z = \mathcal{R}$, and $\pi(c_0^{\mathcal{R}_z}) = i(\eta)$, etc. Let t prove $x = (c_3^{\mathcal{R}_z})_g$. It is clear that $x \in p[i(U)]$, as witnessed by (z, t, π).

Conversely, suppose $x \in p[i(U)]$, as witnessed by (z, t, π). Let g be the $\mathcal{R}_z$-generic at $i(\eta)$ coded into t. Let $y \in \mathbb{R}$ be arbitrary. We have $c_0^{R_z} = \pi^{-1}(\eta)$, etc. Let us assume $\mathcal{R}_z$ is chosen to be transitive. We have that $(\mathcal{R}_z, F)$ is (ω_1+1)-iterable by Σ^π, the π-pullback of Σ.[17] Let

$$j\colon \mathcal{R}_z \to \mathcal{P}$$

[17] This standard fact holds for F-strategies on F-premice, as well.

be an iteration by Σ^π so that $\text{crit}(j) > \pi^{-1}(i(\eta))$ and

$$y \in \mathcal{P}[g][h]$$

for some h on $\text{Col}(\omega, j(\pi^{-1}(i(\eta))))$. We have the commutative diagram

$$\begin{array}{ccccc} \mathcal{M} & \xrightarrow{i} & \mathcal{N} & \xrightarrow{\ell} & Q \\ & & \pi\uparrow & & \uparrow\sigma \\ & & \mathcal{Q}_z & \underset{j}{\longrightarrow} & \mathcal{P} \end{array}$$

Here l is an iteration map by Σ with critical point $> i(\eta)$. Now $\mathcal{P} \models \varphi[\pi^{-1}(i(\eta)), j(\pi^{-1}(i(\delta))), j(\pi^{-1}(i(T))), c_3^{R_z}, c_4^{R_z}]$. (Note here j does not move $\pi^{-1}(i(\eta))$, and hence does not move $c_3^{R_z}$ or $c_4^{R_z}$, the term for x and the condition in g forcing its properties.) It follows we have an f such that

$$(x, y, f) \in [j(\pi^{-1}(i(T)))].$$

But then

$$(x, y, \sigma \circ f) \in [l(i(T))].$$

Because T was an $(\mathcal{M}, \Sigma)$-term for A, we get $(x, y) \in A$, as desired.

A completely parallel argument gives an $(\mathcal{M}, \Sigma)$ Suslin-term for $\exists^{\mathbb{R}} A$ at η. Indeed, one need only change $\forall y$ to $\exists y$ in the formula φ above, in order to get a definition of the tree required. We omit further detail. □

We remark that $(\mathcal{M}, \Sigma)$ may Suslin capture A at δ, without Suslin capturing $\exists^{\mathbb{R}} A$ at δ. For example:

Exercise 1.4.21 *Show that $M_2^\sharp$, with its unique iteration strategy, Suslin captures every Π^1_3 set at its bottom Woodin cardinal. However, it does not Suslin capture every Σ^1_4 set at its bottom Woodin cardinal.*

Exercise 1.4.22 * *The hypotheses of Lemma* **??** *do not imply that $(\mathcal{M}, \Sigma)$ has a Suslin understanding of $\neg A$ at any η.*

One way $(\mathcal{M}, \Sigma)$ might have a tree for A at η is by understanding a scale on A.

Definition 1.4.23 *Let $\vec{\psi}$ be a scale; then we say $(\mathcal{M}, \Sigma)$ understands (captures) $\vec{\psi}$ at η just in case $(\mathcal{M}, \Sigma)$ understands (captures) the relation $R(i, x, y) \Leftrightarrow \psi_i(x) \leq \psi_i(y)$ at η*

The demand in 1.4.23 is that $(\mathcal{M}, \Sigma)$ understand or capture the sequence of prewellordering associated to the scale. In many situations of importance later, $(\mathcal{M}, \Sigma)$ will capture the individual prewellorderings associated to $\vec{\psi}$, without being able to understand $\vec{\psi}$ itself. The simplest example is $\mathcal{M} = 0^\sharp$, and $\vec{\psi}$ the Martin-Solovay scale on Π^1_2:

Exercise 1.4.24 *Let $(\leq_n : n < \omega)$ be the prewellorders from the Martin-Solovay scale on Π^1_2. Show that for each $n < \omega$, $0^\#$ captures $\leq_n$. Show that $0^\#$ does not understand $/ \leq_n : n < \omega)$.*

Lemma 1.4.25 *Let $(\mathcal{M}, F)$ be a countable r-premouse, and Σ an $(\omega_1 + 1)$-iteration strategy for $(\mathcal{M}, F)$. Suppose $(\mathcal{M}, \Sigma)$ understands a scale on A at η, and $\mathcal{M} \models \mathsf{ZFC}^- + \eta^+$ exists. Then $(\mathcal{M}, \Sigma)$ has a tree for A at η.*

1.5 Projective correctness and M_n^F

Given a set A of reals, there is an analog of the projective hierarchy over A:

Definition 1.5.1 *Let A and B be relations on $\mathbb{R}$. Suppose $x \in \mathbb{R}$, and $0 \leq n < \omega$; then $B \in \Sigma^1_n(A, x)$ iff B is definable over $(V_{\omega+1}, \in, A)$ by a Σ_n formula, from the parameters V_ω, x. Put $\Sigma^1_n(A) = \Sigma^1_n(A, \emptyset)$. A relation is analytical in A if it is $\Sigma^1_n(A)$ for some n, and projective in A if it is $\Sigma^1_n(A, x)$ for some n and x.*

So the $\Sigma^1_0(A)$ relations are built up from A and $\neg A$ by Boolean combinations and number quantifiers. After that, we generate the analytical in A relations by real quantification.

The results of the last section clearly imply that an $(\mathcal{M}, \Sigma)$ which understands A at η will understand all $\Sigma^1_n(A)$ sets at γ, provided there are enough Woodin cardinals of $\mathcal{M}$ in the interval $(\gamma, \eta]$. For us, the exact number of Woodin cardinals needed is not very important, but here it is.

Theorem 1.5.2 *Let $(\mathcal{M}, F)$ be a countable r-premouse, and Σ an $(\omega_1 + 1)$-iteration strategy for $(\mathcal{M}, F)$. Let $\delta_0 < ... < \delta_n$ be Woodin in $\mathcal{M}$, where $n \geq 0$, and suppose $(\mathcal{M}, \Sigma)$ captures A at δ_n; then*

(1) if $B \subseteq \mathbb{R}$ is $\Sigma^1_n(A)$, then $(\mathcal{M}, \Sigma)$ captures B at δ_0;

(2) if $B \subseteq \mathbb{R}$ is $\Sigma^1_{n+1}(A)$, then $(\mathcal{M}, \Sigma)$ understands B at all $\eta < \delta_0$;

(3) if $b \subseteq \omega$ is $\Sigma^1_{n+1}(A)$ in some countable ordinal, then $b \in \mathcal{M}$.

PROOF. Parts (1) and (2) follow immediately from 1.4.17 and 1.4.18. For part (3), suppose $\alpha < \omega_1$ and $B \subseteq \omega \times \mathbb{R}$ in $\Sigma^1_{n+1}(A)$ are such that whenever $x \in \mathrm{WO}$ and $|x| = \alpha$,

$$n \in b \Leftrightarrow B(n, x),$$

for all n. Here WO is some standard set of codes for countable ordinals. Let $i \colon \mathcal{M} \to \mathcal{N}$ come from iterating at the least measurable cardinal of $\mathcal{M}$ long enough that $\alpha < i(\delta_0)$. By part (2), $\mathcal{N}$ understands B at $\alpha + 1$. This implies $b \in \mathcal{N}[g]$, whenever g is $\mathcal{N}$-generic over $\mathrm{Col}(\omega, \alpha)$, and thus $b \in \mathcal{N}$. Hence $b \in \mathcal{M}$, as desired. □

The counterpart to 1.5.2 for Suslin capturing is

Theorem 1.5.3 *Let $(\mathcal{M}, F)$ be a countable r-premouse, and Σ an $(\omega_1 + 1)$-iteration strategy for $(\mathcal{M}, F)$. Let $\delta_0 < ... < \delta_n$ be Woodin in $\mathcal{M}$, where $n \geq 0$, and suppose $(\mathcal{M}, \Sigma)$ Suslin captures A at δ_n; then*

(1) if $B \subseteq \mathbb{R}$ is positive $\Sigma^1_n(A)$ or positive $\Pi^1_n(A)$, then $(\mathcal{M}, \Sigma)$ Suslin captures B at δ_0; moreover $B \neq \emptyset$ implies $B \cap \mathcal{M} \neq \emptyset$, and

(2) if $B \subseteq \mathbb{R}$ is positive $\Sigma^1_{n+1}(A)$ or positive $\Pi^1_{n+1}(A)$, then $(\mathcal{M}, \Sigma)$ has trees for B at all $\eta < \delta_0$.

Here the positive $\Sigma^1_n(A)$ relations are those definable over $(V_{\omega+1}, \in, A)$ by a Σ_n formula whose predicate symbol for A has only positive occurences, and similarly for positive $\Pi^1_n(A)$. Theorem 1.5.3 follows routinely from Lemma 1.4.20.

Now let F be a model operator defined on HC, and suppose that F determines itself on generic extensions. Let F^* be the set of reals coding F, as in definition 1.4.9. Let $A \subseteq \mathbb{R}$ be projective in F^*, then by 1.4.11 and 1.5.3, if Σ is an $\omega_1 + 1)$-iteration strategy for $(\mathcal{M}, F)$, where $\mathcal{M}$ has enough Woodin cardinals, then $(\mathcal{M}, \Sigma)$ Suslin captures A. It will be important later that in a certain context, ω_1-iterability is enough, so we explain this now.

Definition 1.5.4 *Let $n < \omega$. An r-premouse $(\mathcal{M}, R)$ is* n-small *if for any $\mathcal{P} \trianglelefteq \mathcal{M}$, it is not the case that there are Woodin cardinals $\delta_1 < ... < \delta_n$ of $\mathcal{P}$ such that $\mathcal{P}|\delta_n$ is the longest proper initial segment of $\mathcal{P}$. An r-premouse satisfies "I am $M_n^{\dot{F}}$" if it is not n-small, but all its proper initial segments are n-small.*

For example, let $G(x) = x^\sharp$ for all sets x. Let $(\mathcal{M}, G)$ be an $\omega_1 + 1$ iterable r-premouse satisfying "I am $M_n^{\dot{F}}$". Then $\mathcal{M}$ is essentially what is usually called $M_n^\sharp$.[18] In fact, we do not need full (ω_1+1)-iterability here, as the following lemma shows.

Lemma 1.5.5 *Let F be a model operator defined in the HC-cone above y, and suppose F condenses well, and relativises well. Let $n < \omega$, and suppose that for all countable transitive x such that $y \in x$, there is an ω_1-iterable r-premouse $(\mathcal{M}, F)$ over x such that $\mathcal{M} \models$ "I am $M_n^{\dot{F}}$". Let x be countable transitive such that $y \in x$, and $(\mathcal{R}, F)$ and $(\mathcal{S}, F)$ be sound, ω_1-iterable r-premice over x which project to x. Suppose all proper initial segments of $\mathcal{R}$ or $\mathcal{S}$ are n-small; then $\mathcal{R} \trianglelefteq \mathcal{S}$ or $\mathcal{S} \trianglelefteq \mathcal{R}$.*

Proof. By induction on n.

Let $\mathcal{R}$ and $\mathcal{S}$ be given. For any z, let $M_n^F(z)$ be the unique sound ω_1-F-iterable model of "I am $M_n^{\dot{F}}$" over z. Let Σ and Γ be ω_1-iteration strategies for $(\mathcal{R}, F)$ and $(\mathcal{S}, F)$. Then Σ and Γ are M_n^F-guided by our induction hypothesis. (If $b = \Sigma(\mathcal{T})$, then $\mathcal{Q}(b, \mathcal{T})$ has all its proper initial segments n-small, and $\mathcal{Q}((b, \mathcal{T})$ is ω_1-iterable, so $\mathcal{Q}(b, \mathcal{T}) \trianglelefteq M_n^F((M(\mathcal{T})).)$ Let $\mathcal{M} = M_n^F(z)$, where z is the transitive closure of $\{y, \mathcal{R}, \mathcal{S}\}$. Working in $\mathcal{M}$, we coiterate $\mathcal{R}$ with $\mathcal{S}$ using Σ and Γ to pick branches at limit stages. Note that $\mathcal{M}$ can reconstruct $M_n^F(x)$, for any $x \in \mathcal{M}$ such that $y \in x$, by a full background extender construction, using extenders from its own sequence as the backgrounds. We use here that F relativises well, and that it condenses well. Of course, the reconstructed $M_n^F(x)$ will in general be a definable proper class of $\mathcal{M}$. Thus $\mathcal{M}$ can indeed track the coiteration, until one of the two sides reaches a tree $\mathcal{T}$ which is "maximal", in that $M_n(\mathcal{M}(\mathcal{T})) \models \delta(\mathcal{T})$ is Woodin. If that never happens at a stage $\leq \omega_1^{\mathcal{M}}$, the comparison terminates in $\mathcal{M}$ by the usual argument. But if it does happen at stage α, then the comparison is done at $\alpha + 1$. (Although $\mathcal{M}$ may not be able to find the final branches $b = \Sigma(\mathcal{T} \restriction \alpha)$ and $c = \Gamma(\mathcal{U} \restriction \alpha)$, they do exist, and by maximality of $\mathcal{T} \restriction \alpha$ and $\mathcal{U} \restriction \alpha$,

$$\mathcal{M}_b^{\mathcal{T}} = M_n(\mathcal{M}(\mathcal{T})) = \mathcal{M}_c^{\mathcal{U}}.$$

□

Motivated by this, we define

[18] Only the stratifications differ, and in no important way.

Definition 1.5.6 *Let F be a model operator defined in the HC-cone over y, and suppose F condenses and relativises well. Let $n < \omega$, and z be countable transitive with $y \in z$. We say $M_n^F(z)$* exists *iff*

(1) for all $k < n$ and countable transitive x such that $y \in x$, $M_k^F(x)$ exists, and

(2) there is an ω_1-F-iterable model of "I am $M_n^{\dot{F}}$" over z.

In this case, we let $M_n^F(z)$ be the unique sound model as in (2).

An argument parallel to the proof of Lemma 1.4.4 shows

Lemma 1.5.7 *Suppose $M_n^F(z)$ exists; then $M_n^F(z)$ absorbs reals at each of its Woodin cardinals.*

From this and the proof of 1.5.3 we get

Theorem 1.5.8 *Let G be a model operator defined on an HC-cone which condenses well and determines itself on generic extensions. Let $F(x) = G(x)^\sharp$ for all x. Let $n \geq 1$, and suppose $M_n^F(z)$ exists; then $M_n^F(z)$ Suslin-captures every $\Pi^1_{n+1}(G^*)$ or $\Sigma^1_{n+1}(G^*)$ set of reals at its least Woodin cardinal.*

Corollary 1.5.9 *Suppose $M_n^\sharp(z)$ exists, where $n \geq 1$; then $M_n^\sharp(z)$ Suslin-captures every Π^1_{n+1} set at its bottom Woodin cardinal.*

Recall that $M_n^\sharp$ is the minimal active, $(\omega_1 + 1)$-iterable premouse with n Woodin cardinals. We get that $M_n^\sharp$ understands all Σ^1_2 sets everywhere, understands all Σ^1_3 sets everywhere below its top Woodin cardinal, understands all Σ^1_4 sets everywhere below its second Woodin from the top, and so forth. Precisely:

Corollary 1.5.10 *Let $n \geq 1$, and suppose $M_n^\sharp$ exists, and is (ω_1+1)-iterable via Σ. Let $\delta_0, ..., \delta_{n-1}$ be the Woodin cardinals of $M_n^\sharp$; then for $1 \leq k \leq n$*

(1) if $B \subseteq \mathbb{R}$ is Σ^1_2, then $(\mathcal{M}, \Sigma)$ understands B at all η;

(2) if $B \subseteq \mathbb{R}$ is Σ^1_{k+2}, then $(\mathcal{M}, \Sigma)$ understands B at all $\eta < \delta_{n-k}$;

(3) for $b \subseteq \omega$,

$$b \in M_n^\sharp \Leftrightarrow b \text{ is } \Sigma^1_{n+2} \text{ in some countable ordinal.}$$

PROOF. Part (1) is Shoenfield's theorem, and part (2) follows from (1) and 1.5.2, with A the universal Σ^2_1 set of reals.)

That $M_n^\sharp$ contains all reals b which are Σ^1_{n+2}-definable from countable ordinals follows at once from Theorem 1.5.2, part (3), applied with A the universal Σ^1_2 set of reals.

If x is the α^{th} real in $M_n^\sharp$ then x is Σ^1_{n+2} in α using the comparison theroem. To get this optimal definability bound, one must carefully compute the complexity of the Q-structure guided iteration strategies for proper intial segments of $M_n^\sharp$ which project to ω. We refer the reader to [**?**] for details. □

It is possible to give an exact description of those sets of reals which are understood by $M_n^\sharp$ and its unique $(\omega_1 + 1)$-iteration strategy, at each η.

1.6 CMIP theory

Countable iterability and K^c

Definition 1.6.1 *A premouse $\mathcal{M}$ is* countably iterable *iff every countable elementary submodel of $\mathcal{M}$ is $(\omega, \omega_1, \omega_1+1)$-iterable. In this case, $\mathcal{M}$ is also called a* mouse.

The ω in $(\omega, \omega_1, \omega_1 + 1)$ iterability refers to the degrees of ultrapowers allowed. We shall drop reference to it in the future, with the understanding that one takes ultrapowers of the largest degree possible, and the caveat that the author has not always considered the issue carefully. The ω_1 refers to the fact one can stack normal trees ω_1 times. In the sequel, an iteration tree in general is almost always a stack of normal trees in which maximal ultrapowers are taken, even shifting between normal trees.

In the sequel we shall sometimes speak of *partial* iteration strategies, with the obvious meaning. We make the convention here that an iteration strategy Σ is only defined on those iteration trees $\mathcal{T}$ such that $\mathcal{T}$ has limit length and is itself a play by Σ. If Σ is defined on all such trees, it is total.

Countable iterability is enough to compare countable mice. A countably iterable premouse must have the solidity and universality properties of iterable mice, as these are first order. Finally, if $\mathcal{M}$ and $\mathcal{N}$ agree to a common cutpoint η, project to η, and are countably iterable above η, then one is an initial segment of the other.

Let Ω be strongly inaccessible. A K^c-construction below Ω is determined by a sequence

$$\langle \mathcal{N}_\alpha \mid \alpha \leq \Omega \rangle,$$

essentially what is called by Jensen an *ms-array.* The precise details depend on what background condition one demands for the last extender of an $\mathcal{N}_\alpha$; for official purposes, we follow CMIP. As long as each $\mathcal{C}_k(\mathcal{N}_\alpha)$ is countably iterable, the construction does not lead to fine-structural pathology, and one can set

$$K^c = \mathcal{N}_\Omega.$$

If μ is a normal measure on Ω, then again assuming only countable iterability, one has

$$(\alpha^+)^{K^c} = \alpha^+, \text{ for } \mu\text{-a.e. } \alpha,$$

for any maximal K^c construction (i.e. one which adds extenders wherever it can, perhaps with some μ-measure zero set of forbidden critical points for total extenders),

Question. Must every $\mathcal{C}_k(\mathcal{N}_\alpha)$ occurring in a K^c-construction be countably iterable?

The good (i.e. affirmative) answer is known for $\mathcal{N}_\alpha$ which are tame, or even domestic. It is not known significantly beyond that, for example, it is open below a Woodin limit of Woodins. One seems to need some form of UBH, even to handle countable trees, and certainly to handle the length ω_1 trees which one must face as part of countable iterability.

Question 2. Can one prove that K^c computes successor cardinals correctly on a stationary class without assuming that Ω is measurable?

In 2007, Jensen and Steel found a way of developing a theory of true K without assuming the measurable. This also solved the "test problem" of the equiconsistency of one Woodin cardinal with the existence of a saturated ideal on ω_1.

Core model inductions generally inductivley produce sufficiently closed inner models M admitting M-measures over some Ω which serve as local universes; K^c and K are then constructed inside M.

Uncountable iterability and K

In order to construct K from K^c, one needs that K^c is $(\omega, \Omega+1)$-iterable. Assuming this much iterability, and that Ω is measurable, one can make the

Definition 1.6.2 *K is the transitive collapse of the intersection of all thick hulls of K^c.*

Theorem 1.6.3 *Suppose that K exists. One then has:*

(1) K is $(\omega, \Omega+1)$-iterable.

(2) (Generic absoluteness) $K^V = K^{V[g]}$, whenever g is V-generic over a poset of size less than Ω.

(3) (Weak covering a.e.) $\alpha^{+K} = \alpha^+$ for μ-a.e. α, for any normal μ on Ω.

(4) (Weak covering) For any K-cardinal $\kappa \geq \omega_2^V$, $\mathrm{cof}(\kappa^{+K}) \geq |\kappa|$. Thus $\kappa^{+K} = \kappa^+$, whenever κ is a singular cardinal of V. ([20]. This may use a little more iterabilty for K^c than we have stated.)

(5) (Inductive definition) $K \cap HC$ is definable over $(J_{\omega_1}(\mathbb{R}), \in, \mathcal{I})$, where $\mathcal{I}$ is the collection of all countably iterable countable premice.

(6) (Embeddings of K) K is rigid, i.e., any elementary $j\colon K \to K$ is the identity. K is the unique universal weasel which embeds into all other iterable universal weasels. K^c is an iterate of K.

See [27] for a survey of what's known about K.

In order to prove $(\omega, \Omega+1)$ iterability for K^c, one needs to have an iteration strategy on countable iteration trees which is sufficiently absolutely definable that it can be extended to uncountable trees. The main technique is based on $\mathcal{Q}$-structures.

Mouse operators and $\mathcal{Q}$-structures

We shall often consider relativised mice, that is, mice built by starting with some set x, then constructing from an extender sequence with all critical points $> \mathrm{rk}(x)$. We tacitly assume here that x is transitive, and usually assume that it is equipped with a wellorder. We could always code such an x as $\sup(A) \cup \{A\}$, for A a set of ordinals. Let us call such an x *self-wellordered*, or swo. The main exception to this rule is the case of mice over the reals, where $x = V_{\omega+1}$ must be considered without a wellorder.

Definition 1.6.4 *An* r-premouse *is a premouse over some swo'd set x.*

The theory of r-premice and mice is a trivial variant of the lightface theory.

Definition 1.6.5 *A* mouse operator on Z *is a function $\mathcal{N}$ assigning to each swo $x \in Z$ a countably iterable x-premouse $\mathcal{N}(x)$ such that $\mathcal{N}(x)$ is pointwise definable from members of $x \cup \{x\}$. We say that $\mathcal{N}$ is* first order *just in case there is a theory T in the language of r-premice (so having a symbol $\dot{x}$ for x) such that for all $x \in Z$, $\mathcal{N}(x)$ is the least countably iterable x-mouse satisfying T.*

Countable iterability goes down under Skolem hulls, so we get:

Lemma 1.6.6 (Condensation) *Let $\mathcal{N}$ be a first order mouse operator on Z, and suppose $\pi\colon \mathcal{P} \to \mathcal{N}(x)$ is fully elementary in the language of relativised mice; then $\mathcal{P} = \mathcal{N}(\pi^{-1}(x))$.*

A very important first order mouse operator is the $\mathcal{Q}$-structure function:

Definition 1.6.7 *Let $\mathcal{M}$ be an r-premouse. A* $\mathcal{Q}$-structure for $\mathcal{M}$ *is an r-premouse $\mathcal{Q}$ such that*

(a) $\mathcal{M} \trianglelefteq^* \mathcal{Q}$ (i.e. $\mathcal{M}$ is a cutpoint initial segment of $\mathcal{Q}$),

(b) $\mathcal{Q}$ defines a minimal failure of $o(\mathcal{M})$ to be Woodin via the extenders of $\mathcal{M}$, and

(c) $\mathcal{Q}$ is countably iterable above $o(\mathcal{M})$.

A comparison argument shows there is at most one $\mathcal{Q}$ structure for $\mathcal{M}$. We denote it $\mathcal{Q}(\mathcal{M})$ if it exists. It is easy to see that $\mathcal{T} \mapsto \mathcal{Q}(\mathcal{M})$ is a first order mouse operator on its domain. Let us call this operator $\mathcal{Q}^t$. An important special case is $\mathcal{M} = \mathcal{M}(\mathcal{T})$, for $\mathcal{T}$ an iteration tree of limit length on some r-premouse. In this case we write $\mathcal{Q}(\mathcal{T})$ for $\mathcal{Q}^t(\mathcal{M}(\mathcal{T}))$. Thus $\mathcal{Q}(\mathcal{T})$, when it exists, defines a minimal failure of $\delta(\mathcal{T})$ to be Woodin via extenders of the common part model $\mathcal{M}(\mathcal{T})$.

One can sometimes use $\mathcal{Q}$-structures to determine an iteration strategy on an r-premouse:

Definition 1.6.8 *Let $\mathcal{T}$ be an iteration tree of limit length on an r-premouse, and b a cofinal branch of $\mathcal{T}$. Then $\mathcal{Q}(b, \mathcal{T})$ is the first initial segment of $\mathcal{M}_b^{\mathcal{T}}$*

defining a failure of $\delta(\mathcal{T})$ to be Woodin. We let $\mathcal{Q}(b,\mathcal{T})$ be undefined if there is no such initial segment.

Definition 1.6.9 *Σ^t is the following partial iteration strategy (for arbitrary tame r-premice):*

$$\Sigma^t(\mathcal{T}) = \text{ unique } b \text{ such that } \mathcal{Q}(b,\mathcal{T}) = \mathcal{Q}(\mathcal{T}).$$

We write $\Sigma^t_{\mathcal{M}}$ for the restriction of Σ^t to iteration trees based on $\mathcal{M}$.

One can show:

1. If $\mathcal{T}$ is an iteration tree on a tame r-premouse $\mathcal{M}$, then there is at most one cofinal b such that $\mathcal{Q}(\mathcal{T})$ and $\mathcal{Q}(b,\mathcal{T})$ are defined, and $\mathcal{Q}(\mathcal{T}) = \mathcal{Q}(b,\mathcal{T})$, moreover

2. if in addition $\mathcal{M} \models$ "There are no Woodin cardinals", or even just projects below its bottom Woodin, (above the set over which it is built, in each case) then for any cofinal b of $\mathcal{T}$, $\mathcal{Q}(b,\mathcal{T})$ exists, so that

3. if in addition, $\mathcal{M}$ is ω_1+1-iterable, then Σ^t is its unique ω_1+1-iteration strategy.

Tameness is important here; the natural extension of these notions to nontame r-premice would involve $\mathcal{Q}$-phalanxes. For pretty much the rest of these notes, we shall stick to the tame case. (The superscript in Σ^t is for "tame".)

We shall use mouse operators to keep track of the complexity of the $\mathcal{Q}$-structures determining $\Sigma^t_{\mathcal{M}}$:

Definition 1.6.10 *Let $\mathcal{N}$ be a mouse operator, and let $\mathcal{M}$ be a tame premouse. We say that $\Sigma^t_{\mathcal{M}}$ is $\mathcal{N}$-guided on Y just in case whenever $\mathcal{T} \in Y$ is a tree of limit length played by $\Sigma^t_{\mathcal{M}}$, then $\Sigma^t_{\mathcal{M}}(\mathcal{T})$ is defined, and letting $b = \Sigma^t_{\mathcal{M}}(\mathcal{T})$, we have $\mathcal{Q}(b,\mathcal{T}) \trianglelefteq \mathcal{N}(\mathcal{M}(\mathcal{T}))$.*

Of course, $\Sigma^t_{\mathcal{M}}$ is $\mathcal{Q}^t$-guided on its domain, simply by definition.

The following basic condensation property of Σ^t is often used.

Lemma 1.6.11 (Condensation for Σ^t) *Let $\mathcal{T}$ be an iteration tree played according to $\Sigma^t_{\mathcal{M}}$, and let $\pi\colon S \to V_\theta$ be such that $\pi(\mathcal{P}) = \mathcal{M}$ and $\pi(\mathcal{U}) = \mathcal{T}$; then $\mathcal{U}$ is played according to $\Sigma^t_{\mathcal{P}}$.*

Proof. This follows immediately from condensation for the mouse operator $\mathcal{Q}^t$. □

The following lemma encapsulates the main way one extends iteration strategies for countable trees on tame mice so as to act on uncountable trees.

Lemma 1.6.12 ($\mathcal{Q}$-reflection) *Let $\mathcal{M}$ be tame, and Z be transitive and rudimentarily closed, with $HC \subseteq Z$. Let $\mathcal{N}$ be a first order mouse operator defined on all $\mathcal{M}(\mathcal{T})$ such that $\mathcal{T} \in Z$ is played according to $\Sigma^t_{\mathcal{M}}$. Suppose that whenever $\mathcal{P}$ is countable and elementarily embeddable into $\mathcal{M}$, then $\Sigma^t_{\mathcal{P}}$ is an $\mathcal{N}$-guided strategy on HC. Then $\Sigma^t_{\mathcal{M}}$ is an $\mathcal{N}$-guided strategy on Z.*

Proof. Let $\mathcal{T} \in Z$ be a tree of limit length on $\mathcal{M}$ which is played by Σ^t, so that $\mathcal{N}(\mathcal{M}(\mathcal{T}))$ exists by hypothesis. We must show that there is a cofinal branch b of $\mathcal{T}$ such that $\mathcal{Q}(b, \mathcal{T}) \trianglelefteq \mathcal{N}(\mathcal{M}(\mathcal{T}))$. Let $\pi \colon S \to V_\theta$, where S is countable transitive, θ is large, $\pi(\mathcal{P}) = \mathcal{M}$, and $\pi(\mathcal{U}) = \mathcal{T}$. Let g be S-generic for the collapse of $\mathcal{P}$ and $\mathcal{U}$ to be countable. By 1.6.11, $\mathcal{U}$ is by Σ^t, and by mouse condensation,

$$\pi^{-1}(\mathcal{N}(\mathcal{M}(\mathcal{T})) = \mathcal{N}(\mathcal{M}(\mathcal{U})).$$

But $\Sigma^t_{\mathcal{P}}$ is $\mathcal{N}$-guided on HC, so letting $b = \Sigma^t_{\mathcal{P}}(\mathcal{U})$, $\mathcal{Q}(b,\mathcal{U}) \trianglelefteq \mathcal{N}(\mathcal{M}(\mathcal{U}))$. Since $S[g]$ is Σ^1_1 absolute in V, we get

$$S[g] \models \exists c(\mathcal{Q}(c,\mathcal{U}) \trianglelefteq \mathcal{N}(\mathcal{M}(\mathcal{U}))),$$

which then implies that $b \in S[g]$ by the uniqueness of such c. This is true for all g, and hence $b \in S$. That is, $S \models \exists b(\mathcal{Q}(b,\mathcal{U}) \trianglelefteq \mathcal{N}(\mathcal{M}(\mathcal{U})))$, and hence $V_\theta \models \exists b(\mathcal{Q}(b, \mathcal{T}) \trianglelefteq \mathcal{N}(\mathcal{M}(\mathcal{T})))$, and we are done. □

Here is a variant of the $\mathcal{Q}$-reflection Lemma 1.6.12.

Lemma 1.6.13 *Let $\mathcal{M}$ be tame. Suppose that*

(a) For any countable $\mathcal{P}$ embeddable into $\mathcal{M}$, $\Sigma^t_{\mathcal{P}}$ is an ω_1 iteration strategy for $\mathcal{P}$, and

(b) For any tree $\mathcal{T} \in V_\Omega$ on $\mathcal{M}$ of limit length which is played by Σ^t, $\mathcal{Q}(\mathcal{T})$ exists.

Then $\Sigma^t_{\mathcal{M}}$ is an Ω-iteration strategy for $\mathcal{M}$.

Lemma 1.6.13 is just the special case of Lemma 1.6.12 in which $\mathcal{N}$ is the operator $\mathcal{Q}^t$ itself, and $Z = V_\Omega$.

The existence of K

First, we have the result of CMIP:

Theorem 1.6.14 *Let Ω be measurable, and suppose there is no premouse of height Ω with a Woodin cardinal; then K^c is $(\omega, \Omega+1)$-iterable via Σ^t, and hence K exists.*

This is an easy consequence of the $\mathcal{Q}$-reflection lemmas. Hypothesis (a) of 1.6.13 holds because K^c must be tame, and hypothesis (b) holds because $L_\Omega[\mathcal{M}(\mathcal{T})] \models \delta(\mathcal{T})$ is not Woodin, and $L[\mathcal{M}(\mathcal{T})]$ is trivially iterable above $\delta(\mathcal{T})$.

One can use the $\mathcal{Q}$-reflection Lemmas 1.6.12 and 1.6.12 to go beyond one Woodin cardinal. In the case of finitely many Woodins, the central result is:

Theorem 1.6.15 *Let Ω be measurable. Suppose that for all $x \in V_\Omega$, $M_n^\sharp(x)$ exists and is $(\omega, \Omega+1)$-iterable. Then exactly one of the following holds:*

(1) for all $x \in V_\Omega$, $M_{n+1}^\sharp(x)$ exists and is $(\omega, \Omega+1)$-iterable,

(2) for some $x \in V_\Omega$, $K^c(x)$ is n-small, has no Woodin cardinals, and is $(\omega, \Omega+1)$-iterable. (Hence $K(x)$ exists, is n-small, and has no Woodin cardinals.)

Proof. It is easy to see that (1) and (2) are mutually exclusive.

Assume that (1) fails, and let x be such that there is no fully iterable $M_{n+1}^\sharp(x)$. Then there must be an $\eta < \Omega$ such that the maximal $K^c(x)$ construction done with all background extenders having critical point above η fails to reach $M_{n+1}^\sharp(x)$. The reason is that any fully sound premouse projecting to x which is reached by such a construction is η-iterable, for we can carry out the proof of countable iterability in $V^{\mathrm{Col}(\omega,\xi)}$, with $\xi < \eta$. Our background extenders prolong to $V^{\mathrm{Col}(\omega,\xi)}$, and the strategy we get in $V^{\mathrm{Col}(\omega,\xi)}$ pulls back to V by uniqueness.

So let $K^c(x)$ come from a construction which does not reach $M_{n+1}^\sharp(x)$.

Claim. $K^c(x) \models$ there are no Woodin cardinals.

Proof. Otherwise, let δ be the largest Woodin of $K^c(x)$. (There must certainly be a largest one, as otherwise $K^c(x)$ reaches $M_{n+1}^\#(x)$.) We can then compare $K^c(x)$ with $M_n^\sharp(K^c(x)|\delta)$, using the $\Omega+1$-iterability of the latter to pick branches on both sides. (There is another Lowenheim-Skolem argument to see that the $\mathcal{Q}$-structures provided by the iterates of $M_n^\sharp(K^c(x)|\delta)$ are realized by branches on the $K^c(x)$ side. Cf. Exercise 1.6.16.) Because $K^c(x)$ is universal, the $M_n^\sharp(K^c(x)|\delta)$ side comes out strictly shorter, which implies that $K^c(x)$ did indeed reach $M_{n+1}^\sharp(x)$. Contradiction! □

To see that $K^c(x)$ is $(\omega, \Omega+1)$-iterable, we simply apply the $\mathcal{Q}$-reflection Lemma 1.6.12, with our mouse operator $\mathcal{N}$ given by $\mathcal{N}(y) = M_n^\sharp(y)$. It is clear from the fact that that $K^c(x)$ has no Woodin cardinals and does not reach $M_{n+1}^\sharp(x)$ that for any countable $\mathcal{P}$ embeddable into $K^c(x)$, $\Sigma_{\mathcal{P}}^t$ is $\mathcal{N}$-guided on HC. Thus Σ^t is an $\mathcal{N}$-guided strategy on all of V_Ω, as desired. □

Exercise 1.6.16 *Show that in the proof of the above Claim in the proof of Theorem 1.6.15, $K^c(x)$ may be compared with $M_n^\#(K^c(x)|\delta)$, so that $M_n^\#(K^c(x)|\delta)$ ends up as an initial segment of $K^c(x)$.*

1.7 Universally Baire iteration strategies

In this section, we show

Theorem 1.7.1 (Steel, Woodin 1990) *Suppose Ω is measurable, and every set of reals in $L(\mathbb{R})$ is $< \Omega$-weakly homogeneous; then for all $x \in V_\Omega$, $M_\omega^\sharp(x)$ exists and is $\Omega + 1$-iterable.*

The proof pre-dated the core model induction. It is interesting because it shows that the main thing the core model induction is giving us is something like a universally Baire representation of Σ^t restricted to countable trees. Granted such a representation, there is no need for an induction.

Our proof needs weak homogeneity, rather than just universal Baireness, however. It seems to be open whether theorem 1.7.1 remains true if “weakly homogeneous” is replaced by “universally Baire”.

An immediate corollary of 1.7.1 is

Corollary 1.7.2 *If every set in $L(\mathbb{R})$ is weakly homogeneous, then $\mathsf{AD}^{L(\mathbb{R})}$ holds.*

The existence of a strongly compact implies all sets of reals in $L(\mathbb{R})$ are weakly homogeneous by work of Woodin from the mid 80’s. So

Corollary 1.7.3 *If there is a strongly compact cardinal, then* $\mathsf{AD}^{L(\mathbb{R})}$ *holds.*

We now prove 1.7.1. Our presentation simplifies the original proof somewhat.

Lemma 1.7.4 *Let* $\langle x_\alpha \mid \alpha < \omega_1 \rangle$ *be a sequence of distinct reals. Put*

$$R(y, n, m) \Leftrightarrow (y \in WO \text{ and } x_{|y|}(n) = m).$$

Then R *is not* ω_1*-universally Baire.*

Proof. Let (T, U) be an ω_1-absolutely complementing pair such that $p[T] = R$. Let $\pi \colon M \to V_\theta$, where M is transitive and countable, θ is large, and $\pi((\bar{T}, \bar{U})) = (T, U)$. We have $\pi(\langle x_\alpha \mid \alpha < \omega_1^M \rangle) = \langle x_\alpha \mid \alpha < \omega_1 \rangle$, and we have that whenever $x_\beta \in M$, then $\beta < \omega_1^M$.

Now let g be M-generic for $\mathrm{Col}(\omega, \omega_1^M)$, and let $y \in (M[g] \cap \mathrm{WO})$ be such that $|y| = \omega_1^M$. Put

$$(n, m) \in x \Leftrightarrow (y, n, m) \in p[\bar{T}].$$

Since $p[\bar{T}] \subseteq p[T]$, we have $x \subseteq x_{\omega_1^M}$. But if $(n, m) \notin x$, then $(y, n, m) \in p[\bar{U}]$ because $\bar{T}$ and $\bar{U}$ are absolute complements over M, and since $p[\bar{U}] \subseteq p[U]$, we get that $(n, m) \notin x_{\omega_1^M}$. Thus $x_{\omega_1^M} \in M[g]$. This is true for all g, so $x_{\omega_1^M} \in M$. But then $x_{\omega_1^M} = x_\beta$ for some $\beta < \omega_1^M$, a contradiction. □

Lemma 1.7.5 *Under the hypotheses of the theorem,* $\mathbb{R}^\sharp$ *is* $< \Omega$*-universally Baire. Moreover, if* (T, U) *is a pair of* $< \Omega$*-absolutely complementing pair of trees witnessing this, then*

$$V[G] \models p[T] = \mathbb{R}^\sharp,$$

whenever G *is* V*-generic for a poset of size* $< \Omega$.

Proof. $\mathbb{R}^\sharp$ exists because we have a measurable cardinal. $\mathbb{R}^\sharp = \bigcup_{n<\omega} T_n$, where T_n is the type with real parameters of the first n indiscernibles. Since $T_n \in L(\mathbb{R})$, it is $< \Omega$-weakly homogeneous. But the class of $< \Omega$-weakly homogeneous sets is closed under countable unions.

In order to show the "moreover" part, it suffices to produce just one pair (T, U) of $< \Omega$-absolutely complementing trees with the property, since all such pairs determine the same set of reals in every size $< \Omega$ generic extension. Let T be $< \Omega$-weakly homogeneous with $p[T] = \mathbb{R}^\sharp$, and let U be the Martin-Solovay tree for $\mathbb{R} \setminus p[T]$. Supposing toward contradiction that $V[G] \models p[T] \neq \mathbb{R}^\sharp$, for some G is V-generic for a poset of size $< \Omega$, we get in V

$$\pi\colon M \to V_\theta, \text{ such that } \pi((R,S)) = (T,U),$$

where M is countable transitive. Letting g be M-generic containing a condition forcing $p[T] \neq \mathbb{R}^\sharp$, we have

$$M[g] \models p[R] \neq \mathbb{R}^\sharp.$$

It is easy to see, using that (R,S) are absolute complements over M, $p[R] \subseteq p[T]$, and $p[S] \subseteq p[U]$, that $p[R] \cap M[g] = \mathbb{R}^\sharp \cap M[g]$. This yields that all the properties of $\mathbb{R}^\sharp$ hold of $p[R]$ in $M[g]$, except possibly the witness condition.

The witness condition asserts that if $\exists v\phi \in \mathbb{R}^\sharp$, where ϕ involves indiscernibles and real parameters $\vec{x}$, then there is a term τ involving indiscernibles *and real parameters* $\vec{y}$ such that $\phi(\tau) \in \mathbb{R}^\sharp$. To verify it of $p[R]^{M[g]}$, it suffices to show that if $x \in \mathbb{R} \cap M[g]$, then

$$\exists y \in \mathbb{R}(\psi, x, y) \in p[T] \Rightarrow \exists y \in \mathbb{R} \cap M[g](\psi, x, y) \in p[R].$$

So assume we have such an x. We now use the full Martin-Solovay construction, which gives a tree W such that $p[W] = \forall^{\mathbb{R}}\neg p[T]$ holds in all size $< \Omega$ extensions of V. Let Q be such that $\pi(Q) = W$. Suppose toward contradiction that it is not the case that $\exists y \in \mathbb{R} \cap M[g](\psi, x, y) \in p[R]$. We then get that from the elementarity of π that $(\psi, x) \in p[Q]$, which implies that $(\psi, x) \in p[W]$, so that $\forall y(\psi, x, y) \notin p[T]$, a contradiction. □

It is in verifying the witness condition part of $p[T] = (\mathbb{R}^*)^{V[G]}$ that we need weak homogeneity, and not just universal Baireness. The resulting generic absoluteness of the theory of $L(\mathbb{R})$ is used below.

Now fix Ω as in the hypotheses of 1.7.1. Fix also $x \in V_\Omega$. We are done if we can show that every maximal-above-some-η $K^c(x)$ construction reaches $M_\omega^\sharp(x)$, so fix a $K^c(x)$ which does not. It follows that there is a cardinal cutpoint ξ of $K^c(x)$ above which $K^c(x)$ has no Woodin cardinals.

Claim. $K^c(x)$ is $(\omega, \Omega+1)$ iterable above ξ.

Proof. We shall need the following basic facts: Let $\mathcal{Q}$ be a countable, ω small mouse with no Woodin cardinals over z, and $\mathcal{Q}$ project to z; then

(a) If $\mathcal{Q}$ is ω_1+1-iterable, then $\Sigma^t_{\mathcal{Q}} \restriction \mathrm{HC} \in L(\mathbb{R})$, and

(b) If $L(\mathbb{R}) \models \mathcal{Q}$ is ω_1-iterable, then $\mathcal{Q}$ is Ω-iterable in V.

For (a), see [41, §7]. Part (b) uses that any such strategy is determined by choosing the unique b such that $\mathcal{Q}(b, \mathcal{T})$ is ω_1-iterable in $L(\mathbb{R})$ (again, [41, §7]), together with the generic absoluteness of the theory of $L(\mathbb{R})$ given by 1.7.5.

To prove the Claim, is enough by 1.6.13 to see that whenever $\mathcal{T} \in V_\Omega$ is an iteration tree of limit length on $K^c(x)$ above ξ, then $\mathcal{Q}(\mathcal{T})$ exists. Let $\mathcal{T}$ be given, and let $\pi\colon M \to V_\theta$ be elementary, where M is countable transitive, and $\pi(\mathcal{U}) = \mathcal{T}$. By 1.6.11, $\mathcal{U}$ is by $\Sigma^t_{\mathcal{P}}$, where $\pi(\mathcal{P}) = K^c(x)$. It follows that $\mathcal{Q}(\mathcal{U})$ exists, and has an ω_1 iteration strategy in $L(\mathbb{R})$. Let g be generic for a poset of size $< \pi^{-1}(\Omega)$ over M, and make $\mathcal{U}$ and some initial segment of $\mathcal{P}$ on which $\mathcal{U}$ is based countable. Then $\mathcal{Q}(\mathcal{U})$ is definable from $\mathcal{U}$ over $(L(\mathbb{R}))^{M[g]}$ as the unique ω_1-iterable $\mathcal{Q}$-structure for $\mathcal{U}$. We use here the correctness of $(L(\mathbb{R}))^{M[g]}$ which follows from $(\mathbb{R}^*)^{M[g]} = \mathbb{R}^* \cap M[g]$. It follows that $\mathcal{Q}(\mathcal{U})$ is in M, and is ω_1-iterable in $L(\mathbb{R})^M$. But then $\mathcal{Q}(\mathcal{U})$ is $\pi^{-1}(\Omega)$-iterable in M, and from this we easily get that $\pi(\mathcal{Q}(\mathcal{U})) = \mathcal{Q}(\mathcal{T})$, so that indeed $\mathcal{Q}(\mathcal{T})$ exists. □

But now, letting $y = K^c(x)|\xi$, we see that $K(y)$ exists. Weak covering and the $L(\mathbb{R})$-definability of $K(y)$ imply that in some $V[G]$ for $G < \Omega$-generic over V, we have an uncountable sequence of distinct reals in $L(\mathbb{R})$. (See [41, 7.3,7.4].) But 1.7.5 then implies this sequence is ω_1-universally Baire in $V[G]$, contrary to our first lemma. □

Exercise 1.7.6 *Show that if every OD*$(\mathbb{R})$ *set of reals is weakly homogeneous, then there is a nontame mouse. (We do not know whether the hypothesis is consistent. Replacing OD*$(\mathbb{R})$ *with definability by a certain long game quantifier, one gets a consistent hypothesis which yields a nontame mouse.)*

Exercise 1.7.7 *Let* $c\colon HC \to \mathbb{R}$ *be some nice coding of hereditarily countable sets by reals. Let* Σ *be an* ω_1*-iteration strategy for a countable premouse* $\mathcal{M}$*, and suppose that*

$$I = \{c(\mathcal{T}) \mid \mathcal{T} \text{ is according to } \Sigma\}$$

is κ*-weakly homogeneous. Show that* Σ *extends to a* κ*-iteration strategy for* $\mathcal{M}$*.*

Chapter 2

The projective case

In this chapter we illustrate the use of Theorem 1.6.15 by showing how to get inner models with finitely many Woodin cardinals from various hypotheses. The proofs will comprise the first ω steps of the core model induction. Given a hypothesis, say φ, we aim to prove that in the light of φ the second alternative of the K-existence Dichotomy 1.6.15 cannot hold true. However, we need not have a measurable cardinal around; and even if so, its presence is often of no real help, as we may have to work more "locally." Hence whereas the key method is always the same, each φ comes with its own set of details with respect to the application of the core model induction technique.

Those applications fall into different classes according to where the extenders witnessing large cardinal properties in inner models come from. There are two main (nontrivial) sources for such extenders: (a) hull embeddings as in the proof of the Covering Theorem [20], and (b) embeddings coming from generic ultrapowers through ideals. Arguments which exploit (a) often need to make a reference to the following key result of Schimmerling and Zeman.

Theorem 2.0.8 (Schimmerling, Zeman, [30]) *Let $L[E]$ be a countably iterable extender model. If κ is not subcompact in $L[E]$, then $L[E] \models \square_\kappa$.*

Moreover, arguments of both types (a) and (b) often incorporate the local definability of K, cf. item (5) of Theorem 1.6.3.

We are now about to present a few sample arguments along these lines. For each application, we always choose an infinite cardinal ν and work "locally" to inductively prove that H_ν is closed under $M_n^\#$ for every $n < \omega$. (Sometimes, we may even work "globally" and have $\nu = \infty$.) There are often more hypotheses which the induction has to carry along.

Let X be swo'd, and let $n < \omega$. Recall that $M_n^\#(X)$ is the least sound X-mouse which projects to X and which is not n-small. By $M_n^{\#^\#}(X)$ we mean the least *active* sound X-mouse which projects to X and is closed under $x \mapsto M_n^\#(x)$. We'll frequently use $M_n^{\#^\#}(X)$ as a local universe in which to apply Theorem 1.6.15. Typical steps of inductions discussed in this chapter now are:

$$H_\nu \text{ is closed under } M_n^\# \qquad (\#)_{\nu,n}$$

$$H_\nu \text{ is closed under } M_n^{\#^\#} \qquad (\#^\#)_{\nu,n}$$

We need to prove $(\#)_{\nu,0}$ and $(\#)_{\nu,n} \implies (\#^\#)_{\nu,n}$ and $(\#^\#)_{\nu,n} \implies (\#)_{\nu,n+1}$ for all $n < \omega$. Often, there is are intermediate steps: for instance we might work with two (or even more) infinite cardinals $\bar\nu < \nu$ and show that $(\#)_{\bar\nu,0}$ and $(\#)_{\bar\nu,n} \implies (\#)_{\nu,n}$, $(\#)_{\nu,n} \implies (\#^\#)_{\bar\nu,n}$, $(\#^\#)_{\bar\nu,n} \implies (\#^\#)_{\nu,n}$, and $(\#^\#)_{\nu,n} \implies (\#)_{\bar\nu,n+1}$ for all $n < \omega$. The steps $(\#)_{\bar\nu,n} \implies (\#)_{\nu,n}$, and $(\#^\#)_{\bar\nu,n} \implies (\#^\#)_{\nu,n}$ are instances of *mouse reflection* and we deal with it first.

We shall give the arguments for $\neg\Box_\kappa$ and the existence of a saturated or even ω_1-dense ideal on ω_1 in great detail, as we think of them as prototype arguments. Moreover, the existence of an ω_1-dense ideal on ω_1 will play a role in later chapters of this book. On the other hand, the presentation of the other applications will be somewhat sketchy, and in these cases the reader will have to consult the literature for further detail.

2.1 Mouse reflection.

The following proof as well as many others needs the "lower part closure" of a given set x. Should the next mouse over x exists, then it must certainly be an initial segment of the lower part closure of x.

Definition 2.1.1 *Let x be swo. Let $\langle \mathcal{L}_\xi | \xi \leq \mathrm{OR}\rangle$ be defined as follows. Set $\mathcal{L}_0$ be the rud closure of x. Having defined $\mathcal{L}_\xi$, let $\mathcal{L}_{\xi+1}$ be the (model theoretic) union of all mice $\mathcal{M} \trianglerighteq \mathcal{L}_\xi$ such that $\mathcal{M}$ is sound, projects to $o(\mathcal{L}_\xi)$, and such that $o(\mathcal{L}_\xi)$ is a cutpoint of $\mathcal{M}$. Having defined $\mathcal{L}_\xi$ for all $\xi < \lambda$, where λ is a limit ordinal or $\lambda = \mathrm{OR}$, we let $\mathcal{L}_\lambda$ be the (model theoretic) union of all $\mathcal{L}_\xi$, $\xi < \lambda$. We finally set* $\mathrm{Lp}(x) = \mathcal{L}_{\mathrm{OR}}$. $\mathrm{Lp}(x)$ *is called the* lower part closure *of x.*

It is easy to see that $\mathrm{Lp}(x)$ is a mouse which does not have any total extenders. It is obtained as the "maximal" stack of mice over x with no total extenders.

Lemma 2.1.2 *Let J be a mouse operator which relativizes well. Then $Lp(x)$ is closed under J.*

Proof. Let $y \in Lp(x)$, say $y \in Lp(x)|\kappa$, where κ is a cardinal of $Lp(x)$. Then $J(Lp(x)|\kappa) \lhd Lp(x)$ by the construction of $Lp(x)$, so that $J(y) \in Lp(x)$ by Definition 3.1.5 (1). □

Definition 2.1.3 *For cardinals $\kappa < \lambda$ we say that* Mouse Reflection holds at (κ, λ) *if for every $A \in H_\kappa$ and for every mouse operator J over A, if J is total on H_κ, then J is also total on H_λ. If $\lambda = \kappa^+$, then we simply say that* Mouse Reflection holds at κ.

The statements $(\#)_{\kappa,n} \implies (\#)_{\lambda,n}$ and $(\#)_{\kappa,n} \implies (\#)_{\lambda,n}$ are therefore instances of (κ, λ)-mouse reflection.

Lemma 2.1.4 *Let κ be a singular cardinal, and suppose $\square_\kappa$ to fail. Then Mouse Reflection holds at κ with respect to mouse operators which relativize well.*

Proof. Fix $A \in H_\kappa$, and let J be a mouse operator over A which relativizes well. Let $B \in H_{\kappa^+}$. We need to see that $J(B)$ exists.

Let $W = Lp(B)$. Set $\eta = \kappa^{+W}$. By $\neg\,\square_\kappa$ and Theorem 2.0.8, $\eta < \kappa^+$, so that $\mathrm{cf}^V(\eta) < \kappa$. Let $X \prec H_{\kappa^+}$ be such that $B \in X$, $X \cap \eta$ is cofinal in η, and $\mathrm{Card}(X) = \mathrm{cf}^V(\eta) \cdot \aleph_1$. Let $\pi \colon \bar{H} \cong X$, and let $\bar{B} = \pi^{-1}(B)$ and $\bar{W} = \pi^{-1}(Lp(B)|\eta) = Lp(\bar{B})^{\bar{H}}$. Write $\bar{\kappa} = \pi^{-1}(\kappa)$ and $\bar{\eta} = \pi^{-1}(\eta) = \bar{W} \cap \mathrm{OR}$. Notice that $\bar{W}|\bar{\kappa} \in H_\kappa$, so that $J(\bar{W}|\bar{\kappa})$ exists.

We claim that $J(\bar{W}|\bar{\kappa}) \lhd \bar{W}$. Suppose not. There is then a least initial segment $\mathcal{P} \trianglelefteq J(\bar{W}|\bar{\kappa})$ such that $\mathcal{P} \rhd \bar{W}$ and $\rho_\omega(\mathcal{P}) < \bar{\eta}$. Say $\rho_{n+1}(\mathcal{P}) < \bar{\eta} \leq \rho_n(\mathcal{P})$. Let $E = E_{\pi \restriction \bar{W}}$ be the (long) extender derived from $\pi \restriction \bar{W}$, and let $\tilde{\mathcal{P}} = \mathrm{Ult}_n(\mathcal{P}; E)$ be the (fine) ultrapower of $\mathcal{P}$ by E. By the key technique of [20], we may have picked X in such a way that $\tilde{\mathcal{P}}$ is in fact a mouse. But then $W|\eta \lhd \tilde{\mathcal{P}}$ (as $X \cap \eta$ is cofinal in η) and $\rho_\omega(\tilde{\mathcal{P}}) \leq \kappa$. This contradicts the construction of $Lp(B)$.

We must therefore have that $J(\bar{W}|\bar{\kappa}) \lhd \bar{W}$. But then $\pi(J(\bar{W}|\bar{\kappa})) \lhd W$ and in fact $\pi(J(\bar{W}|\bar{\kappa})) = J(W|\kappa)$. Because J relativizes well, this now shows that $J(B)$ exists. □

The weak reflection principle $\mathsf{WRP}_{(2)}(\kappa)$ says that if $S, T \subset [\kappa]^\omega$ are both stationary, then there is some $\alpha < \kappa$ such that both $S \cap [\alpha]^\omega$ and $T \cap [\alpha]^\omega$ are stationary in $[\alpha]^\omega$.

Lemma 2.1.5 *Let κ be regular. If $\mathsf{WRP}_{(2)}(\kappa)$ holds, then Mouse Reflection holds at κ.*

PROOF. Let $B \in H_\kappa$, and let J be a mouse operator over B which is total on H_κ. Let us suppose that $B = \emptyset$. If A is a set of ordinals, then

$$J(A) = \mathrm{Hull}_{\Sigma_1}^{J(A)}(A)$$

(if $J(A)$ exists), so that we may identify $J(A)$ with its theory consisting of elements of the form $(\ulcorner\varphi\urcorner, \alpha_1, \ldots, \alpha_k) \in \omega \times [\sup(A)]^{<\omega}$ (where φ is a Σ_1 formula from the languange associated with $J(A)$).

Let us now fix $A \subset \kappa$. If $X \subset \kappa$, then we shall write π_X for the transitive collapse of X. Let $\vec{\alpha} = (\ulcorner\varphi\urcorner, \alpha_1, \ldots, \alpha_k) \in \omega \times [\sup(A)]^{<\omega}$ be given, and set

$$S_{\vec{\alpha}} = \{X \in [\kappa]^\omega \mid (\ulcorner\varphi\urcorner, \pi_X(\alpha_1), \ldots, \pi_X(\alpha_k)) \in J(\pi_X(A))\}$$

and

$$T_{\vec{\alpha}} = \{X \subset [\kappa]^\omega \mid (\ulcorner\varphi\urcorner, \pi_X(\alpha_1), \ldots, \pi_X(\alpha_k)) \notin J(\pi_X(A))\}.$$

Claim. One of $S_{\vec{\alpha}}$, $T_{\vec{\alpha}}$ contains a club.

PROOF. Otherwise both $S_{\vec{\alpha}}$ and $T_{\vec{\alpha}}$ are stationary in $[\kappa]^\omega$. By $\mathsf{WRP}_{(2)}(\kappa)$, pick $\delta < \kappa$ such that both $S_{\vec{\alpha}} \cap [\delta]^\omega$ and $T_{\vec{\alpha}} \cap [\delta]^\omega$ are stationary in $[\delta]^\omega$. We may pick δ such that $\alpha_1 \ldots \alpha_k < \delta$.

Say $(\ulcorner\varphi\urcorner, \alpha_1, \ldots, \alpha_k) \in J(A \cap \delta)$. Let θ be sufficiently big, and let $X \prec H_\theta$ be countable such that $\alpha_1, \ldots, \alpha_k$, δ, $J(A \cap \delta) \in X$. As $T_{\vec{\alpha}} \cap [\delta]^\omega$ is stationary in $[\delta]^\omega$, we may also arrange that $X \cap \delta \in T_{\vec{\alpha}}$. Let $\sigma\colon H \cong X$, where H is transitive, so that $\sigma^{-1} \restriction \delta = \pi_{X\cap\delta}$. Therefore, $\sigma^{-1}(A \cap \delta) = \pi_{X\cap\delta}''(A\cap\delta)$ and $\sigma^{-1}(J(A\cap\delta)) = J(\pi_{X\cap\delta}''(A\cap\delta))$. But $X\cap\delta \in T_{\vec{\alpha}}$, so that $(\ulcorner\varphi\urcorner, \pi_{X\cap\delta}(\alpha_1), \ldots, \pi_{X\cap\delta}(\alpha_k)) \notin J(\pi_{X\cap\delta}''(A \cap \delta))$. This is a contradiction! $\square$

Let us now set

$$\mathcal{P} = \{\vec{\alpha} = (\ulcorner\varphi\urcorner, \alpha_1, \ldots, \alpha_k) \in \omega \times [\sup(A)]^{<\omega} \mid S_{\vec{\alpha}} \text{ contains a club }\}.$$

By what we showed so far, $\mathcal{P}$ is a reasonable candidate for (the theory) of $J(A)$. We now need to see that (the structure coded by) $\mathcal{P}$ is a mouse.

For this, let $\tau\colon \bar{\mathcal{P}} \to \mathcal{P}$ be such that $\bar{\mathcal{P}}$ is countable. There is a club C in $[\kappa]^\omega$ such that for each $X \in C$ and for each $\vec{\alpha} \in \operatorname{ran}(\tau)$, $X \in S_{\vec{\alpha}}$ iff $\vec{\alpha} \in \mathcal{P}$. Let $\delta < \kappa$ be such that $C \cap [\delta]^\omega$ is stationary in $[\delta]^\omega$, and let θ be sufficiently big, and let $X \prec H_\theta$ be countable such that δ, $J(A \cap \delta) \in X$, and $X \cap \delta \in C$. Let $\sigma\colon H \cong X$, where H is transitive. Then for each $\vec{\alpha} \in \operatorname{ran}(\tau)$, $\vec{\alpha} \in \mathcal{P}$ iff $X \cap \delta \in C$ iff $\sigma^{-1}(\vec{\alpha}) \in J(\sigma^{-1}(A \cap \delta))$, so that $\sigma^{-1} \circ \tau\colon \bar{\mathcal{P}} \to J(A \cap \delta)$ is sufficiently elementary to verify that $\bar{\mathcal{P}}$ is an $\omega_1 + 1$ iterable premouse. $\square$

2.2 From J to $J^{\#}$.

Definition 2.2.1 *Let B be a set, and let $J(B)$ be a mouse operator over B. The mouse operator $J^{\#}$ is then defined as follows. For an A such that $B \in L[A]$ we let $J^{\#}(B)$ be the least initial segment $\mathcal{P}$ of $Lp(B)$ such that $\mathcal{P}$ is active and closed under J.*

The following Lemma is an abstract version of $(\#)_{\kappa^+,n} \Longrightarrow (\#^{\#})_{\kappa,n}$.

Lemma 2.2.2 *Let κ be a singular cardinal, and suppose $\square_\kappa$ to fail. Let $B \in H_\kappa$, and let J be a mouse operator over B which is total on H_{κ^+}. Then $J^{\#}$ is total on H_κ.*

Proof. Fix $A \in H_\kappa$. We need to see that $J^{\#}(A)$ exists. Let $W = Lp(A)$, so that $W|\kappa^+$ is closed under J. Set $\eta = \kappa^{+W}$. By $\neg\, \square_\kappa$ and Theorem 2.0.8, $\eta < \kappa^+$, so that $\mathrm{cf}^V(\eta) < \kappa$. The argument for Lemma 2.1.4 shows that we may pick some $X \prec H_{\kappa^+}$ such that $B \in X$, $X \cap \eta$ is cofinal in η, $\mathrm{Card}(X) = \mathrm{cf}^V(\eta) \cdot \aleph_1$, and if $\pi\colon \bar{H} \cong X$, then there is no $\pi^{-1}(\kappa)$-sound mouse $\mathcal{P}$ such that $\mathcal{P} \triangleright \pi^{-1}(W|\eta)$ and $\rho_\omega(\mathcal{P}) < \pi^{-1}(\eta)$. Let us write $\bar{W} = \pi^{-1}(W|\eta)$.

Set $\delta = \mathrm{crit}(\pi) < \kappa$. By the Condensation Lemma, $\bar{W}|\delta^{+\bar{W}} \triangleleft W$. We claim that in fact $\delta^{+\bar{W}} = \delta^{+W}$. Suppose not and let $\bar{\mathcal{P}} \triangleleft W$ be least such that $\bar{W}|\delta^{+\bar{W}} \triangleleft \bar{\mathcal{P}}$ and $\rho_\omega(\bar{\mathcal{P}}) \leq \delta$. The coiteration of $\bar{W}$, $\bar{\mathcal{P}}$ does not move $\bar{\mathcal{W}}$ (as $\mathcal{W}$ is a lower part model), and hence produces a $\pi^{-1}(\kappa)$-sound mouse $\mathcal{P}$ such that $\mathcal{P} \triangleright \pi^{-1}(W|\eta)$ and $\rho_\omega(\mathcal{P}) < \pi^{-1}(\eta)$, which gives a contradiction.

We therefore have $\bar{W}|\delta^{+\bar{W}} = W|\delta^{+W}$, so that from π we may derive a (short) W-extender E at (δ, δ^{++W}) such that $(W|\delta^{++W}, E)$ is a premouse. The argument of [20] shows that we may have picked X in such a way that $(W|\delta^{++W}, E)$ is now in fact a mouse (cf. also [29]). $\square$

2.3 From $J^{\#}$ to M_1^J.

In the middle of our induction, we often would like to make sense of the core model up to some cardinal $\nu \geq \aleph_2$ in situations where H_ν is sufficiently closed. The idea is to produce a "stable K." (Cf. Theorem 1.3.22.) The stable K idea may be used to show $(\#^{\#})_{\lambda,n} \Rightarrow (\#)_{\kappa,n+1}$ for $\lambda < \kappa$ under the right circumstances.

Definition 2.3.1 *Let κ be an uncountable cardinal. A* cone of elements of H_κ *is a set $\mathcal{C}$ for which there is some $A_0 \in H_\kappa$ (the* base of $\mathcal{C}$*) such that*

$$\mathcal{C} = \{A \in H_\kappa \colon A_0 \in L[A]\}.$$

A statement $\varphi(v)$ holds for a cone of elements of H_κ *if there is a cone $\mathcal{C}$ of elements of H_κ such that $\varphi(A)$ holds true for every $A \in \mathcal{C}$.*

Definition 2.3.2 *Let κ be an uncountable cardinal. Let J be a mouse operator such that $J^{\#}$ is total on H_{κ^+}. Let $B \in H_\kappa$, and assume that $M_1^J(B)$ does not exist. We then say that* there is a stable $K(B)$ up to κ^+ *if and only if for every $\alpha < \kappa^+$ there is a cone $\mathcal{C}$ of $A \in H_{\kappa^+}$ such that for all A, $D \in \mathcal{C}$,*

$$K(B)^{J^{\#}(A)}|\alpha = K(B)^{J^{\#}(D)}|\alpha.$$

[7, Theorem 3.11] now shows that if the continuum is small, then a stable K exists. (Cf. Theorem 1.3.22.) More precisely:

Theorem 2.3.3 ([7, Theorem 3.11]) *Let κ be a cardinal such that $2^{\aleph_0} \leq \kappa$. Let J be a mouse operator such that $J^{\#}$ is total on H_{κ^+}. Then for every $B \in H_\kappa$ exactly one of the following holds.*
(1) $M_1^J(B)$ exists, or else
(2) there is a stable $K(B)$ up to κ^+.

The first key ingredient of the proof of Theorem 2.3.3 is a result of Steel which basically is Theorem 2.3.3 for $\kappa = 2^{\aleph_0}$ (cf. [28]). The second key ingredient of the proof of Theorem 2.3.3 is the following result, cf [7].

Lemma 2.3.4 (Schindler) *Suppose that $K(X)$ exists for some X. Then the following holds true. For all $\kappa \geq \aleph_2 \cdot \mathrm{Card}(TC(\{X\}))$, if κ is a cardinal of $K(X)$, then $K(X)||\kappa^{+K(X)}$ is just the stack of collapsing mice for $K(X)||\kappa$.*

2.4 PFA and the failure of □.

Theorem 2.4.1 (Schimmerling for $n = 1$; Steel, Woodin, independently, for $n > 1$) *If* PFA *holds, then for every X and for every $n < \omega$, $M_n^\#(X)$ exists.*

Schimmerling's result, i.e., Theorem 2.4.1 for $n = 1$, was produced in [26]. A much stronger theorem than Theorem 2.4.1 can in fact be obtained by working with K^c rather than a core model induction, cf. [10]. Theorem 2.4.1 also follows from Theorem 2.4.3 in the light of the following Theorem of Todorcevic.

Theorem 2.4.2 (Todorcevic) *If* PFA *holds, then for all $\kappa \geq \omega_1$, $\Box_\kappa$ fails.*

Theorem 2.4.3 (Schimmerling, Steel) *If $\Box_\kappa$ fails, where κ is a singular cardinal such that $2^{\aleph_0} \leq \kappa$, then for every $X \subset \kappa$ and for every $n < \omega$, $M_n^\#(X)$ exists.*

Steel has actually shown that the hypothesis of Theorem 2.4.3, where $2^{\aleph_0} \leq \kappa$ is strengthened to stating that κ is a strong limit cardinal, implies that AD holds in $L(\mathbb{R})$ (cf. [39]). It is not known how to significantly strengthen the conclusion in Theorem 2.4.3.

Proof of Theorem 2.4.3. By a simultaneous induction, we shall verify $(\#)_{\kappa,n}$, $(\#)_{\kappa^+,n}$, $(\#^\#)_{\kappa,n}$, and $(\#^\#)_{\kappa^+,n}$ to hold for every $n < \omega$.

To verify $(\#)_{\kappa,0}$, let us start with the trivial mouse operator $J = \mathrm{id}$. By Lemma 2.2.2, H_κ is closed under $J^\#$, i.e., under the usual sharp operator $B \mapsto B^\#$.

The implications $(\#)_{\kappa,n} \implies (\#)_{\kappa^+,n}$ and $(\#^\#)_{\kappa,n} \implies (\#^\#)_{\kappa^+,n}$ are given by Lemma 2.1.4. To verify the implication $(\#)_{\kappa^+,n} \Longrightarrow (\#^\#)_{\kappa,n+1}$, we apply Lemma 2.2.2 to the mouse operator $J = M_n^\#$.

Finally, the implication $(\#^\#)_{\kappa^+,n} \Longrightarrow (\#)_{\kappa,n+1}$ is shown as follows. Fix $B \in H_\kappa$, and apply Lemma 2.3.3 to the mouse operator $J = M_n^\#$. If $M_1^J(B) = M_{n+1}^\#(B)$ does not exist, then Lemma 2.3.3 tells us that there is then a stable $K(B)$ of height κ^+, which we denote by $K(B)$.

By Theorem 2.0.8, $\kappa^{+K(B)} < \kappa^+$. Set $\eta = \kappa^{+K(B)}$. Let $C \subset \kappa$ be such that for every $D \subset \kappa$ with $C \in L[D]$,

$$K(B)|\eta = K(B)^{M_n^{\#^\#}(D)}|\eta$$

and

$$\eta = \kappa^{+K(B)^{M_n^{\#\#}(D)}}.$$

Let $D \subset \kappa$ be such that $C \in L[D]$ and $\eta < \kappa^{+L[D]}$. By Theorem 1.6.3 (4),

$$\eta < \kappa^{+L[D]} \leq \kappa^{+M_n^{\#\#}(D)} = \kappa^{+K(B)^{M_n^{\#\#}(D)}}.$$

This is a contradiction! □

2.5 Successive cardinals with the tree property

The following application is somewhat off the side, as the extenders are produced neither by hull embeddings nor by generic ultrapowers but rather by ultrapower embeddings generated from cardinals with the tree property.

Theorem 2.5.1 (Foreman, Magidor, Schindler, [6, Theorem 1.1]**)** *Suppose that $(\delta_n : n < \omega)$ is a strictly increasing sequence of cardinals with supremum λ such that for every $n < \omega$, both δ_n and δ_n^+ have the tree property. Suppose also that $2^{\aleph_0} < \lambda$. Then for all bounded $X \subset \lambda$ and for all $n < \omega$, $M_n^\#(X)$ exists.*

It is not known if the conclusion of Theorem 2.5.1 can be strengthened significantly. In particular, it is open whether its hypothesis implies $\mathsf{AD}^{L(\mathbb{R})}$.

The key fact here is the following.

Lemma 2.5.2 *([6, Lemma 2.1]) If η has the tree property and if M is an inner model such that η is inaccessible in M and* $\mathrm{cf}(\eta^{+M}) = \mathrm{cf}(\eta^{++M}) = \eta$, *then there is a countably complete elementary embedding $\pi\colon M \to N$ with critical point η such that N is transitive and π is discontinuous at η^{+M}.*

Exercise 2.5.3 *Prove Lemma 2.5.2.*

The proof of Theorem 2.5.1 will be similar in its structure to the proof of Theorem 2.4.3 and it will be fairly immediate from the following two lemmas.

Lemma 2.5.4 *Let δ and δ^+ have the tree property, and let J be a mouse operator over $B \in H_\delta$ which is total on H_{δ^+}. Then $J^\#$ is total on H_δ.*

PROOF. Let $A \in H_\delta$, where $B \in L[A]$, and let $M = Lp(A)$. By [9, p. 283], both δ and δ^+ have to be inaccessible in $Lp(A)$; cf. Exercise 2.5.5. Also, if $\mathrm{cf}^V(\delta^{+M}) < \delta$ or $\mathrm{cf}^V(\delta^{++M}) < \delta$, then $J^\#(A)$ exists by the argument for Lemma 2.2.2. Therefore, we may let $\pi\colon M \to N$ be as in Lemma 2.5.2.

We must have that $M|\delta^{+M} = N|\delta^{+N}$ by condensation and the definition of $Lp(A)$. From π, we may therefore derive an extender on M which, in much the same way as in the proof of Lemma 2.2.2, witnesses that $J^\#(A)$ exists. □

Exercise 2.5.5 *Assume* GCH *to hold in V. Let κ be a cardinal, and suppose $\square_\kappa$ to hold. Show that there is then a special κ^+-Aronszajn tree T, i.e., there is a κ^+-Aronszajn tree T together with some $s\colon T \to \kappa$ such that if $a \leq_T b$, then $s(a) \neq s(b)$. In particular, T will still be a κ^+-Aronszajn tree in any outer model $W \supset V$ unless W collapses κ^+ to κ.*

Lemma 2.5.6 *Let δ and δ^+ have the tree property, and let J be a mouse operator over $B \in H_\delta$ such that $J^\#$ is total on H_{δ^+}. Then M_1^J is total on H_δ*

PROOF. (Cf. Exercise 2.5.7.) Let $A \in H_\delta$, where $B \in L[A]$. Assuming that $M_1^J(A)$ doesn't exist, we can exploit Theorem 2.3.3 to build a stable $K(A)$ of height δ^+. By Excercise 2.5.5 and by Theorem 1.6.15 (4), δ is inaccessible in $K(X)$, there is no largest cardinal in $K(X)$, and $\mathrm{cf}^V(\delta^{+K(X)}) = \mathrm{cf}^V(\delta^{++K(A)}) = \delta$. Let $\pi\colon K(A) \to N$ be an elementary embedding as in Lemma 2.5.2. We have a contradiction with [37, Theorem 8.14 (3)]). □

Exercise 2.5.7 *Provide the details to the proof of Lemma 2.5.6.*

From the hypothesis of Theorem 2.5.1 we may now prove $(\#)_{\lambda,0}$ and $(\#)_{\lambda,n} \Longrightarrow (\#^\#)_{\lambda,n}$ and $(\#^\#)_{\lambda,n} \Longrightarrow (\#)_{\lambda,n+1}$ for every $n < \omega$ by the same methods as in the preceding section. This provides a proof of Theorem 2.5.1.

2.6 Pcf theory

Theorem 2.6.1 (Gitik, Schindler, [7, Theorem 1.4]**)** *Suppose that κ is a singular cardinal of uncountable cofinality such that the set*

$$\{\alpha < \kappa \ : \ 2^\alpha = \alpha^+\}$$

is stationary as well as costationary in κ. Then for all bounded $X \subset \kappa$ and for all $n < \omega$, $M_n^\#(X)$ exists.

Theorem 2.6.2 (Schindler, [7, Theorem 1.1]**)** *Suppose that $\aleph_\omega$ is a strong limit cardinal and $2^{\aleph_\omega} > \aleph_{\omega_1}$. Then for every bounded $X \subset \aleph_{\omega_1}$ and for all $n < \omega$, $M_n^\#(X)$ exists.*

Proof of Theorem 2.6.1. From the hypothesis of Theorem 2.6.1, we get that for every club $C \subset \kappa$ there is some strictly increasing sequence $(\kappa_n : n < \omega)$ with supremum $\tilde{\kappa}$ in C such that $\mathrm{pcf}(\{\kappa_n^+ : n < \omega\}) \geq \tilde{\kappa}^{++}$. The idea is that if the core model exists, then this can't be the case by a covering argument.

We prove by induction on n that for every bounded subset X of κ and for every $n < \omega$, $M_n^\#(X)$ exists. Fix X and n. First, for all bounded $Y \subset \kappa$, we get $M_n^{\#\#}(Y)$ by a covering argument. (This is easy; notice that the hypothesis implies that SCH fails cofinally often below κ.) Now suppose that $K(X)^{M_n^{\#\#}(Y)}$ exists for all Y above X. We may then use Lemma 2.3.4 to define a stable $K(X)$ of height κ in a straightforward way.

We may now pick the club $C \subset \kappa$ in such a way that we'll have the following for the strictly increasing sequence $(\kappa_n : n < \omega)$ which is given to us. For every $f \in \prod_{n<\omega} \kappa_n^+$, a covering argument produces some $\mathcal{M} \triangleleft K(X)||\tilde{\kappa}^+$ projecting to $\tilde{\kappa}$ such that for each $n < \omega$, $f(\kappa_n^+)$ (which is $< \kappa_n^+$) is contained (as a subset) in $\mathrm{Hull}^{\mathcal{M}}(\kappa_n \cup \{p\})$ (for some parameter p). But this clearly implies that $\mathrm{cf}(\prod_{n<\omega} \kappa_n^+) = \tilde{\kappa}^+$. Contradiction! □

The proof of Theorem 2.6.2 is similar.

It is not known how to get $AD^{L(\mathbb{R})}$ from one of the hypotheses of Theorems 2.6.1 or 2.6.2.

2.7 All uncountable cardinals are singular.

We now turn to two applications which exploit forcing absoluteness. The first one is a result for which we have to work in ZF rather than ZFC.

Theorem 2.7.1 (Busche, Schindler, [1, Lemma 4.2]**)** *Suppose that all uncountable cardinals are singular. Then for every set X of ordinals and every $n < \omega$, $M_n^\#(X)$ exists.*

[1, Theorem 1.5] actually shows that the hypothesis of Theorem 2.7.1 proves that AD holds in the $L(\mathbb{R})$ of a set-forcing extension of HOD.

The following is the key new technical ingredient. The reader may find a proof of Lemma 2.7.2 for the concrete case of a projective J in [1, Lemma 4.1].

Part of the (unstated) hypotheses of Lemma 2.7.2 is that V is a model of choice. In the course of the proof of Theorem 2.7.1Lemma, 2.7.2 will be applied basically to HOD.

Lemma 2.7.2 *Let B be a set, and let J be a mouse operator over B which condenses well and determines itself on set generic extensions. Let A be such that $B \in L[A]$, and let $\mathcal{M}$ be an A-premouse which is countably iterable via a strategy which is guided by J. Let $\mathbb{P} \in V$ be a partial order, and let G be $\mathbb{P}$-generic over V. Then in $V[G]$, $\mathcal{M}$ is iterable w.r.t. set-sized trees via a strategy which is guided by the extension of J to $V[G]$.*

PROOF. We shall also write J for the extension of J to $V[G]$. Suppose that in $V[G]$ there is some tree $\mathcal{T} = \dot{\mathcal{T}}^G$ on $\mathcal{M}$ which is guided by J such that either $\mathcal{T}$ has a last ill-founded model or else there is no cofinal branch through $\mathcal{T}$ such that an initial segment of $\mathcal{M}_b^{\mathcal{T}}$ is a $\mathcal{Q}$-structure for $\mathcal{M}(\mathcal{T})$ which is provided by (an initial segment of) $J(\mathcal{M}(\mathcal{T}))$. Let $p \in \mathbb{P}$ force this statement about $\check{\mathcal{M}}$ and $\dot{\mathcal{T}}$.

Let θ be large enough, and let $\pi \colon H \to H_\theta$ be elementary, where H is countable and transitive and $\mathcal{M}$, p, $\mathbb{P}$, $\dot{\mathcal{T}} \in \mathrm{ran}(\pi)$. Let us write $\bar{\mathcal{M}} = \pi^{-1}(\mathcal{M})$, $\bar{p} = \pi^{-1}(p)$, $\bar{\mathbb{P}} = \pi^{-1}(\mathbb{P})$, and $\bar{\dot{\mathcal{T}}} = \pi^{-1}(\dot{\mathcal{T}})$. Let $g \in V$ be $\bar{\mathbb{P}}$-generic over H, where $\bar{p} \in h$, and let $\bar{\mathcal{T}} = \bar{\dot{\mathcal{T}}}^g$.

It is not hard to see that by the fact that J condenses well, $\bar{\mathcal{T}} \in V$ is now guided by J, and that there is therefore some cofinal branch b through $\bar{\mathcal{T}}$ such that an initial segment of $\mathcal{M}_b^{\bar{\mathcal{T}}}$ is a $\mathcal{Q}$-structure for $\mathcal{M}(\bar{\mathcal{T}})$ which is provided by (an initial segment of) $J(\mathcal{M}(\bar{\mathcal{T}}))$. This $\mathcal{Q}$-structure is an element of $H[g]$, as J determines itself on set generic extensions. Moreover, b exists and is definable in $H[g]^{\mathrm{Col}(\omega,\rho)}$ (for some $H[g]$-cardinal ρ), so that in fact $b \in H[g]$, and b is the unique cofinal branch through $\bar{\mathcal{T}}$ such that an initial segment of $\mathcal{M}_b^{\bar{\mathcal{T}}}$ is a $\mathcal{Q}$-structure for $\mathcal{M}(\bar{\mathcal{T}})$ which is provided by (an initial segment of) $J(\mathcal{M}(\bar{\mathcal{T}}))$. This contradicts what $\bar{p}$ is supposed to force over H. □

PROOF of Theorem 2.7.1: By an induction, we'll verify that $(\#)_{\infty,n}$ and $(\#^{\#})_{\infty,n}$ holds for every $n < \omega$.

The argument for $(\#)_{\infty,0}$ is a simplified variant of the argument for $(\#)_{\infty,n} \implies (\#^\#)_{\infty,n}$. In the first case we work with the trivial mouse operator $J = \mathrm{id}$, and in the second case we work with $J = M_n^\#$. We fix A and want to see that $J^\#(A)$ exists.

Let us suppose that $J^\#(A)$ does not exist. Then $W = Lp(A)^{\mathsf{HOD}_{A\cup\{A\}}} = (L^J(A))^{\mathsf{HOD}_{A\cup\{A\}}}$. Let $\kappa > \sup(A)$ be an uncountable cardinal (in V). We have that $\mathrm{cf}^V(\kappa) = \mathrm{cf}^V(\kappa^{+W}) = \omega$, so that we may pick some cofinal $X \subset \kappa$ of order type ω and some cofinal $Y \subset \kappa^{+W}$ also of order type ω. We know that $X \oplus Y$ is Vopenka-generic over $\mathsf{HOD}_A[X,Y] \subset V$. By Lemma 2.7.2, W is still iterable in $\mathsf{HOD}_A[X,Y]$, κ is a singular cardinal in $\mathsf{HOD}_A[X,Y]$, and

$$\kappa^{+W} < \kappa^{+\mathsf{HOD}_A[X,Y]}.$$

We may then run the argument for Lemma 2.2.2 to show that $J^\#(A)$ exists after all.

The implication $(\#^\#)_{\infty,n} \implies (\#)_{\infty,n+1}$ is shown as follows. Let $J = M_n^\#$. Fix A. If $M_1^J(A) = M_{n+1}^\#(A)$ does not exist, then Lemma 2.3.3 tells us that inside HOD_A there is then a stable $K(A)$ of height OR, which we denote by $K(A)$. We now apply the argument from the preceeding paragraph to $W = K(A)$ to derive a contradiction. Notice that we may again apply Lemma 2.7.2 to deduce that W is still iterable in any set generic extension of HOD_A. □

It is shown in [?] that the hypothesis of Theorem ?? implies that AD holds in $L(\mathbb{R})$.

2.8 $L(\mathbb{R})$ absoluteness

Theorem 2.8.1 (Steel, Woodin) *Suppose that the (lightface) theory of $L(\mathbb{R})$ cannot be changed by set-sized forcing. Then for every X and for all $n < \omega$, $M_n^\#(X)$ exists.*

Woodin has shown that the hypothesis of Theorem 2.8.1 implies that $AD^{L(\mathbb{R})}$ holds.

Proof of Theorem 2.8.1. By applying the hypothesis to adding ω_1 Cohen reals, we see that there can't be a well-ordering of $\mathbb{R}$ in $L(\mathbb{R})$. The idea now is that if K were to exist, then such a well-ordering would have to exist after all.

As in the proof of Theorem 2.7.1, we shall inductively verify $(\#)_{\infty,n}$ and $(\#^\#)_{\infty,n}$ to hold for every $n < \omega$.

The argument for $(\#)_{\infty,0}$ is again a simplified variant of the argument for $(\#)_{\infty,n} \implies (\#^\#)_{\infty,n}$. In the first case we work with the trivial mouse operator $J = \mathrm{id}$, and in the second case we work with $J = M_n^{\#\#}$. We fix A and want to see that $J^\#(A)$ exists.

Let us suppose that $J^\#(A)$ does not exist. Then $W = Lp(A) = L^J(A)$. Let κ be a singular cardinal above the rank of A. If $\kappa^{+W} < \kappa^+$, then we may use the argument for Lemma 2.2.2 to show that $J^\#(A)$ exists after all. Let us thus assume that $\kappa^{+W} = \kappa^+$.

Let g be $\mathrm{Col}(\omega, \kappa)$-generic over V, so that $\omega_1^{V[g]} = \kappa^{+W}$. There is a further set generic forcing extension $V[g, G]$ of $V[g]$ in which there is a real x such that inside $V[g, G]$, a well-ordering of the reals may be defined from the parameters x and $W|\omega_1^{V[g]}$. (Force with $\mathrm{Col}(\omega_1, 2^{\aleph_0}) * \mathbb{Q}$, where $\mathbb{Q}$ uses an ω_1-sequence of pairwise almost disjoint reals from $W[g]$ to code a subset of ω_1 listing all the reals by a single real.) However, by Lemma 2.7.2, W is still iterable in $V[g, G]$ via a strategy which is guided by the extension of J to $V[g, G]$, and the extension of J to $V[g, G]$ is definable in the $L(\mathbb{R})$ of $V[g, G]$. This implies that there is a well-ordering of the reals in $L(\mathbb{R})^{V[g,G]}$. Contradiction!

The implication $(\#^\#)_{\infty,n} \implies (\#)_{\infty,n+1}$ is shown as follows. Let $J = M_n^\#$. Fix A. If $M_1^J(A) = M_{n+1}^\#(A)$ does not exist, then Lemma 2.3.3 tells us that there is then a stable $K(A)$ of height OR, which we denote by $K(A)$. We now apply the argument from the preceeding paragraph to $W = K(A)$ to produce $V[g, G]$ and derive a contradiction. Notice that we may again apply Lemma 2.7.2 to deduce that W is still iterable in any set generic extension of V. Theorem 1.6.15 (5) implies that we may again define a well-ordering of the reals in $L(\mathbb{R})^{V[g,G]}$. □

2.9 The unique branches hypothesis

The following is shown in [38].

Theorem 2.9.1 (Steel) *Suppose that there is a non-overlapping iteration tree $\mathcal{T}$ on V with cofinal well-founded branches $b \neq c$. Then for every bounded subset $X \in \mathcal{M}(\mathcal{T})$ of $\delta(\mathcal{T})$ and for all $n < \omega$, $M_n^\#(X)$ exists.*

This result is shown in [38]. It is actually shown there that its hypothesis gives the existence of $M_\omega^\#$.

PROOF of Theorem 2.9.1. Let $\mathcal{T}$ be a non-dropping iteration tree on V with cofinal well-founded branches $b \neq c$. Set $\delta = \delta(\mathcal{T})$, which is an ordinal of cofinality ω.

Lemma 2.9.2 (Woodin) *The set $\mathcal{P}(\delta) \cap \mathcal{M}_b^{\mathcal{T}} \cap \mathcal{M}_c^{\mathcal{T}}$ has size δ.*

The reason for this is that every $X \subset \delta$ in $\mathcal{M}_b^{\mathcal{T}} \cap \mathcal{T}_c^{\mathcal{T}}$ is coded by some bounded subset of δ in a fashion which comes out of the proof that δ is Woodin in $\mathcal{M}_b^{\mathcal{T}} \cap \mathcal{T}_c^{\mathcal{T}}$.

Another fact is that δ is either singular or measurable in $\mathcal{M}_b^{\mathcal{T}}$ as well as in $\mathcal{M}_c^{\mathcal{T}}$.

Let us now fix n, where we assume that $\mathcal{M}(\mathcal{T})$ is closed under $Y \mapsto M_n^\#(Y)$. Let us consider $X \in \mathcal{M}(\mathcal{T})$. If $M_{n+1}^\#(X)$ does not exist, then we may produce $W = K(X)^{\mathcal{M}(\mathcal{T})}$ by knitting together $K(X)^{\mathcal{M}(\mathcal{T})||\Omega}$ for $\mathcal{M}(\mathcal{T})$-measurables $\Omega < \delta$ (there are cofinally in δ many such). We may also let $\mathcal{P} \trianglerighteq W$ be the stack of all δ-sound premice end-extending W which project to δ and are countably iterable above δ. We also have versions of $\mathcal{P}$ inside $\mathcal{M}_b^{\mathcal{T}}$ and $\mathcal{M}_c^{\mathcal{T}}$, respectively; call them $\mathcal{P}_b$ and $\mathcal{P}_c$. W.l.o.g., $\mathcal{P}_b \trianglelefteq \mathcal{P}_c \trianglelefteq \mathcal{P}$. Because δ is Woodin in $\mathcal{M}_b^{\mathcal{T}} \cap \mathcal{M}_c^{\mathcal{T}}$, δ is Woodin in $\mathcal{P}_b$. A covering argument shows that $\mathcal{P}_b \cap \mathrm{OR} = \delta^{+\mathcal{M}_b^{\mathcal{T}}}$, and similarily for c.

Let $\mu \leq \delta$ be least such that $\pi_{0b}^{\mathcal{T}}(\mu) \geq \delta$. If $\pi_{0b}^{\mathcal{T}}(\mu) > \delta$, or else if $\pi_{0b}^{\mathcal{T}}$ is discontinuous at μ, then we may then produce inside $\mathcal{M}_b^{\mathcal{T}}$ (by a "lift up") a δ-sound premouse $\mathcal{Q}$ end-extending W which project to δ, is countably iterable above δ, and in fact kills the Woodinness of δ. This is nonsense.

So $\pi_{0b}^{\mathcal{T}}(\mu) = \delta$ and $\pi_{0b}^{\mathcal{T}}$ is continuous at μ. We then use Lemma 2.9.2 to get a contradiction. □

2.10 A homogeneous presaturated ideal on ω_1, with CH.

Our last set of results involves ideals on ω_1.

Theorem 2.10.1 (Steel) *Suppose that* CH *holds true. If there is a homogeneous presaturated ideal on ω_1, then* AD *holds in $L(\mathbb{R})$.*

In this section, we'll just prove:

Theorem 2.10.2 *Suppose that* CH *holds true. If there is a homogeneous presaturated ideal on ω_1, then for every $n < \omega$, H_{ω_2} is closed under $M_n^\#$.*

PROOF of Theorem 2.10.2: Let I be a homogeneous presaturated ideal on ω_1. If we force with I, we get a generic V-ultrafilter G and some

$$j\colon V \to M,$$

where M is transitive, $\mathrm{crit}(j) = \omega_1^V$, $j(\omega_1^V) = \omega_2^V$, and ${}^{<\omega_2^V}M \cap V[G] \subset M$ (so that $\omega_2^V = \omega_1^{V[G]} = \omega_1^M$). Let us fix j for the rest of this proof.

Definition 2.10.3 *Let J be a mouse operator which is total on H_{ω_2} and definable in V. We say that J has the* extension property *(*e.p.*, for short) iff $j(J)$ extends J and $j(J) \restriction \mathrm{HC}^{V[G]}$ is definable in $V[G]$. We say that ω_1 is J-inaccessible to the reals iff ω_1 is inaccessible in every $L^J_{\omega_2}(x)$, where x is a real.*

Let us now consider the following two statements.

Claim 1. Let J be a mouse operator which is definable in V and such that H_{ω_2} is closed under J. Suppose that J condenses well, J has the extension property, and ω_1 is J-inaccessible to the reals. Then H_{ω_2} is closed under $J^\#$, $J^\#$ condenses well, $J^\#$ has the extension property, and ω_1 is $J^\#$-inaccessible to the reals.

Claim 2. Let J be a mouse operator which is definable in V and such that H_{ω_2} is closed under J. Suppose that J condenses well, J has the extension property, and ω_1 is J-inaccessible to the reals. Then H_{ω_2} is closed under M_1^J, M_1^J condenses well, M_1^J has the extension property, and ω_1 is M_1^J-inaccessible to the reals.

Claim 1 and Claim 2 may be used in the obvious fashion to show inductively that for all $n < \omega$, $(\#)_{\omega_2,n}$ and $(\#^\#)_{\omega_2,n}$.

The proof of Claim 1 is a simplified version of the proof of Claim 2, so that we shall only give a proof of the latter. Let us suppose the hypothesis of Claim 2 to be satisfied. By Claim 1, H_{ω_2} is closed under $J^\#$, $J^\#$ condenses well, $J^\#$ has the extension property, and ω_1 is $J^\#$-inaccessible to the reals. Let us first aim at producing $M_1^J(x)$ for reals x.

So let us fix $x \in \mathbb{R}$.

Let $N = J^\#(\mathbb{R})$, which is an $\mathbb{R}$-premouse. (Here we use CH.) Let H be $Col(\omega_1, \mathbb{R})$-generic over N. (We may actually pick $H \in V$.) By the K^J-existence dichotomy, either $M_1^J(x)$ exists in N, or else $K(x)^{N[H]}$ exists. Suppose the latter to be true, and write $K = K(x)^{N[H]}$.

We claim that $j(K) \in V$. (This is where the homogeneity of I is used.) K is definable in $N[H]$, and hence in N, as $Col(\omega_1, \mathbb{R})$ is homogeneous. So

$j(K)$ is definable in $j(N)$. By the homogeneity of I, it therefore suffices to verify that $j(N)$ is definable in $V[G]$.

We claim that inside $V[G]$, $j(N)$ is the unique $\mathbb{R}^{V[G]}$-premouse $\mathcal{M}(\mathbb{R}^{V[G]})$ which satisfies all the relevant first order properties in order to be a candidate for $j(N)$ and which is such that whenever $\sigma\colon \bar{\mathcal{M}}(r) \to \mathcal{M}(\mathbb{R}^{V[G]})$, where $\bar{\mathcal{M}}(r)$ is countable, then $\bar{\mathcal{M}}(r)$ is $\omega_1^{V[G]}$ iterable via a strategy guided by $j(J^{\#}) \restriction \mathrm{HC}^{V[G]}$. This is true because this property holds of $\mathcal{M}(\mathbb{R}^{V[G]})$, and if $\sigma\colon \bar{\mathcal{M}}(r) \to \mathcal{M}(\mathbb{R}^{V[G]})$ and $\sigma\colon \bar{\mathcal{M}}'(r) \to j(N)$ are such that $\bar{\mathcal{M}}(r)$ and $\bar{\mathcal{M}}'(r)$ are both countable r-premice which are $\omega_1^{V[G]}$-iterable via a strategy guided by $j(J^{\#}) \restriction \mathrm{HC}^{V[G]}$, then they may be compared inside the model

$$L_{\omega_1}^{J^{\#}}(\bar{\mathcal{M}}(r), \bar{\mathcal{M}}'(r))$$

by the fact that ω_1 is $J^{\#}$-inaccessible to the reals. But now $j(J^{\#}) \restriction \mathrm{HC}^{V[G]}$ is definable in $V[G]$, and hence $j(N)$ is definable in $V[G]$.

Let us write $\kappa = \omega_1^V$. The fact that $j(K) \in V$ easily implies that κ is inaccessible in K: if $\kappa = \mu^{+K}$, then $\omega_2^V = j(\kappa) = \mu^{+j(K)}$, so that κ would not be a cardinal in V. This in turn implies that $\kappa^{+j(K)} < \omega_2^V = \omega_1^{V[G]}$: if we had $\kappa^{+j(K)} = \omega_2^V = \omega_1^{V[G]} = \omega_1^M$, then ω_1^V would be a successor in K by the elementarity of j. Notice also that $K|\kappa = j(K)|\kappa$ implies that $j(K)|j(\kappa) = j(j(K))|j(\kappa)$, and hence

$$j(j(K))|\kappa^{+j(j(K))} = j(K)|\kappa^{+j(K)}.$$

Notice here that because $j(K) \in V$, $j(j(K))$ makes sense.

Now let E be the extender at κ, ω_2^V derived from $j \restriction j(K)$. Notice that for every $\alpha < \omega_2^V$, $E \restriction \alpha \in M$ because of ${}^{<\omega_2^V} M \cap V[G] \subset M$.

Claim $(*)$**.** For every $\alpha < \omega_2^V$, $E \restriction \alpha \in j(K)$.

This claim easily gives that κ is Shelah in $j(K)$ by the following argument. Let $f\colon \kappa \to \kappa$, $f \in j(K)$. Pick some $\alpha > j(f)(\kappa)$, $\alpha < \omega_1^M$. If

$$k\colon Ult(j(K), E \restriction \alpha) \to j(j(K))$$

is the natural factor map, then $\mathrm{crit}(k) \geq \alpha$ and hence $i_{E\restriction\alpha}(f)(\kappa) \leq \alpha$.

However, there is no Shelah cardinal in K. Contradiction!

In order to verify Claim $(*)$, we need to see that

$$j(N) \models ((j(K), \mathrm{Ult}(j(K)), E \restriction \alpha), \alpha) \text{ is } j(\Omega)\text{−iterable}$$

where Ω is the largest measurable cardinal of N. By reflection, we'd otherwise have, in $j(N)$, $\sigma\colon \bar{K} \to \mathrm{Ult}(j(K), E \restriction \alpha)$ with $\sigma \restriction \alpha = \mathrm{id}$, $\bar{K}$ is countable, and

$$j(N) \models ((j(K), \bar{K}), \alpha) \text{ is not } \omega_1\text{−iterable.}$$

We have the factor map $k\colon \mathrm{Ult}(j(K), E \restriction \alpha) \to j(j(K))$ with $k \restriction \alpha = \mathrm{id}$, so that

$$k \circ \sigma\colon \bar{K} \to j(j(K))$$

is such that $k \circ \sigma \restriction \alpha = \mathrm{id}$.

Set $\psi = k \circ \sigma$. We have that ψ, $\bar{K} \in M$. Let $\psi = [\xi \mapsto \psi_\xi]_G$, $\bar{K} = [\xi \mapsto K_\xi]_G$, and $\alpha = [\xi \mapsto \alpha_\xi]_G$. We need to see that for G-almost all ξ,

$$N \models ((K, K_\xi), \alpha_\xi) \text{ is } \omega_1\text{−iterable.}$$

By absoluteness, for G-almost all ξ, in $j(N)$ there is some $\psi'_\xi\colon K_\xi \to j(K)$ such that $\psi'_\xi \restriction \alpha_\xi = \mathrm{id}$. But for any such ξ we then have in N some $\bar{\psi}\colon K_\xi \to K$ such that $\bar{\psi} \restriction \alpha_\xi = \mathrm{id}$. □ Claim $(*)$

We have verified that inside $J^\#(\mathbb{R})$, $M_1^J(x)$ exists. This premouse is now ω_1-iterable (in $J^\#(\mathbb{R})$ and hence in V) via a strategy which is guided by $J^\#$. By reflection, as H_{ω_2} is closed under $J^\#$, this premouse is then also ω_2-iterable via a strategy which is guided by $J^\#$. This shows that $M_1^J(x)$ exists in V and is ω_2-iterable there.

We are left with having to prove that if $X \subset \omega_1$, then $M_1^J(X)$ exists and is ω_2-iterable. Fix $X \subset \omega_1$. By the elementarity of j and by what was shown so far, inside M, $M_1^J(X)$ exists and is ω_2-iterable. By the argument for showing $j(N) \in V$ we have that in fact $M_1^J(X)$ is definable in $V[G]$ from the parameter $X \in V$, so that $M_1^J(X) \in V$. Moreover, countable premice which embed into $M_1^J(X)$ are ω_1-iterable via a strategy guided by J, so that by reflection those premice are also ω_2-iterable via a strategy guided by J. □

2.11 An ω_1-dense ideal on ω_1

If there is an ω_1-dense ideal on ω_1, then forcing with $\mathrm{Col}(\omega, \omega_1)$ produces a V-generic filter and an elementary embedding

$$j\colon V \to M \subset V[G],$$

where M is transitive, $\mathrm{crit}(j) = \omega_1^V$, and ${}^{\omega}M \cap V[G] \subset M$ (which implies $j(\omega_1^V) = \omega_1^M = \omega_1^{V[G]} = \omega_2^V$. Woodin showed that the existence of an ω_1-dense ideal on ω_1 may be forced over $L(\mathbb{R})$, assuming that $L(\mathbb{R})$ is a model of AD (cf. [47]). The following Theorem, also shown by Woodin, produces the converse.

Theorem 2.11.1 (Woodin) *If there is an ω_1-dense ideal on ω_1, then AD holds in $L(\mathbb{R})$.*

In this section, we shall prove the following statement.

Theorem 2.11.2 *If there is an ω_1-dense ideal on ω_1, then for every n, H_{ω_2} is closed under $M_n^{\#}$.*

Proof. We'll proceed in exactly the same fashion as in the proof of Theorem 2.10.2, with certain necessary adjustments of course. Again, it suffices to verify Claims 1 and 2 as they were formulated in the course of the proof of Theorem 2.10.2. As there, the proof of Claim 1 is a simplified version of the proof of Claim 2, so that we shall only give a proof of the latter. Let us suppose the hypothesis of Claim 2 to be satisfied. By Claim 1, H_{ω_2} is closed under $J^{\#}$, $J^{\#}$ condenses well, $J^{\#}$ has the extension property, and ω_1 is $J^{\#}$-inaccessible to the reals. Let us first aim at producing $M_1^J(x)$ for reals x.

So let us fix $x \in \mathbb{R}$. We shall make use of the concept of "α-strongness" from [22].

Let $\mathcal{I}$ be the collection of all countable mice. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a Skolem function for $(J_{\omega_1}(\mathbb{R}), \in, \mathcal{I})$, i.e., whenever φ is a formula, $\vec{z} \in \mathbb{R}$, and $(J_{\omega_1}(\mathbb{R}), \in, \mathcal{I}) \models \exists x \in \mathbb{R}\ \varphi(x, \vec{z})$ holds true, then $f(\ulcorner\varphi\urcorner, \vec{z})$ is a witness. Let $A_0 \subset \omega_1$ be such that in $J_{\omega_1}[A_0]$, every ordinal is at most countable. For any $A \subset \omega_1$, let $\bar{N}_A = J_{\omega_1}[A, A_0, f]$. Of course, $\bar{N}_A$ is closed under f. Also, $\bar{N}_A$ has the same reals as $L[\bar{N}_A]$ by an easy Skolem hull argument, so that $\bar{N}_A$ also has the same reals as $(\bar{N}_A)^{\#}$. Let us write $N_A = (\bar{N}_A)^{\#}$ for any $A \subset \omega_1$, and let us write $N = N_{\emptyset}$.

We need to see that $M_1^J(x)$ exists. Let us suppose that the $K^c(x)$ of no N_A reaches $M_1^{\#}$. We may then construct $K(x)^{N_A}$ for any $A \subset \omega_1$. Let us suppress x and write K^{N_A} for $K(x)^{N_A}$ in what follows.

We have that $J_{\omega_1}(\mathbb{R}^{N_A})$ and $J_{\omega_1}(\mathbb{R})$ have the same theory with real parameters for any $A \subset \omega_1$. Therefore,

$$K^{N_A}|\omega_1^V = K^N|\omega_1^V$$

for every $A \subset \omega_1$ by Theorem 1.6.15 (5). Moreover, for all countable premice $\mathcal{N}$, for every A with $\mathcal{N} \in N_A$ and for every $\alpha < \omega_1$, $\mathcal{N}$ is α-strong in N_A if and only if $\mathcal{N}$ is α-strong in V, if and only if $\mathcal{N}$ is α-strong in M (using the elementarity of j) if and only if $\mathcal{N}$ is α-strong in $V[G]$ (as M and $V[G]$ have the same reals). This implies that if $\mathcal{N}$ has size at most $\aleph_1$, then "$\mathcal{N}$ is $< \omega_1$-strong" is absolute between $N_{\mathcal{N}}$, V, M, and $V[G]$.

Subclaim 1. Let $\mathcal{M} \in N_A$ be ω_1-strong in N_A. Then $\mathcal{M}$ is also ω_1-strong in $V[G]$.

PROOF. Suppose not. Working in $V[G]$, pick a $< \omega_1$-strong premouse $\mathcal{N}$ such that the phalanx $(\mathcal{N}, \mathcal{M}, \omega_1)$ is not iterable. Pick $\tau \in V^{\mathrm{Col}(\omega,\omega_1)}$ with $\tau^G = \mathcal{N}$. We may assume τ to be "nice," so that it may be easily coded by a subset T of ω_1. Let us write $N_{A,\tau}$ for N_B, where $B = A \oplus T$. As G is also $\mathrm{Col}(\omega, \omega_1)$-generic over $N_{A,\tau}$, we have that $\mathcal{N} = \tau^G \in N_{A,\tau}[G]$. (We here use that the ideal be ω_1-dense to have that the forcing which produces j as well as names for elements of H_{ω_2} can be found in some N_B.)

Let $p \Vdash \tau$ is $< \omega_1^V$-strong. For all $q \leq p$, we may pick some $G_q \in N_{A,\tau}[G] \subset V[G]$ which is $\mathrm{Col}(\omega, \omega_1)$-generic over V and such that $N_{A,\tau}[G_q] = N_{A,\tau}[G]$. For each such q, τ^{G_q} will be $< \omega_1$-strong in $V[G]$. We also know that $K^{N_{A,\tau}}$ is $< \omega_1$-strong in $V[G]$. Therefore, for all $q \leq p$ and all $\alpha < \omega_1$,

$$(K^{N_{A,\tau}}, \tau^{G_q}, \alpha)$$

is iterable.

Let us fix $\alpha < \omega_1$ for a moment. We my then simultaneously compare all $(K^{N_{A,\tau}}, \tau^{G_q}, \alpha)$, $q \leq p$, with $K^{N_{A,\tau}}$. As $K^{N_{A,\tau}} = K^{N_{A,\tau}[G]} = K^{N_{A,\tau}[G_q]}$ for every $q \leq p$, this comparison produces one model $\mathcal{N}_\alpha^*$ such that for all $q \leq p$ there is an embedding

$$i \colon \tau^{G_q} \to \mathcal{N}_\alpha^*$$

with $\mathrm{crit}(i) \geq \alpha$ coming from the comparison. By homogeneity, $\mathcal{N}_\alpha^*$ is in $N_{A,\tau}$. In particular, there is now an embedding

$$i_\alpha \colon \mathcal{N} \to \mathcal{N}_\alpha^*$$

with $\mathrm{crit}(i) \geq \alpha$. We have that

$$(K^{N_{A,\tau}}, \mathcal{N}_\alpha^*, \alpha) \text{ is iterable,} \tag{2.1}$$

as every iteration of this phalanx may be construed as a continuation of an iteration of $(K^{N_{A,\tau}}, \mathcal{N}, \alpha)$.

Now if the phalanx $(\mathcal{N}, \mathcal{M}, \omega_1)$ is not iterable, then there is some $\alpha < \omega_1$ such that $(\mathcal{N}, \mathcal{M}, \alpha)$ is not iterable, which in turn implied that $(\mathcal{N}^*_\alpha, \mathcal{M}, \alpha)$ cannot be iterable. However, $\mathcal{N}^*_\alpha$ is $< \alpha$-strong in $N_{A,\tau}$ by (6.2.2, so that $(\mathcal{N}^*_\alpha, \mathcal{M}, \alpha)$ is in fact iterable. Contradiction! □

We have shown that "$\mathcal{N}$ is ω_1-strong" is absolute between $N_{A,\tau}$, V, and $V[G]$.

Now let S be the stack of all ω_1-strong $\mathcal{N} \rhd K^N|\omega_1$ which are sound and project to ω_1. We have that $S^V[G] = S^V$ using the homogeneity of $\mathrm{Col}(\omega, \omega_1)$. The generic embedding may easily be used to see that $o(S) < \omega_2^V$. Therefore, we may pick some $A \subset \omega_1$ with $S \in N_A$ and $\mathrm{Card}(o(S)) = \aleph_1$ in N_A. We shall now look at

$$j \upharpoonright N_A \colon N_A \to j(N_A)$$

. As in the proof of Theorem 2.10.2, the following Subclaim produces the desired contradiction.

Subclaim 2. Let $\alpha < \omega_2^V$, and let F be the (ω_1^V, α)-extender on $j(K^{N_A})$ derived from j. Then $F \in j(K^{N_A})$.

Proof. Notice that $F \in j(N_A)$ by Kunen's argument. To verify Subclaim 2, it thus suffices to prove that the phalanx

$$(K^{j(N_A)}, \mathrm{Ult}(K^{j(N_A)}, F), \alpha)$$

is iterable inside $j(N_A)$. But this is shown in [2]. □

The rest is as in the proof of Theorem 2.10.2. □

2.12 Open problems.

We aim to finish this chapter by stating a few open problems at the level of Projective Determinacy.

Greg Hjorth has shown that Π^1_2 Wadge determinacy is equivalent with Π^1_2 determinacy.

(1) Does Wadge determinacy for projective sets imply PD?

Steel has shown that if all projective sets are Lebesgue measurable and have the Baire property, and if all Π^1_3 relations can be uniformized by Π^1_3 functions, then Π^1_2 determinacy holds.

(2) If all projective sets are Lebesgue measurable and have the Baire property, and if all Π^1_{2n+1} relations can be uniformized by Π^1_{2n+1} functions, does PD hold?

There is a scenario for answering both questions in the affirmative by an induction as in this chapter. However, the argument will probably need a proof of the right correctness conjectures for the projective core models, cf. for instance [31].

Chapter 3

The witness dichotomy in $L(\mathbb{R})$

3.1 Core model theory for one J-Woodin

What we have done so far is start with some transitive model U closed under the mouse operator $J(x) = M_n^\sharp(x)$, and use core model theory to show that U is closed under the "one J-Woodin" operator $J^w(x) = M_{n+1}^\sharp(x)$.[1] We have done this by showing that the alternative to this closure is that there is a J-closed core model, cf. Theorem 1.6.15, then invoking whatever strong hypothesis we are considering to rule out the existence of such a core model.

This is the basic pattern of the core model induction at all its successor steps. The limit steps are devoted to constructing an operator J which is suitable as a basis for the next successor step. In order to go further, we must abstract what it is about J which makes it suitable. We begin the abstraction process in this section.

As usual, if ν is an infinite cardinal, then we shall write H_ν for $\{x \mid |\mathrm{TC}(x)| < \nu\}$. We should say here that we are going to use this notion also when ν is singular, in which case $H_\nu = \bigcup_{\mu<\nu} H_\mu$.

Definition 3.1.1 *Let ν be an infinite cardinal, and let $A \in H_\nu$. Then a* mouse operator over A on H_ν *is a function J such that for some Q-formula ψ,*

$$J(B) = \textit{least } \mathcal{P} \trianglelefteq \mathrm{Lp}(B) \textit{ such that } \mathcal{P} \models \psi[A, B]$$

[1] There was an intermediate step, in which we obtained closure under the operator $J^\sharp(x) = M_n^{\sharp^\sharp}(x)$.

for all $B \in H_\nu$ such that $A \in B$. (J must be defined at all such B.) We call J a (ν, A)-mo. The (ν, A)-mo J is called tame *iff for every $B \in \text{dom}(J)$, $J(B)$ is a tame B-premouse.*

Assuming that they are defined on all of H_ν, the $M_n^\sharp$ operators are $(\nu, 0)$-mouse operators, for each $n \leq \omega$. The weakest operator beyond all the $M_n^\sharp$ operators, for n finite, is $x \mapsto \bigcup_n M_n^\sharp(x)$. This would be the operator we would construct at step ω of the core model induction, as a basis for the steps $\omega + n$, for $n < \omega$. A core model induction which remains in $L(\mathbb{R})$ will never reach the $M_\omega^\sharp$ operator.

A (ν, A)-mo may not be a first order mouse operator in the sense we defined it earlier, because we are allowing a name for A, and not just B, in the theory which fixes $J(B)$. In particular, a (ν, A)-mo will only satisfy condensation for hulls containing $\text{TC}(A \cup \{A\})$. A $(\nu, 0)$-mo is a first order mouse operator, but the converse may fail because we have imposed a special restriction on the domains. The domain of a (ν, A)-mo is the "cone above A" in H_ν.

Definition 3.1.2 *Letting J be a (ν, A)-mo, we define the* one J-Woodin operator J^w *by*

$$\begin{aligned} J^w(B) &= M_1^J(B) = \textit{ least } \mathcal{P} \trianglelefteq \text{Lp}(B) \textit{ s.t. } \exists \delta \\ & \quad\ \mathcal{P} \models \delta \textit{ is Woodin, and } \mathcal{P} = J(\mathcal{P}|\delta). \end{aligned}$$

So if J^0 is the sharp operator, and $J^{n+1} = (J^n)^w$, then J^n is the $M_n^\sharp$ operator.

Exercise 3.1.3 *Show that if δ is the Woodin cardinal of $J^w(B)$ and if $\eta < \delta$, then $J(J^w(B)|\eta) \lhd J^w(B)$.*

Definition 3.1.4 *Let J be a (ν, A)-mo, and let $\mathcal{M}$ be a premouse over A; then $\mathcal{M}$ is J–*closed above *η iff whenever $\eta < \xi < \nu$ and ξ is a cardinal of $\mathcal{M}$, then $J(\mathcal{M}|\xi) \trianglelefteq \mathcal{M}$. We say $\mathcal{M}$ is J-*closed *iff $\mathcal{M}$ is J-closed above* 0.

Definition 3.1.5 *A (ν, A)-mo J* relativises well *iff*

(1) there is a formula $\theta(u, v, w, z)$ such that whenever $B, C \in \text{dom}(J)$, $B \in C$, and N is a transitive model of ZFC^- such that $J(C) \in N$, then $J(B) \in N$ and $J(B)$ is the unique $x \in N$ such that $N \models \theta[x, A, B, J(C)]$, and

(2) if $B \in \mathrm{dom}(J)$ and η is a cutpoint of $J(B)$, then $J(J(B)|\eta)$ is not a proper initial segment of $J(B)$.

Suppose that $n \leq \omega$, and that the $M_n^\sharp$ operator is defined on H_ν. It is then a $(\nu, 0)$-mo which relativises well. There are many more examples.

The next two lemmas motivate definition 3.1.5. Definition 3.1.5 (1) easily gives:

Lemma 3.1.6 *Let J be a (ν, A)-mo which relativises well, and let $\mathcal{M}$ be a J-closed premouse over A such that $\mathcal{M}|\eta \models \mathsf{ZFC}^-$ for arbitraily large $\eta < o(\mathcal{M})$; then*

(a) for all $x \in \mathrm{dom}(J) \cap \mathcal{M}$, $J(x) \in \mathcal{M}$, and

(b) $J \restriction \mathcal{M}$ is definable over $\mathcal{M}$.

Definition 3.1.7 *Let J be a (ν, A)-mo, and let $B \in \mathrm{dom}(J)$. An iteration tree $\mathcal{T}$ on $J(B)$ is J–*guided *iff for every limit ordinal $\lambda < \mathrm{lh}(\mathcal{T})$,*

$$\mathcal{Q}([0, \lambda]_\mathcal{T}, \mathcal{T}) \trianglelefteq J(\mathcal{M}(\mathcal{T} \restriction \lambda)).$$

We say that $J(B)$ has a J-guided ν-iteration strategy iff for every J-guided iteration tree $\mathcal{T}$ on $J(B)$ of limit length $\lambda < \nu$ there is a (unique) cofinal branch b through $\mathcal{T}$ such that $J(\mathcal{M}(\mathcal{T})) \trianglelefteq \mathcal{M}_b^\mathcal{T}$.

Lemma 3.1.8 *Let J be a tame (ν, A)-mo which relativises well; then for any $B \in \mathrm{dom}(J)$, $J(B)$ has a J-guided ν-iteration strategy.*

Proof. Let $\mathcal{T}$ be a J-guided tree on $J(B)$ of limit length $< \nu$. Note that $J(\mathcal{M}(\mathcal{T}))$ exists. Taking a countable Skolem hull of V as usual (cf. the proof of Lemma 1.6.12), with $\mathcal{P}, \mathcal{U}, \mathcal{R}$ being the images of $J(B), \mathcal{T}, J(\mathcal{M}(\mathcal{T}))$ under collapse, we get that $\mathcal{Q}(b, \mathcal{U}) \trianglelefteq \mathcal{R}$, where b is the good branch of $\mathcal{U}$. This is because clause (2) of Definition 3.1.5 is a first order property of $J(B)$ (relative to A, B). This property (relative to the collapses of A and B) holds in $\mathcal{P}$, and by inspecting it, we see that it also holds in $\mathcal{Q}(b, \mathcal{U})$, and that therefore $\mathcal{Q}(b, U) \trianglelefteq \mathcal{R}$. □

The following is our basic core model dichotomy theorem. (Cf. Theorem 1.6.15.)

Theorem 3.1.9 **(K^J–existence dichotomy)** *Let Ω be measurable, and J an (Ω, B)-mo which relativises well. Suppose $K^c(B)$, as constructed with τ-complete background extenders from V_Ω, is tame, and suppose $\xi < \Omega$ is the strict sup of the Woodin cardinals of $K^c(B)$, with $\xi = 0$ if there are none; then*

(a) $K^c(B)$ is J-closed above ξ, and

(b) if there is no τ-iterable $M_\omega^\sharp$, then either

(i) $M_1^J(B) \trianglelefteq K^c(B)$ (so that $M_1^J(B)$ exists and is τ-iterable), or

(ii) $K^c(B)$ is $(\omega, \Omega+1)$-iterable, so that $K(B)$ exists.

Proof. For (a): Compare $J(K^c(B)|\xi)$ with $K^c(B)$. The $J(K^c(B)|\xi)$ side provides the $\mathcal{Q}$-structures identifying the good branch on the $K^c(B)$ side, so this can be done. We have the branches at Ω by measurability. The $J(K^c(B)|\gamma)$ side comes out shorter because $K^c(B)$ is universal.

For (b): If $K^c(B)$ has a Woodin cardinal, then it has a largest one δ. (This is where we use that $M_\omega^\sharp$ does not exist.) But then $J(K^c(B)|\delta) \trianglelefteq K^c(B))$ by (a), so $K^c(B)$ reaches a one-J-Woodin level. We leave it as an exercise to show that $M_1^J(B) \trianglelefteq K^c(B)$ (cf. Exercise 3.1.10).

So we may assume $K^c(B)$ has no Woodin cardinals, and fails to reach $M_1^J(B)$. In this case, J guides an (ω, Ω)- iteration strategy for $K^c(B)$.

□

Exercise 3.1.10 *Show that $M_1^J(B) \trianglelefteq K^c(B)$ in the proof of Theorem 3.1.9 (b).*

Exercise 3.1.11 *Show that under the hypotheses of 3.1.9, if J relativises well, then $K^c(B)$ is fully J-closed. We shall not need this fact, however.*

We sometimes write $K^{c,J}(B)$ for $K^c(B)$, and $K^J(B)$ for $K(B)$, for such J-closed $K^c(B)$ or $K(B)$. When dealing later with *hybrid mouse operators* J, one must close the analog of $K^{c,J}$ under J by explicitly telling the model how J acts on its levels. At the current level of abstraction, $K^{c,J}(B)$ is an ordinary B-mouse, simply equal to $K^c(B)$, and the notation just reminds us it is J-closed, and can define J restricted to itself.

Theorem 3.1.9 uses the hypothesis that $M_\omega^\sharp$ does not exist in a fairly inessential way. If we had defined $K^{c,J}$ by explicitly closing its levels under J, we could have avoided this hypothesis. We chose to defer a discussion of the explicit-closure construction until we are dealing with hybrid J, so that we need it. The price is that we have to *prove* that our K^c is J-closed, and in that proof, the assumption that K^c has a largest Woodin, if it has any, shows up.

By applying our K^J-dichotomy inside suitably chosen J-closed inner models, we get a "stable K^J" dichotomy theorem which requires no large cardinal hypothesis on V whatsoever. The following version will suffice for our global-strength applications.

Suppose κ is uncountable, $\kappa = |V_\kappa|$, J is a (κ, A)-mo, and $B \in \text{dom}(J)$. We say then that $K^J(B)^{V_\kappa}$ *exists* iff for any $\alpha < \kappa$ such that $B \in V_\alpha$, there is a model $N = N_\alpha$ of ZFC such that $V_\alpha \in N$, and an $\Omega > \alpha$ such that Ω is measurable in N, and

$$N \models K^{c,J}(B) \text{ is } (\omega, \Omega + 1)\text{-iterable.}$$

In this case, $(K^J(B))^N$ exists, of course, by [37]. More importantly,

$$K^J(B)^{N_\alpha}|\alpha = K^J(B)^{N_\beta}|\alpha,$$

for all $\beta \geq \alpha$, because the inductive definition of K is local. When $K^J(B)^{V_\kappa}$ exists, then it *is* the limit of these approximating K's. Note that $K^J(B)^{V_\kappa}$ has all the local properties of $K(B)$; for example, it computes successors of singular cardinals ξ such that $\text{rk}(B) < \xi < \kappa$ correctly.

Theorem 3.1.12 (Stable-K^J dichotomy) *Let $\kappa > \omega$ be such that $\kappa = |V_\kappa|$, and let J be a (κ, A)-mo which relativises well. Suppose that there is no κ-iterable $M_\omega^\sharp$, and let $B \in \text{dom}(J)$; then either*

(a) $M_1^J(B)$ exists, and is κ-iterable, or

(b) for some $C \in \text{dom}(J)$, $K^J(C)^{V_\kappa}$ exists, and is the minimal J-closed model of height κ over C.

In order to use these dichotomy theorems to obtain more than one Woodin cardinal, we need also the following.

Exercise 3.1.13 *Let J be a (ν, A)-mo which relativises well, and suppose that J^w is also a (ν, A)-mo. Shaow that J^w relativises well.*

3.2 The coarse mouse witness condition W^*_α

The induction variable in a core model induction represents the degree of correctness of the mice one can construct, or what is roughly equivalent, the complexity of their iteration strategies. The description of how these two measures move in tandem is at the heart of the method.

The useful measure of correctness and complexity here is descriptive set theoretic. We saw in Lemma 1.6.12 that we should expect to obtain uncountable iterability from sufficiently generically absolute ω_1-iterability for countable structures. An ω_1-iteration strategy for a countable premouse is a set of reals, so our measure of "sufficient absoluteness" should be a measure of definability for sets of reals. For this reason, even when one is trying to show some very large transitive set X is closed under mouse operators, one generally moves to a generic extension $V[g]$ with $X \subseteq \mathrm{HC}^{V[g]}$, and shows $\mathrm{HC}^{V[g]}$ is appropriately closed. In the ensuing discussion, we shall basically assume we are already in such a $V[g]$.

Our plan then is to construct mice which are correct for some given level Γ of the Wadge hierarchy, via an induction on those levels. For now, we shall remain within $L(\mathbb{R})$, but eventually we shall consider Γ which are beyond $L(\mathbb{R})$. Descriptive set theory is used to organize the induction, and in particular, the next Γ to consider is the next scaled pointclass.

There will actually be two hypotheses of the existence of correct mice, or *mouse witnesses*. In the first, the witnessing mouse is coarse-structural, and in the second, it is an honest fine-structural mouse. We begin with the coarse-structural witness condition.

Definition 3.2.1 *Let $U \subseteq \mathbb{R}$, and $k < \omega$. Let N be countable and transitive, and suppose $\delta_0, ..., \delta_k, S,$ and T are such that*

(a) $N \models \mathsf{ZFC} \wedge \delta_0 < ... < \delta_k$ are Woodin cardinals,

(b) $N \models S, T$ are trees which project to complements of one another after the collapse of δ_k to be countable, and

(c) there is an ω_1+1-iteration strategy Σ for N such that whenever $i\colon N \to P$ is an iteration map by Σ and P is countable, then $p[i(S)] \subseteq U$ and $p[i(T)] \subseteq \mathbb{R} \setminus U$.

Then we say that N is a coarse (k, U)-Woodin mouse*, as witnessed by $S, T, \Sigma, \delta_0, ..., \delta_k$.*

Using the extender algebra, it is easy to see that in the situation of the preceeding definition,

$$U = \bigcup\{p[i(S)] \mid i\colon N \to P \text{ is a countable iteration by } \Sigma\}.$$

Let $A \subset \mathbb{R}$, say (in general, think of A as $\subset \mathbb{R}^i$ for some $i < \omega$). Recall that A is (α–) *Suslin* if A is the projection of a tree T on $\omega \times \alpha$ (written $A = p[T]$). A is said to have a *scale* if there is a sequence $(\varphi_n\colon n < \omega)$ of norms on A, i.e., $\varphi_n\colon A \to \mathrm{OR}$ for all $n < \omega$, such that whenever $(x_k : k < \omega)$ is a sequence of reals in A converging to x such that for each $n < \omega$, $\varphi_n(x_k)$ is eventually constant as $k \to \infty$, say with eventual value α_n, then $x \in A$ and $\varphi_n(x) \leq \alpha_n$ for each $n < \omega$. Often, the following *tree from a scale* construction is very helpful. If $(\varphi_n\colon n < \omega)$ is a scale on A, then we may set

$$(s, (\alpha_n\colon n < lh(s))) \in T \text{ iff } \exists x \supset s\ \forall n < lh(s) \varphi_n(x) = \alpha_n.$$

Then T witnesses that A is Souslin, i.e., $A = p[T]$, and if $x \in A$, then T_x has an honest leftmost branch f_x (i.e., $\forall g \in [T_x] \forall n < \omega f_x(n) \leq g(n)$). f_x is defined just by $f_x(n) = \varphi_n(x)$ for $n < \omega$.

Let Γ be a pointclass, i.e., a collection of sets of reals. A is then said to have a Γ–scale if for every $x \in A$, the relation (in y, n)

$$y \in A \wedge f_y(n) \leq f_x(n)$$

is in $\Delta(x)$, uniformly in x.[2] Finally a pointclass Γ is said to have the scale property if every $A \in \Gamma$ admits a Γ–scale.

Our inductive hypothesis now is

(W^*_α) Let U be a subset of $\mathbb{R}$, and suppose there are scales $\vec{\phi}$ and $\vec{\psi}$ on U and $\mathbb{R} \setminus U$ respectively such that $\vec{\phi}^*, \vec{\psi}^* \in J_\alpha(\mathbb{R})$, where $\vec{\phi}^*$ and $\vec{\psi}^*$ are the sequences of prewellorders associated to the scales. Then for all $k < \omega$ and $x \in \mathbb{R}$ there are N, Σ such that

(1) $x \in N$, and N is a coarse (k, U)-Woodin mouse, as witnessed by Σ, and

(2) $\Sigma \restriction \mathrm{HC} \in J_\alpha(\mathbb{R})$.

[2] I.e., there is $\leq^1 \in \Gamma$ and $\leq^2 \in \breve{\Gamma}$ such that for every $x \in A$, $\forall y\ \forall n\ ((y \in A \wedge f_y(n) \leq f_x(n)) \leftrightarrow (x, y, n) \in \leq^1)$ and $\forall y\ \forall n\ ((y \in A \wedge f_y(n) \leq f_x(n)) \leftrightarrow (x, y, n) \in \leq^2)$

We emphasize that in W^*_α, it is the *sequences* $\vec{\phi}^*, \vec{\psi}^*$ which are in $J_\alpha(\mathbb{R})$, not just the individual prewellorders in the sequences.

In the end, the mice we construct to verify W^*_α will not be particularly coarse; they will either be ordinary mice constructed from fine extender sequences, or *hybrid* mice, constructed from a fine extender sequence and an iteration strategy. In either case they will have a fine structure, and be suitable for building core models. We shall use the core model theory of [37] to construct them.

One can think of W^*_α as asserting, for the given U, that there is a *mouse operator* $x \mapsto \mathcal{M}_x$, defined on $x \in \mathbb{R}$ such that $\mathcal{M}_x$ is a (k, U)-Woodin mouse over x.

3.3 Scales in $L(\mathbb{R})$

One proves the Witness Dichotomy 3.6.1, cf. below, by considering the least α for which W^*_α fails, and analyzing the situation well enough to obtain alternative (b). The W^*_α assert that, given $U \subseteq \mathbb{R}$, as soon as a scale on U appears, an iteration strategy for a coarse mouse having a forcing term for such a scale appears. Thus it is useful to know how scales appear. Under appropriate determinacy hypotheses, there is in the Wadge hierarchy of $L(\mathbb{R})$ (and beyond) a tight correspondence between the appearance of scales on sets which did not previously admit them, and certain failures of reflection. This correspondence is analysed in detail in [35] and [40]. Our proof of Theorem 3.6.1 breaks into cases which reflect that analysis.

Definition 3.3.1 *An ordinal β is* critical *just in case there is some set $U \subseteq \mathbb{R}$ such that U and $\mathbb{R} \setminus U$ admit scales in $J_{\beta+1}(\mathbb{R})$, but U admits no scale in $J_\beta(\mathbb{R})$.*

(Once again, we are identifying a scale with the *sequence* of its pre-wellorders here.) Clearly, we need only show that $W^*_{\beta+1}$ holds whenever β is critical, in order to conclude that W^*_α holds for all α.

It follows from [35] that if β is critical, then $\beta + 1$ is critical. Moreover, if β is a limit of critical ordinals, then β is critical if and only if $J_\beta(\mathbb{R})$ is not an admissible set.

Lemma 3.3.2 (Steel, [35]) *Suppose W^*_β to hold. Let β be critical. Then one of the following holds true.*

(1) $\beta = \eta + 1$, for some critical η;

(2) β is a limit of critical ordinals, and either

(a) $\mathrm{cof}(\beta) = \omega$*, or*

(b) $\mathrm{cof}(\beta) > \omega$*, but $J_\beta(\mathbb{R})$ is not admissible;*

(3) $\alpha = \sup(\{\eta < \beta \mid \eta$ is critical $\})$ is such that $\alpha < \beta$, and either

(a) $[\alpha, \beta]$ is a weak Σ_1 gap, or

(b) $\beta - 1$ exists, and $[\alpha, \beta - 1]$ is a strong Σ_1 gap.

In this case, we also have that

(i) every set of reals in $J_\beta(\mathbb{R})$ has a scale such that each one of the associated prewellorderings belongs to $J_\beta(\mathbb{R})$, and

(ii) if $n < \omega$ is least with $\rho_n(J_\beta(\mathbb{R})) = \mathbb{R}$ and if A is a boldface $\Sigma_n(J_\beta(\mathbb{R}))$ set of reals, then we may write $A = \bigcup_{k<\omega} A_k$, where each A_k is in $J_\beta(\mathbb{R})$.

Moreover, β is critical iff

(1) β is inadmissible and β begins a gap, or

(2) β ends a proper weak gap.

If β is inadmissible and begins a gap, then the pointclasses $\Sigma_{2n+1}^{J_\beta(\mathbb{R})}$ and $\Pi_{2n+2}^{J_\beta(\mathbb{R})}$ have the scale property, and if β ends a proper weak gap, then the pointclasses $\Sigma_{2n+k}^{J_\beta(\mathbb{R})}$ and $\Pi_{2n+k}^{J_\beta(\mathbb{R})}$ have the scale property, where $k < \omega$ is least with $\rho_k(J_\beta(\mathbb{R})) = \mathbb{R}$.

Definition 3.3.3 *We shall call Γ a* scaled Σ-pointclass in $L(\mathbb{R})$ *if and only if for some n, $\Gamma = \Sigma_{2n+1}^{J_\beta(\mathbb{R})}$, where β is inadmissible and begins a gap, or else for some n, $\Gamma = \Sigma_{2n+k}^{J_\beta(\mathbb{R})}$, where β ends a proper weak gap and $k < \omega$ is least with $\rho_k(J_\beta(\mathbb{R})) = \mathbb{R}$.*

In each case, we prove 3.6.1 by constructing a (ν, A)-hmo J such that, were J^w, $(J^w)^w$,..., etc., to exist, then collectively they would yield mice which verify $W^*_{\beta+1}$. Thus, we must find a way of feeding truth at the bottom of the Levy hierarchy of $J_\beta(\mathbb{R})$ into mice. In cases 1 and 2(a), this is fairly easy: $\Sigma_1^{J_\beta(\mathbb{R})}$ is the class of countable unions of sets belonging to $J_\beta(\mathbb{R})$, so we can just put together countably many mice given by our induction hypothesis. The case 2(b) is somewhat harder. We shall call the cases 1, 2(a), and 2(b) the *inadmissible cases*, and we shall give them a thorough treatment in the next chapter.

Case 3, the end-of-gap (in scales) case, is the most subtle. In this case, the set coding truth at the bottom of the Levy hierarchy over $J_\beta(\mathbb{R})$ which we feed into our mice will be an *iteration strategy* Σ for a mouse $\mathcal{M}$ with a Woodin cardinal which is Lp^α-full, in a sense we shall explain. The structures which witness the truth of $W^*_{\beta+1}$ will be *hybrid* Σ*-mice*, mice over $\mathcal{M}$ constructed from an extender sequence as usual, while simultaneously closing under Σ. We shall go further into case 3 in the second next chapter.

In $L(\mathbb{R})$, every set is ordinal definable from a real parameter. If $z \in \mathbb{R}$ and α is an ordinal, then we shall write $x \in \mathrm{OD}^\alpha(z)$ iff there is a formula φ and some $\vec{\beta} \in \alpha$ such that for all $\bar{x}$,

$$\bar{x} = x \Leftrightarrow J_\alpha(\mathbb{R}) \models \varphi[\bar{x}, \vec{\beta}, z].$$

We also write $x \in \mathrm{OD}^{<\alpha}$ iff $x \in \mathrm{OD}^\beta(z)$ for some $\beta < \alpha$.

Here is a basic reflection property which we shall need. It will be referred to as "Martin Reflection" in what follows.

Theorem 3.3.4 (Martin [15]) *Assume W^*_β, where β is critical and case 3 holds at β. Then for any $x, y \in \mathbb{R}^g$, if $x \in OD^\gamma(y)$ for some $\gamma < \beta$, then $x \in OD^\gamma(y)$ for some $\gamma < \alpha$.*

Here is an interesting consequence of W^*_α.

Lemma 3.3.5 *If W^*_α holds, then $J_\alpha(\mathbb{R}) \models \mathsf{AD}$.*

Proof. Suppose that $J_\alpha(\mathbb{R}) \models \neg\mathsf{AD}$. Let $\gamma + 1 \leq \alpha$ be least such that

$$J_{\gamma+1}(\mathbb{R}) \models \text{ there is a non-determined set of reals.}$$

As "there is a non-determined set of reals" is Σ_1, $\gamma + 1$ begins a gap. (This observation is due to Kechris and Solovay.)

Let us first assume that γ is not critical. Then, setting $\delta = \sup(\{\eta < \gamma + 1 \mid \eta \text{ is critical }\})$, $[\delta, \gamma]$ is a strong gap by Lemma 3.3.2. But then by the key argument of [12], every set of reals in $J_{\gamma+1}(\mathbb{R})$ is determined which contradicts our choice of γ.

We therefore have that γ is critical. Let us fix $U \in J_{\gamma+1}(\mathbb{R}) \setminus J_\gamma(\mathbb{R})$. By Lemma 3.3.2, there is some coarse $(1, U)$-Woodin mouse N, as witnessed by

S, T, δ_0, δ_1, and Σ. We have

$$N \models p[S] \text{ is homogeneously Suslin },$$

and hence $p[S]$ is determined in N (This is a result of Martin, Steel, and Woodin, cf. [45]). Let

$$N \models \tau \text{ is a winning strategy for } p[S].$$

We may assume without loss of generality that N believes τ wins for I. We claim that in V, τ wins the game with payoff U for I. For suppose y is a play for II defeating τ; then we can iterate N by Σ, yielding $i\colon N \to P$, with y generic over P at $i(\delta_0)$. Since $\tau(y) \notin U$, and $i(S), i(T)$ are absolute complements over P, $\tau(y) \in p[i(T)]$. By absoluteness of wellfoundedness, $P \models \exists y \ \ \tau(y) \in p[i(T)]$. This contradicts the elementarity of i. □

3.4 The mouse set theorem

We now prove the *Mouse Set Theorem* for $L(\mathbb{R})$, Theorem 3.4.7.

Definition 3.4.1 *Let Γ be a pointclass, and let x be a real. Then*

$$C_\Gamma(x) = \{y \in \mathbb{R} : \exists \xi < \omega_1 \ y \textit{ is } \Gamma(x,z) \textit{ for all } z \textit{ coding } \xi\}.$$

More precisely, $y \in C_\Gamma(x)$ iff there is a countable ordinal ξ and some $A \in \Gamma$ such that for all z coding ξ we have that $\bar{y} = y$ iff $(\bar{y}, x, z) \in A$ (iff $\forall y'[(y', x, z) \in A \Rightarrow \bar{y} = y']$; i.e., practically, if Γ is closed under $\exists^{\mathbb{R}}$ then y is $\breve{\Gamma}$ in a countable ordinal, too, and thus Δ in a countable ordinal, where $\Delta = \Gamma \cap \breve{\Gamma}$).

It is easy to verify that if $J_\alpha(\mathbb{R}) \models \mathsf{AD}$ (or just if $J_\alpha(\mathbb{R})$ knows that there are only countably many $\mathrm{OD}^{<\alpha}(x)$-reals), and if $\Gamma = \Sigma_1^{J_\alpha(\mathbb{R})}$, then

$$C_\Gamma(x) = \mathrm{OD}^{<\alpha}(x).$$

It is also very useful to extend $x \mapsto C_\Gamma(x)$ to countable transitive sets.

The countable transitive set a coded by the real x is the unique a such that $\phi\colon (a, \in) \cong (\omega, R)$ where nRm iff $(n,m) \in x$ (if it exists). If $b \subset a$ then we write b_x for the real ϕb, and say that b_x *codes* b *relative to* x.

Definition 3.4.2 *Let Γ be a pointclass, and let a be a countable transitive set. Then $C_\Gamma(a)$ denotes the set of all $b \subset a$ such that $b_x \in C_\Gamma(x)$ for all $x \in \mathbb{R}$ coding a.*

Lemma 3.4.3 *Let a be any countable set, and let $\mathcal{D}$ be a countable family of open dense subsets of $\mathrm{Col}(\omega, a)$. Then the set of all $G \in {}^\omega a$ being $\mathcal{D}$–generic is comeager.*

Proof: Let $\mathcal{D} = \{D_i : i < \omega\}$. We may view D_i as a dense set in (the space) ${}^\omega a$ (by confusing it with $\{g \in {}^\omega a : g \restriction n \in D_i$ for all sufficiently large $n\}$. Let $C_i = {}^\omega a \setminus D_i$. Clearly, C_i is closed, i.e., $\overline{C_i} = C_i$, and so $D_i = {}^\omega a \setminus \overline{C_i}$. Thus every C_i is nowhere dense. But G is $\mathcal{D}$–generic iff $G \in \bigcap_{i<\omega} D_i$. $\square$

Lemma 3.4.4 *Let $\mathcal{C} \subset {}^\omega\omega$ be comeager. Let $p_0, p_1 \in {}^{<\omega}\omega$ be such that $\mathrm{lh}(p_0) = \mathrm{lh}(p_1)$. Then there is $\alpha \in {}^\omega\omega$ such that $\{p_0{}^\frown\alpha, p_1{}^\frown\alpha\} \subset \mathcal{C}$.*

Proof: Let $\mathcal{D}_i, i < \omega$, be dense open sets such that $\bigcap_{i<\omega} D_i \subset \mathcal{C}$. By a simple induction, we may pick $q_0, q_0', q_1, q_1', \ldots \in {}^{<\omega}\omega$ such that for all $n < \omega$

$$\{q_0{}^\frown q_0{}^\frown q'_0{}^\frown \ldots {}^\frown q_n{}^\frown x : x \in {}^\omega\omega\} \subset D_n \text{ and}$$
$$\{p_1{}^\frown q_0{}^\frown q'_0{}^\frown \ldots {}^\frown q_n{}^\frown q'_n{}^\frown x : x \in {}^\omega\omega\} \subset D_n.$$

But then, if we put $y = q_0{}^\frown q'_0{}^\frown \ldots$, we get that $\{p_0{}^\frown y, p_1{}^\frown y\} \subset \bigcap_{i<\omega} D_i \subset \mathcal{C}$. $\square$

The following condensation result will be helpful in what follows.

Lemma 3.4.5 *Let $\mathcal{M}$ be a coarse (k, U)-Woodin mouse, as witnessed by S, T, Σ, δ_0, ..., δ_k. Suppose that U is a universal Γ-set of reals, where Γ is a scaled Σ-pointclass in $L(\mathbb{R})$. Let $N \in V_{\delta_0}^{\mathcal{M}}$.*[3] *Let $\pi\colon \bar{\mathcal{M}} \to \mathcal{M}$, where N, S, $T \in \mathrm{ran}(\pi)$. Then $C_\Gamma(\pi^{-1}(N)) \subset \bar{\mathcal{M}}$.*

Proof: The set $A = \{(x, y) : \forall w \in C_\Gamma(x) \exists i \in \omega\ w = (y)_i\}$ is in $\breve{\Gamma}$, and so there is k_0 such that for all x, y,

$$(k_0, x, y) \in \mathbb{R} \setminus U \iff \forall w \in C_\Gamma(x) \exists i \in \omega w = (y)_i.$$

Let us first prove the statement for $\pi = \mathrm{id}$. The proof in this case is similar to the proof of Lemma 1.4.19. Let $a \in C_\Gamma(N)$. To show that $a \in N$ it suffices to prove that $a \in \mathcal{M}^{\mathrm{Col}(\omega,N)}$ (independently from the generic). Let

[3] We typically think of N as being a rank initial segment of $\mathcal{M}$.

$x \in \mathbb{R} \cap \mathcal{M}^{\mathrm{Col}(\omega,N)}$ code N, and let $y \in \mathbb{R} \cap V$ be such that $(x,y) \in A$. We may iterate $\mathcal{M}^{\mathrm{Col}(\omega,N)}$ at δ_0 so as to make y generic over the iterate $\mathcal{M}^*$, i.e., such that if

$$i\colon \mathcal{M}^{\mathrm{Col}(\omega,N)} \to \tilde{\mathcal{M}}$$

is the iteration map, then $y \in \tilde{\mathcal{M}}^{\mathrm{Col}(\omega,i(\delta_0))}$. By absoluteness and elementarity, there is then some $y' \in \mathcal{M}^{\mathrm{Col}(\omega,N)}$ such that $(x,y') \in p[T]$. But then a is easily definable from N, x, and y' over $\mathcal{M}^{\mathrm{Col}(\omega,N)}$, so that $a \in \mathcal{M}^{\mathrm{Col}(\omega,N)}$.

Let us now drop the hypothesis that $\pi = \mathrm{id}$. By what we showed so far, we may pick $\sigma, \rho \in \mathcal{M}^{Col(\omega,N)}$ such that

$$\Vdash^{\mathcal{M}}_{Col(\omega,N)} \sigma \in \mathbb{R} \text{ codes } a \wedge \{(\rho)_i : i < \omega\} = \mathbb{R} \cap \mathcal{M}[\sigma].$$

We'll then have that

$$\Vdash^{M}_{Col(\omega,N)} (k_0, \sigma, \rho) \in p[T],$$

Let $\pi(\overline{\sigma}, \overline{\rho}, \overline{T}) = \sigma, \rho, T$. Then by the elementarity of π,

$$\Vdash^{\bar{M}}_{Col(\omega,\overline{N})} (k_0, \overline{\sigma}, \overline{\rho}) \in \overline{T}.$$

Thus if g is $Col(\omega, \overline{N})$–generic over $\bar{\mathcal{M}}$ then

$$(k_0, \overline{\sigma}^g, \overline{\rho}^g) \in p[\overline{T}] \subset p[T] \subset A.$$

But this implies that $C_\Gamma(\overline{\sigma}^g) \subset \overline{M}[g]$, and therefore if $a \in C_\Gamma(\bar{N})$, then $a \in \bar{\mathcal{M}}$. □

The Mouse Set Theorem 3.4.7 will follow from:

Theorem 3.4.6 (Mouse Capturing Theorem) *Suppose W^*_α holds; then for all reals x, the following are equivalent, for all reals y:*

(a) y is $OD^{<\alpha}(x)$, and

(b) there is a (fine-structural) x-mouse $\mathcal{M}$ such that $y \in \mathcal{M}$ and $\mathcal{M}$ has an ω_1-iteration strategy in $J_\alpha(\mathbb{R})$.

Proof. Let us first show (b) $\Longrightarrow$ (a). We need to see that if $\mathcal{M}$, $\mathcal{N}$ are x-mice with ω_1-iteration strategies in $J_\alpha(\mathbb{R})$, then $\mathcal{M}$ and $\mathcal{N}$ can be successfully compared. Let $\Sigma \in J_\alpha(\mathbb{R})$ be an ω_1-iteration strategy for $\mathcal{M}$, and let $\Gamma \in J_\alpha(\mathbb{R})$ be an ω_1-iteration strategy for $\mathcal{N}$. The point is that

$$\omega_1^{J_\kappa[\mathcal{M},\mathcal{N},\Sigma,\Gamma]} < \omega_1^V,$$

because otherwise there would be a sequence $\langle x_\xi | \xi < \omega_1^V \rangle \in J_\alpha(\mathbb{R})$ of pairwise distinct reals, which contradicts $J_\alpha(\mathbb{R}) \models \mathsf{AD}$ (and which in turn follows from W_α^* via Lemma 3.3.5). The comparison of $\mathcal{M}$ with $\mathcal{N}$, performed inside $J_\kappa[\mathcal{M},\mathcal{N},\Sigma,\Gamma]$, must therefore terminate.

Now if y is as in (b), then there is some ξ such that we have $\bar{y} = y$ iff

$$J_\alpha(\mathbb{R}) \models \exists \mathcal{M}(\mathcal{M} \text{ is an } \omega_1\text{–iterable } x\text{–premouse,}$$

$$\text{and } \bar{y} \text{ is the } \xi^{\text{th}} \text{ real of } \mathcal{M}).$$

Let us now show (a) $\Longrightarrow$ (b). Let $\beta < \alpha$ be least such that y is $\mathrm{OD}^\beta(x)$. Then β is the end of a gap, as the statement

$$\exists \bar{\beta} \ (y \text{ is ordinal definable from } x \text{ over } J_{\bar{\beta}}(\mathbb{R}))$$

is a Σ_1 statement about $x \oplus y$ which holds true in $J_{\beta+1}(\mathbb{R})$ but is false in $J_\beta(\mathbb{R})$. As $\beta + 1$ is certainly not admissible, $\beta + 1$ is critical by Lemma 3.3.2 If β were not critical, i.e.,

$$\bar{\alpha} = \sup(\{\eta < \beta + 1 | \eta \text{ is critical }\}) < \beta + 1,$$

then by Lemma 3.3.2 and Martin Reflection 3.3.4 we have that y is $\mathrm{OD}^{<\bar{\alpha}}(x)$, contradicting the choice of β. Therefore, β is in fact critical. By Lemma 3.3.2 once more, there is then some $k < \omega$ such that $\rho_k(J_\beta(\mathbb{R})) = \mathbb{R}$ and all the pointclasses $\Sigma_{2n+k}^{J_\beta(\mathbb{R})}$ and $\Pi_{2n+k+1}^{J_\beta(\mathbb{R})}$, for $n < \omega$, have the scale property.

Let us now suppose that y is $\mathrm{OD}^\beta(x)$ as witnessed by a $\Sigma_{2n+k}^{J_\beta(\mathbb{R})}$ formula. Let us write $\Gamma = \Sigma_{2n+k}^{J_\beta(\mathbb{R})}$ in what follows.

We now first prove the following preliminary version of (b):

Claim. For a Turing cone of reals z, if u is $\mathrm{OD}^\beta(z)$, then there is a (fine structural) z-mouse $\mathcal{M}$ such that $u \in \mathcal{M}$ and $\mathcal{M}$ has an ω_1 iteration strategy in $J_{\beta+1}(\mathbb{R})$.

Proof: By a theorem of Rudominer and Steel, it is enough to show that for any real y, there is an $x \geq_T y$, an ω_1–iterable x–mouse $\mathcal{R}$, and a real

$z \in \mathcal{R}$ such that $z \notin C_\Gamma(x)$ and $\mathcal{R}$ is ω_1–iterable via a strategy in $J_{\beta+1}(\mathbb{R})$. (See [25].) So fix a real y.

Let M be a witness to $W^*_{\beta+1}$, where $k = 1$ and U is a universal Γ set of reals. Let us also assume that $y \in M$. Let $\Omega = OR \cap M$, and $(\mathcal{N}_\eta | \eta \leq \Omega)$ be the levels of the $L[\vec{E}, y]$ construction done inside M. (So y is thrown in at the bottom, and we use full background extenders.) Since M is iterable, all $\mathcal{N}_\eta$ are iterable, and the construction never breaks down.

As δ_0 is Woodin in M, δ_0 is also Woodin in $\mathcal{Q}$, where $\mathcal{Q} = \mathcal{N}_\Omega$. Since $\mathcal{Q}$ has an ω_1–iteration strategy in $J_{\beta+1}(\mathbb{R})$, $\mathcal{Q}$ is tame. It follows that for all sufficiently large $\eta < \delta_0$, η is not Woodin in $\mathcal{Q}$.

Let us work inside M to pick some sufficiently elementary

$$\pi \colon \bar{M} \to M$$

with $S, T \in \mathrm{ran}(\pi)$ such that $\mathrm{ran}(\pi) \cap \delta_0 \in \delta_0$. Set $\eta = \mathrm{ran}(\pi) \cap \delta_0 = \mathrm{crit}(\pi)$. We shall have that η is not Woodin in $\mathcal{Q}$. It follows from Lemma 3.4.5 that $C_\Gamma(\mathcal{Q}|\eta) \subset \bar{M}$, so that η is Woodin with respect to all $A \in C_\Gamma(\mathcal{Q}|\eta)$. Since η is not Woodin in $\mathcal{Q}$, we can pick a least $\xi > \eta$ such that there is a subset b of η which is in $\mathcal{Q}$ but not in $C_\Gamma(\mathcal{Q}|\eta)$. Let us write $\mathcal{P} = \mathcal{Q}|\xi + 1$

Now let $g \colon \omega \to \mathcal{Q}|\eta$ be $\mathcal{Q}$–generic for $Col(\omega, \mathcal{Q}|\eta)$ and such that, setting $x = x_g$, we have $b_x \notin C_\Gamma(x)$; there are in fact comeager many such g. Clearly, $y \leq_T x$ and $b_x \in \mathcal{P}[x]$. It remains only to show that $\mathcal{P}[x]$ can be re–arranged as an x–mouse $\mathcal{R}$. We define $\mathcal{R}$ by adding E to the $\mathcal{R}$–sequence with index α just in case $\eta < \alpha$, α indexes an extender F on the $\mathcal{P}$–sequence, and E is the canonical extension of F to $\mathcal{R}|\alpha = \mathcal{P}|\alpha[x]$ determined by the fact that that this structure is a small forcing extension of $\mathcal{P}|\alpha$. One can prove by induction on β, using the quantifier–by–quantifier definability of the forcing relation over $\mathcal{P}|\eta + \beta$, that $\mathcal{R}|\beta$ has the same projecta and standard parameters as $\mathcal{P}|\eta + \beta$, and hence is ω–sound. (See [32].)

Notice that because M witnesses $W^*_{\beta+1}$ to hold, M is ω_1-iterable via an iteration strategy in $J_{\beta+1}(\mathbb{R}) \subset J_\alpha(\mathbb{R})$, so that $\mathcal{P}$ and hence also $\mathcal{R}$ is ω_1-iterable via an iteration strategy in $J_\alpha(\mathbb{R})$. The Claim is thus shown. $\square$

We now want to show that if $z \in C_\Gamma(x)$, then there is a (fine structural) x-premouse $\mathcal{Q}$ such that $z \in \mathcal{Q}$ and $\mathcal{Q}$ has an ω_1–iteration strategy which is in $J_{\beta+1}(\mathbb{R})$. The proof will straightforwardly relativize to any real, so let us assume that $x = 0$. We thus want to show that if $z \in C_\Gamma$ then there is a (fine structural lightface) premouse $\mathcal{Q}$ such that $z \in \mathcal{Q}$ and $\mathcal{Q}$ has an ω_1–iteration strategy which is projective in Γ. This will suffice, as we may as well choose

$\mathcal{Q} = \mathcal{Q}_z$ so that it projects to ω and is ω–sound. Letting $\eta = OR \cap \mathcal{Q}$, we then have $\mathcal{Q}$ is definable as the unique ω_1–iterable, ω–sound premouse of height η projecting to ω. If we then let $\mathcal{M}$ be the premouse whose proper initial segments are precisely the $\mathcal{Q}$'s, we'll have that $C_\Gamma \subset \mathcal{M}$.

So fix $z \in C_\Gamma$. Let $B \in \Gamma$ and $\xi < \omega_1$ be such that z is unique with $(z, z^*) \in B$ for any z^* coding ξ. We'll have that for all reals x,

$$(C_\Gamma(x); B \cap C_\Gamma(x)) \prec_{\Sigma_1} (\mathbb{R}; B).$$

Here we use that Γ has the scale property, so that each relation in Γ can be uniformized by a function whose graph is in Γ, and hence each non-empty $\Gamma(z)$-set has a member u such that $\{u\}$ is in $\Gamma(z)$. We may and shall in fact assume that B codes the $(2n+k)^{\text{th}}$ reduct, call it M^{2n+k}, of $J_\beta(\mathbb{R})$. Notice that we can express in a Π_2 fashion that $\mathbb{R} =$ the reals of the transitive collapse of $(\mathbb{R}, B)$. Thus $(C_\Gamma(x), B \cap C_\Gamma(x)) \prec_{\Sigma_1} (\mathbb{R}, B)$ will give an embedding $\overline{\pi} : \overline{M} \rightarrow_{\Sigma_1} M^{2n+k}$, which lifts to

$$\pi : J_{\overline{\beta}}(C_\Gamma(x)) \rightarrow_{\Sigma_{2n+k}} J_\beta(\mathbb{R})$$

for some $\overline{\beta} \leq \beta$. Then our given z is definable over $J_{\bar{\beta}}(C_\Gamma(x))$ from a countable ordinal. We'll use this fact below.

By the above Claim, the operator $x \mapsto C_\Gamma(x)$ is fine structural; in fact, we may fix y so that whenever $x \geq_T y$ there is an ω_1–iterable x–mouse $\mathcal{N}_x$ whose ω_1 iteration strategy is projective in Γ and whose reals are those in $C_\Gamma(x)$.

Let $\Gamma_0 = \Sigma_{2n+k+2}^{J_\beta(\mathbb{R})}$, and let M be a coarse Γ_0 Woodin mouse having an ω_1–iteration strategy projective in Γ_0 and such that $y \in M$. Let $\langle \mathcal{N}_\eta | \eta \leq \Omega \rangle$ be the models of the $L[\vec{E}]$ construction, and done over the real $\emptyset$) of M. Just as in the proof of the above Claim, we can fix an $\eta < \Omega$ such that $\mathcal{Q}|\eta$ projects to η and (the natural code for) $\mathcal{Q}$ is not in $C_\Gamma(V_\eta^M)$. Let us choose $\mathcal{Q}$ to be the first level $\mathcal{P}$ of $\mathcal{N}_\Omega$ such that $\mathcal{P}$ projects to η and $\mathcal{P} \notin C_\Gamma(\mathcal{N}_\eta \cup \{y, \mathcal{N}_\eta\})$. Notice that $\mathcal{Q} \models$ "η is Woodin".

Let $\mathbb{P}$ be the every–real–generic poset of $\mathcal{Q}$ (up to η). Here we only use extenders from the $J_\eta^\mathcal{Q}$–sequence which are total and strong out to their lengths to define the identities. Since the $J_\eta^\mathcal{Q}$–sequence has background extenders from V_η^M (which haven't been collapsed in the construction) for these extenders on the $J_\eta^\mathcal{Q}$ sequence, every real in M is $\mathbb{P}$–generic over $\mathcal{Q}$. In particular, y is so generic.

By our choice of $\mathcal{Q}$ there are comeager many $f : \omega \rightarrow J_\eta^\mathcal{Q} \cup \{y\}$ such that $\mathcal{Q}$ is not coded by any real in $C_\Gamma(x_f)$. We can therefore fix such an f which is $Col(\omega, J^{\mathcal{Q}}_\eta \cup \{y\})$ generic over $\mathcal{Q}[y]$. Let $x = x_f$. Clearly, $y \leq_T x$. Also x is $\mathcal{Q}$–generic over a poset of size η in $\mathcal{Q}$, and x codes $J^{\mathcal{Q}}_\eta$, so by the level–by–level definability of forcing we can find an x–premouse $\mathcal{R}$ whose universe is $\mathcal{Q}[x]$. The iterability of $\mathcal{Q}$ guarantees that of $\mathcal{R}$. Since $\mathcal{Q}$ projects to η, $\mathcal{R}$ projects to ω. By our choice of x, the real canonically coding $\mathcal{R}$, its first order theory with parameter x is not in $C_\Gamma(x)$. On the other hand, every proper initial segment of $\mathcal{Q}$ projecting to η is in $C_\Gamma(J^{\mathcal{Q}}_\eta \cup \{y\})$, and therefore every proper initial segment of $\mathcal{R}$ with $\mathcal{N}_x$ we see easily that $\mathbb{R} \cap \mathcal{R} = C_\Gamma(x)$.

We now show that z is ordinal definable over $\mathcal{R}$. This will suffice to finish the proof, since $\mathcal{R}$ is a homogeneous forcing extension of $\mathcal{Q}$ (being an extension via a poset of size η which collapses η to ω), so that we have $z \in \mathcal{Q}$ as desired.

Now recall that z is definable over $J_{\overline{\beta}}(C_\Gamma(x))$ from a countable ordinal. W.l.o.g., $\overline{\beta} \in \mathcal{R}$; this is because the extender sequence of $\mathcal{R}$ is nonempty (since $\Pi^1_2 \subset \Gamma$). Since $C_\Gamma(x)$ is ordinal definable over $\mathcal{R}$, as its set of reals, we have that z is ordinal definable over $\mathcal{R}$, as desired. □

Theorem 3.4.6 now immediately implies the following.

Theorem 3.4.7 (Mouse Set Theorem) *Let α be any ordinal and let x be any real. Suppose W^*_α holds. Then there is some x-premouse $\mathcal{N}$ such that*

$$C_{\Sigma_1(J_\alpha(\mathbb{R}))}(x) = \mathrm{OD}^{<\alpha}(x) = \mathbb{R} \cap \mathcal{N},$$

where $\mathcal{N}$ is ω_1-iterable via a startegy in $\Sigma_1(J_\alpha(\mathbb{R}))$.

Proof. For any $y \in \mathrm{OD}^{<\alpha}$, let $\mathcal{M}_y$ be the least (sound) x–mouse $\mathcal{M}$ with $y \in \mathcal{M}$ which is given by the Mouse Capturing Theorem 3.4.6. Let $\mathcal{N}$ be the union of all $\mathcal{M}_y$ for $y \in \mathrm{OD}^{<\alpha}$. By construction, $\mathrm{OD}^{<\alpha} \subset \mathcal{N}$. On the other hand, if $z \in \mathcal{N} \cap \mathbb{R}$, then $z \in \mathcal{M}_y$ for some $y \in \mathrm{OD}^{<\alpha}$. As $\mathcal{M}_y$ has an iteration strategy in $J_\alpha(\mathbb{R})$, this implies that z is in $\mathrm{OD}^{<\alpha}$. Therefore, $\mathrm{OD}^{<\alpha} \cap \mathbb{R} \subset \mathcal{N}$. □

3.5 The fine structural mouse witness condition W_α

We now want to prove a lightface result on the existence of fine-structural mouse witnesses. We shall call this result W_α. For a technical reason having to do with the real parameters which may enter into the definition of a scale, we are only able to prove W_α in the case that α is a limit ordinal and α begins a (perhaps trivial) gap.

To any Σ_1 formula $\theta(v)$ we associate formulae $\theta^k(v)$ for $k \in \omega$, such that θ^k is Σ_k, and for any γ and any real x,

$$J_{\gamma+1}(\mathbb{R}) \models \theta[x] \Leftrightarrow \exists k < \omega J_\gamma(\mathbb{R}) \models \theta^k[x].$$

Our fine-structural witnesses are as follows.

Definition 3.5.1 *Suppose $\theta(v)$ is a Σ_1 formula (in the language of set theory expanded by a name for $\mathbb{R}$), and z is a real; then a $\langle\theta, z\rangle$-*prewitness *is an ω-sound z-premouse $\mathcal{N}$ in which there are $\delta_0 < ... < \delta_9$, S, and T such that $\mathcal{N}$ satisfies the formulae expressing*

(a) ZFC,

(b) $\delta_0, ..., \delta_9$ are Woodin,

(c) S and T are trees on some $\omega \times \eta$ which are absolutely complementing in $V^{\mathrm{Col}(\omega,\delta_9)}$, and

(d) For some $k < \omega$, $p[T]$ is the Σ_{k+3}-theory (in the language with names for each real) of $J_\gamma(\mathbb{R})$, where γ is least such that $J_\gamma(\mathbb{R}) \models \theta^k[z]$.[4]

*If $\mathcal{N}$ is also $(\omega, \omega_1, \omega_1 + 1)$-iterable, then we call it a $\langle\theta, z\rangle$-*witness.

To be more explicit, the formula expressing (d) states that $p[T]$ is a maximal consistent Σ_{k+3}-theory of a well-founded model of

$$V = L(\mathbb{R}) \wedge \theta^k[z] \wedge \forall\beta J_\beta(\mathbb{R}) \models \neg\theta^k[z]$$

which contains all the reals. It is straightforward to verify that the formula expressing (d) can thus be written in a $\forall^{\mathbb{R}}\exists^{\mathbb{R}}(p[T], p[S])$ way. This in turn implies that it is also true in $\mathcal{N}^{\mathrm{Col}(\omega,\delta_0)}$ that for some $k < \omega$, $p[T]$ is the Σ_{k+3}-theory (in the language with names for each real) of $J_\gamma(\mathbb{R})$, where γ is least such that $J_\gamma(\mathbb{R}) \models \theta^k[z]$.

We should note that this is different from the notion of $\langle\theta, z\rangle$-witness defined in [41]. The witnesses in that sense are mice with infinitely many Woodin cardinals, and so they are too crude for our purposes here.

[4] WHY NOT REQUIRING $\mathrm{Col}(\omega, \delta_0) \Vdash (d)$?

Remark 3.5.2 The more general notion would be that of a Σ^2_1-*witness.* Given a Σ^2_1 formula $\exists A\phi(A, v)$ and real z, a mouse witness to $\exists A\phi(A, z)$ is a mouse $\mathcal{N}$ satisfying the conditions of 3.5.1, with (d) changed to: $\mathcal{N} \models \phi[p[T], z]$. The definition given above amounts to considering only Σ^2_1 formulae of the form $\exists A($ A codes some $J_\gamma(\mathbb{R})$ such that $J_\gamma(\mathbb{R}) \models \theta[z])$. This is all we need for an induction which stays in $L(\mathbb{R})$.

The next lemma justifies "witness."

Lemma 3.5.3 *If there is a* $\langle\theta, z\rangle$*-witness, then* $L(\mathbb{R}) \models \theta[z]$.

Proof. This is an exercise using genericity iterations. The key is that if $\mathcal{N}, T, U$ are as in 3.5.1, then the iteration strategy for $\mathcal{N}$ lets us interpret T on arbitrary reals in an unambiguous way: if $i\colon \mathcal{N} \to \mathcal{N}_0$ and $j\colon \mathcal{N} \to \mathcal{N}_1$ are iteration maps, and x is generic over both $\mathcal{N}_0$ and $\mathcal{N}_1$, then $x \in p[i(T)]$ iff $x \in p[j(T)]$. We get this from comparing $\mathcal{N}_0$ with $\mathcal{N}_1$, and using Dodd-Jensen plus the existence of U.

Now

$$\mathsf{Th} = \bigcup\{p[i(T)] \mid i\colon \mathcal{N} \to \mathcal{N}^* \text{ is countable iteration } \}$$

is easily seen to be a maximal consistent Σ_{k+3}-theory of a well-founded model of

$$V = L(\mathbb{R}) \wedge \theta^k[z] \wedge \forall\beta J_\beta(\mathbb{R}) \models \neg\theta^k[z]$$

which (as any real in V can be made $\mathrm{Col}(\omega, i(\delta_0))$-generic over $\mathcal{N}^*$ for some countable iteration $i\colon \mathcal{N} \to \mathcal{N}^*$) contains all the reals which exist in V. Therefore, Th is the Σ_{k+3}-theory of $J_\gamma(\mathbb{R})$ where γ is least such that $J_\gamma(\mathbb{R}) \models \theta^k[z]$. But this implies that $L(\mathbb{R}) \models \theta[z]$. □

What we show in our core model induction is just that the converse of 3.5.3 holds for $L(\mathbb{R})$, at least for α a limit ordinal. More precisely, we show that for α a limit,

(W_α) If $\theta(v)$ is Σ_1, $z \in \mathbb{R}$, and $J_\alpha(\mathbb{R}) \models \theta[z]$, then there is a $\langle\theta, z\rangle$-witness $\mathcal{N}$ whose associated iteration strategy, when restricted to countable iteration trees, is in $J_\alpha(\mathbb{R})$.

Lemma 3.5.4 *Let* α *be a limit ordinal, and suppose that* W^*_α *holds; then* W_α *holds.*

Proof. This is by the argument for (a) $\Rightarrow$ (b) in the proof of Theorem 3.4.6.

Let us fix a Σ_1-formula $\theta(v)$ and a real z such that $J_\alpha(\mathbb{R}) \models \theta[z]$. Let β be least such that $J_\beta(\mathbb{R}) \models \theta[z]$. We then have that β is a successor ordinal (so $\beta < \alpha$), β begins a gap, and the pointclasses

$$\Gamma_{2n+1} = \Sigma_{2n+1}^{J_\beta(\mathbb{R})}(\{z\})$$

and

$$\Gamma_{2n+2} = \Pi_{2n+2}^{J_\beta(\mathbb{R})}(\{z\})$$

have the scale property. Moreover, if T_n is the tree from the scale on "the" universal Γ_n-set of reals constructed in [35], then T_n is definable over $J_\beta(\mathbb{R})$ from the parameter z. These facts about the T_n's imply that for every $k < \omega$, there are trees S^k and T^k which are definable over $J_\beta(\mathbb{R})$ from the parameter z such that S^k and T^k are complementing and S^k projects to the Σ_{k+3}-theory of $J_{\beta-1}(\mathbb{R})$.

Let us pick $k < \omega$ such that $J_{\beta-1}(\mathbb{R}) \models \theta^k[z]$, and let $n < \omega$ be such that S^k and T^k are both $\Sigma_n^{J_\beta(\mathbb{R})}(\{z\})$-definable.

Now let $\mathcal{R} = \mathcal{Q}[x]$ be the mouse constructed as in the proof of (a) $\Rightarrow$ (b) of Theorem 3.4.6 with the only difference that we want $\mathcal{Q}$ be an initial segment of the $L[E, z]$ construction (rather than the $L[E]$ construction) of M and M should also contain z. We argued that there is some $\bar\beta \in \mathcal{R}$ such that there exists an embedding

$$\pi\colon J_{\bar\beta}(\mathbb{R} \cap \mathcal{R}) \to_{\Sigma_n} J_\beta(\mathbb{R}).$$

By the definability of S^k and T^k, $S^k, T^k \in \operatorname{ran}(\pi)$, and setting $\bar S^k = \pi^{-1}(S^k)$ and $\bar T^k = \pi^{-1}(T^k)$, $\bar S^k$ and $\bar T^k$ are definable over $J_{\bar\beta}(\mathbb{R} \cap \mathcal{R})$ from parameter z. We may conclude that

$$\bar S^k, \bar T^k \in \mathcal{Q}.$$

Without loss of generality, $\mathcal{Q}$ has ten Woodin cardinals, where η is the largest. (Just start with an M with ten Woodin cardinals.) We may then cut off $\mathcal{Q}$ at the least measurable cardinal of $\mathcal{Q}$ above η to get a $\langle\theta, z\rangle$-witness. □

3.6 The witness dichotomy

Although our induction hypothesis W_α^* is often interpreted in some $V[g]$, one must nevertheless periodically go back to V, where the proposition from which one is mining logical strength holds true. We shall see that W_α gives us mice in V over terms τ such that $\tau^g \in \mathbb{R}^{V[g]}$. Those mice can be used in V to form operators J to which our dichotomy theorems 3.1.9 and 3.1.12 on the existence of K^J can be applied. What we show in this part of the argument is the following key result.

Theorem 3.6.1 (Witness Dichotomy) *Let ν be a cardinal, $\nu \geq 2$. Let g be $\mathrm{Col}(\omega, <\nu)$- generic over V, and let $\mathbb{R}^g = \bigcup_{\alpha<\nu} \mathbb{R}^{V[g\restriction\alpha]}$ be the reals of the symmetric collapse.*[5] *Suppose that $L(\mathbb{R}^g) \models \mathsf{DC}$. Then for every ordinal α such that in $L(\mathbb{R}^g)$, α is critical and W^*_β holds true for every $\beta \leq \alpha$, there is in V a mouse operator or a hybrid mouse operator J^0_α which is total on $H_{\aleph_1^{V[g]}}$ and such that exactly one of the following holds true.*

(a) $W^*_{\alpha+1}$ is true in $L(\mathbb{R}^g)$, or else

(b) for some $n < \omega$, J^n_α is not total on $(H_{\aleph_1^{V[g]}})^V$.

We shall prove the Witness Dichotomy in the two succeeding chapters, where (b) will also be made more precise. The meaning of being a "(ν, A)-mo" was defined in Definition 3.1.1, and the meaning of being a "(ν, A)-hmo" (where the "hmo" will stand for "*hybrid* mouse operator") will be defined in Definition 5.6.6. The assertion that J^w does not exists means just that there is some $B \in \mathrm{dom}(J)$ such that there is no countably iterable $M^J_1(B)$.

Notice that we allow $\nu = 1$ in which case $g \in V$ and the Witness Dichotomy applies directly to $L(\mathbb{R})^V$.

Combining Theorem 3.6.1 with Theorem 3.1.12, we get

Theorem 3.6.2 *Let $\kappa > \omega$ be such that $|V_\kappa| = \kappa$. Let g be $\mathrm{Col}(\omega, <\kappa)$-generic over V, let $\mathbb{R}^g = \bigcup_{\alpha<\kappa} \mathbb{R}^{V[g\restriction\alpha]}$ be the reals of the symmetric collapse, and suppose that $L(\mathbb{R}^g) \models \mathsf{DC}$; then either*

(a) for all α, W^*_α holds in $L(\mathbb{R}^g)$, or

(b) for some $A \in V_\kappa$, and some (κ, A)-hmo J, $K^J(A)^{V_\kappa}$ exists.

The assertion that $K^J(A)^{V_\kappa}$ exists is to be understood in the same stable-K sense that we had for pure mouse operators.

Hybrid $K^J(A)^{V_\kappa}$ behaves much like a pure $K^J(A)$, so that, for example, its existence is incompatible with the failure of square at a singular μ such that $\mathrm{rk}(A) < \mu < \kappa$. So we get

[5] If $\nu = 2$, then $g \in V$ and $\mathbb{R}^g = \mathbb{R}^V$.

Corollary 3.6.3 *Let $\kappa > \omega$ be such that $|V_\kappa| = \kappa$, and suppose that for arbitrarily large $\alpha < \kappa$, one of the following holds:*

(1) α is a singular cardinal and $\square_\alpha$ fails,

(2) α and α^+ have the tree property,

(3) there is a generic almost-huge embedding with critical point α given by some ideal in V_κ.

Let g be $\mathrm{Col}(\omega, <\kappa)$- generic over V, let $\mathbb{R}^g = \bigcup_{\alpha<\kappa} \mathbb{R}^{V[g\restriction\alpha]}$ be the reals of the symmetric collapse, and suppose that $L(\mathbb{R}^g) \models \mathsf{DC}$. Then for all α, W^*_α holds in $L(\mathbb{R}^g)$.

Corollary 3.6.4 (Woodin) *If PFA holds, and κ is inaccessible, and g is $\mathrm{Col}(\omega, <\kappa)$-generic over V, then $\mathsf{AD}^{L(\mathbb{R})}$ holds in $V[g]$.*

Remark 3.6.5 From 3.6.3 one gets that the following are equiconsistent over ZFC:

(a) there are infinitely many Woodin cardinals,

(b) there is an uncountable κ such that $|V_\kappa| = \kappa$ and

  (i) for arbitrarily large $\alpha < \kappa$, there is a generic almost-huge embedding with critical point α given by some ideal in V_κ, and

  (ii) $L(\mathbb{R}^g) \models \mathsf{DC}$, where $\mathbb{R}^g$ is the set of reals of a symmetric collapse below κ.

The result would be cleaner if one could omit (ii) from (b). In fact, we do not know whether the hypothesis that $L(\mathbb{R}^g) \models \mathsf{DC}$ can be omitted from 3.6.3. This would amount to proving it from the other hypotheses, as the W^*_α's collectively imply $L(\mathbb{R}^g) \models \mathsf{DC}$.

The Witness Dichotomy summarizes the part of a core model induction which does not involve K directly; the part which keeps track of mouse-correctness, and builds appropriate mouse operators at limit steps. One then combines 3.6.1 with our K^J dichotomies 3.1.9, 3.1.12, and some strong proposition which implies we must be in the "$J^w(B)$ exists" case of 3.1.9 or 3.1.12. The upshot is that W^*_α holds in $L(\mathbb{R}^g)$, for all α.

3.7 Capturing sets of reals over mice

Definition 3.7.1 *Let $\mathcal{A} \subset \mathcal{P}(\mathbb{R})$ be a countable collection of sets of reals. We say that $\mathcal{A}$ is a* self justifying system*, or briefly an* sjs*, if and only if the following holds true. Every $A \in \mathcal{A}$ admits a scale $(\leq_n : n < \omega)$ such that each individual $\leq_n$ belongs to $\mathcal{A}$, too, and such that if $A \in \mathcal{A}$, then $\mathbb{R} \setminus A \in \mathcal{A}$.*

Lemma 3.7.2 *Let $\mathcal{A}$ be a sjs. Let N and M be transitive models of a sufficiently large fragment of ZFC such that $N \in M$. Let $\mathcal{C} \subset {}^{\omega}N$ be a comeager set of $Col(\omega, N)$–generics over M (in particular, N is countable) and suppose that for each $A \in \mathcal{A}$ there is a term $\tau_A \in M$ such that whenever $G \in \mathcal{C}$ then*

$$\tau_A^G = A \cap M[G].$$

Let $\pi : \overline{M} \to M$ be elementary with $\{N\} \cup \{\tau_A : A \in \mathcal{A}\} \subset ran(\pi)$. Let $\pi(\overline{N}, \overline{\tau}_A) = N, \tau_A$. THEN whenever g is $Col(\omega, \overline{N})$–generic over $\overline{M}$, for all $A \in \mathcal{A}$,

$$\overline{\tau}_A^g = A \cap \overline{M}[g].$$

Proof: To commence, fix any $A \in \mathcal{A}$ for a while, and let $(\psi_n : n < \omega)$ be a scale on A such that for every $n < \omega$, if $\leq_n$ is the prewellorder on $\mathbb{R}$ given by ψ_n then $\leq_n \in \mathcal{A}$. Let $\tau_n \in M$ be such that $\tau_n^G = \leq_n \cap M[G]$ for all $G \in \mathcal{C}$. Let ϕ_n be a term in M such that for every G being $Col(\omega, N)$–generic over M, ϕ_n^G is the norm on $A \cap M[G]$ given by τ_n^G. Let U_n be a term for the n^{th} level of the tree associated to these norms, i.e., for all G being $Col(\omega, N)$–generic over M,

$$\dot{U}_n^G = \{(x \restriction n, (\phi_0^G(x), \ldots, \phi_{n-1}^G(x))) : x \in A \cap M[G]\}.$$

Now let $G_h, h = 0, 1$, be any $Col(\omega, N)$–generics over M. Then for any appropriate $\vec{a}$ we have $\vec{a} \in \dot{U}_n^{G_h}$ iff there is some $p_h \in G_h$ forcing $\vec{a} \in \dot{U}_n$. W.l.o.g., $lh(p_0) = lh(p_1)$. Hence by Lemma 3.4.4 we may choose $G_h^* \in \mathcal{C}$ such that for some real y, for every $n < \omega$,$p_0 {}^\frown y \restriction n \in G_0^*$ and $p_1 {}^\frown y \restriction n \in G_1^*$. In particular, we have $M[G_0^*] = M[G_1^*]$, which implies $\tau_n^{G_0^*} = \leq_n \cap M[G_0^*] = \leq_n \cap M[G_1^*] = \tau_n^{G_1^*}$, and so $\dot{U}_n^{G_0^*} = \dot{U}_n^{G_1^*}$. Hence $\vec{a} \in \dot{U}_n^{G_0}$ iff $\vec{a} \in \dot{U}_n^{G_0^*}$ iff $\vec{a} \in \dot{U}_n^{G_1^*}$ iff $\vec{a} \in \dot{U}_n^{G_1}$.

This means that $\dot{U}_n^G$ is independent from G and in the ground model. I.e., there are $U_n \in M$ such that $U_n = \dot{U}_n^G$ for all G being $Col(\omega, N)$–generic over M. Let U be the tree whose n^{th} level is U_n. (Of course, possibly $U \notin M$.)

Claim: Whenever G is $Col(\omega, N)$–generic over $M, A \cap M[G] \subset p[U] \subset A$.

Proof: $A \cap M[G] \subset p[U]$ is obvious from the definition of U. Let $(x, f) \in [U]$. Let G be $Col(\omega, N)$–generic over M. Let $n < \omega$; then the n^{th} level of U is U_n^G, and so we can find a real $x_n \in A$ with $x_n \upharpoonright n = x \upharpoonright n$ and $\forall i < n(\phi_i^G(x_n) = f(i))$. So for any $i, \phi_i^G(x_n)$ is eventually constant as $n \to \omega$. Hence $\psi_i(x_n)$ is eventually constant as $n \to \omega$. But $(\psi_i : i < \omega)$ is a scale on A, thus $x \in A$. This shows $p[U] \subset A$.

We now in particular have that

$$\Vdash_{Col(\omega,N)} \forall x[x \in \tau_A \to (x \upharpoonright n, (\phi_0(x), \dots, \phi_{n-1}(x))) \in U_n].$$

The elementarity of π gives that

$$\Vdash_{Col(\omega,N)} \forall x[x \in \overline{\tau}_A \to (x \upharpoonright n, (\pi^{-1}(\phi_0)(x), \dots, \pi^{-1}(\phi_{n-1})(x))) \in \overline{U}_n],$$

where $\overline{U}_n = \pi^{-1}(U_n)$. Let $\overline{U}$ be the tree whose n^{th} level is $\overline{U}_n$. It is easy to see that $p[\overline{U}] \subset p[U]$ using π. But now if $x \in \overline{\tau}_A^g$ for a $Col(\omega, \overline{N})$–generic g then $x \in p[\overline{U}] \subset p[U] \subset A$, by the above Claim. So $\overline{\tau}_A^g \subset A$.

However, the same reasoning with $\mathbb{R} \backslash A \in \mathcal{A}$ and $\tau_{\mathbb{R} \backslash A}$ instead of A and τ_A shows that $\overline{\tau}_{\mathbb{R}\backslash A}^g \subset \mathbb{R} \backslash A$, and thus in fact $\overline{\tau}_A^g = A \cap \overline{M}[g]$, as $\overline{\tau}_{\mathbb{R}\backslash A}^g = (\mathbb{R} \cap \overline{M}[g]) \setminus \overline{\tau}_A^g$. □

Definition 3.7.3 *Let $A \subset \mathbb{R}$, let $\mathcal{M}$ be a countable premouse, let η be an uncountable cardinal of $\mathcal{M}$, and let $\tau \in \mathcal{M}^{\mathrm{Col}(\omega,\eta)}$. We say that τ* weakly captures A over $\mathcal{M}$ *iff whenever $g \in V$ is $\mathrm{Col}(\omega, \eta)$-generic over $\mathcal{M}$, then*

$$A \cap \mathcal{M}[g] = \tau^g.$$

We may apply Lemma 3.7.2 to $\pi = id$ to immediately get the following.

Corollary 3.7.4 *Let $\mathcal{A}$ be a sjs. Let N and M be transitive models of a sufficiently large fragment of ZFC such that $N \in M$. Let $\mathcal{C} \subset {}^\omega N$ be a comeager set of $Col(\omega, N)$–generics over M (in particular, N is countable) and suppose that for each $A \in \mathcal{A}$ there is a term $\tau_A \in M$ such that whenever $G \in \mathcal{C}$ then*

$$\tau_A^G = A \cap M[G].$$

THEN in fact every τ_A, for $A \in \mathcal{A}$, weakly captures A over M.

The following lemma provides the key method for obtaining terms which capture given sets of reals.

Lemma 3.7.5 *Let z be a real, and let $\mathcal{N}$ be a countable and transitive model of a reasonable fragment of* ZFC^- *such that $z \in \mathcal{N}$. Let κ be a cardinal of $\mathcal{N}$ such that $(H_{\kappa^+})^{\mathcal{N}} \in \mathcal{N}$. Let $[\alpha, \beta^*]$ be a gap, and set $\Gamma = \Sigma_1(J_\alpha(\mathbb{R}))$. Let us also write $\beta = \beta^*$, unless $[\alpha, \beta^*]$ is a strong gap and $\beta^* + 1$ is critical in which case we write $\beta = \beta^* + 1$. Assume that $J_\beta(\mathbb{R}) \models \mathsf{AD}$. Suppose that $C_\Gamma((H_{\kappa^+})^{\mathcal{N}}) \subset \mathcal{N}$. Let $A \subseteq \mathbb{R}$ be $OD_z^{<\beta}$. Then there is a term $\tau \in \mathcal{N}^{\mathrm{Col}(\omega,\kappa)}$ such that for comeager many h which are $\mathrm{Col}(\omega, \mu)$–generic over $\mathcal{N}$,*

$$\sigma^h = A \cap \mathcal{N}[h].$$

Proof. We let $(p, \sigma) \in \tau$ iff $p \in Col(\omega, \kappa)$, $\sigma \in (H_{\kappa^+})^{\mathcal{N}}$ is a $Col(\omega, \kappa)$–standard term for a real, and for comeager many g being $Col(\omega, \kappa)$–generic over $\mathcal{N}$, if $p \in g$, then $\sigma^g \in A$. Trivially, $\tau \subset (H_{\kappa^+})^{\mathcal{N}}$.

Claim 1. $\tau \in \mathcal{N}$.

Proof: Let $x \in \mathbb{R}$ be $Col(\omega, (H_{\kappa^+})^{\mathcal{N}})$-generic over $\mathcal{N}$, i.e.,

$$(\omega, E_x) \cong ((H_{\kappa^+})^{\mathcal{N}}, \in).$$

It is easy to verify that $\tau_x \in OD_{x,z}^{<\beta}$. But $z \in OD_x^{<\beta}$, which can easily be verified as follows. Let m be the preimage of z_0 under the isomorphism $(\omega, E_x) \cong ((H_{\kappa^+})^{\mathcal{N}}, \in)$. Then $k \in z$ iff

$$\exists a \subset \omega (a \text{ represents the set of integers in } (\omega, E_x),$$

$$\text{and if } f : (a, E_x \restriction a) \cong (\omega, \in) \text{ then } f^{-1}(k) E_x m)).$$

Hence $\tau_x \in OD_x^{<\beta}$. This implies that $\tau_x \in C_{\Sigma_1(J_\alpha(\mathbb{R}))}(x) = C_\Gamma(x)$. This is clear if $[\alpha, \beta^*]$ is not strong, because then $[\alpha, \beta] = [\alpha, \beta^*]$ is a gap, and by $J_\beta(\mathbb{R}) \models \mathsf{AD}$ every real in $OD_x^{<\beta}$ is in fact in $C_{\Sigma_1(J_\beta(\mathbb{R}))}(x)$, and thus in $C_{\Sigma_1(J_\alpha(\mathbb{R}))}(x)$. However, if $[\alpha, \beta^*]$ is strong and $\beta = \beta^* + 1$ is critical, then by Martin Reflection 3.3.4 we have that $\tau_x \in OD_x^{<\alpha}$, so that by $J_\alpha(\mathbb{R}) \models \mathsf{AD}$ we also get that $\tau_x \in C_{\Sigma_1(J_\alpha(\mathbb{R}))}(x)$.

We have shown that $\tau_x \in \mathcal{N}[x]$, and thus $\tau \in \mathcal{N}[x]$.

But this is now true for all x, i.e., if x, x' are mutually $Col(\omega, (H_{\kappa^+})^{\mathcal{N}})$-generic over $\mathcal{N}$, then $\tau \in M[x] \cap M[x']$. It follows that $\tau \in \mathcal{N}$. □

Claim 2. τ weakly captures A over $\mathcal{N}$.

Proof: For $p \in Col(\omega, \kappa)$ and σ a term in $\mathcal{N}^{Col(\omega,\kappa)}$ for a real let $C_{p,\sigma} =$

$\{G : p \in G \wedge \sigma^G \in A\}$ and $C'_{p,\sigma} = \{G : p \in G \wedge \sigma^G \notin A\}$. We have $\tau = \tau_A = \{(p, \sigma) : C_{p,\sigma}$ is comeager$\} \in \mathcal{N}$.

We claim that for all σ, $\{p \in Col(\omega, \kappa) : C_{p,\sigma}$ or $C'_{p,\sigma}$ is comeager $\}$ is dense in $Col(\omega, \kappa)$. Fix σ. Let $q \in Col(\omega, \kappa)$. Suppose that $C_{q,\sigma}$ is not comeager. As $C_{q,\sigma}$ has the property of Baire, there is an open set $\mathcal{O}$ such that $(\mathcal{O} \setminus C_{q,\sigma}) \cup (C_{q,\sigma} \setminus \mathcal{O})$ is meager. If $\mathcal{O} = \emptyset$, then $C'_{q,\sigma}$ is comeager. Let us assume that $\mathcal{O} \neq \emptyset$. Then there is some p such that $U_p \setminus C_{q,\sigma}$ is meager, where $U_p = \{g : p \in g\}$. We must have that $p \leq q$, as otherwise $U_p \setminus C_{q,\sigma} = U_p$, which is not meager. But then $C_{p,\sigma}$ is comeager, as $g \notin C_{p,\sigma}$ iff $g \in U_p \setminus C_{p,\sigma}$.

If $C_{p,\sigma}$ or $C'_{p,\sigma}$ is comeager, then let $C^*_{p,\sigma}$ denote the comeager one of them. There are only countably many such p's and σ's so that

$$\mathcal{C} = \bigcap_{p,\sigma} C^*_{p,\sigma}$$

is a comeager set.

Now let $g \in \mathcal{C}$. Then $\sigma^g \in \tau^g \Rightarrow \exists p \in g(p, \sigma) \in \tau \Rightarrow \exists p \in gC_{p,\sigma}$ is comeager $\Rightarrow \sigma^g \in A$. On the other hand, if $\sigma^g \notin \tau^g$, then $\forall p \in g(p, \sigma) \notin \tau$, so $\forall p \in gC_{p,\sigma}$ is not comeager. By densitiy, $\exists p \in gC_{p,\sigma}$ or $C'_{p,\sigma}$ is comeager. Therefore, $\exists p \in gC'_{p,\sigma}$ is comeager, and hence $\sigma^g \notin A$. This shows that $\tau^g = A \cap \mathcal{N}[g]$ for all $g \in \mathcal{C}$.

□ (Claim 2)

□

Definition 3.7.6 *For $\mathcal{N}$, z,κ, and A as in Lemma 3.7.5, $\tau^{\mathcal{N}}_{A,\kappa}$ denotes the unique standard term τ constructed in the proof of Lemma 3.7.5. $\tau^{\mathcal{N}}_{A,\kappa}$ is called the* standard term *(*for A over $\mathcal{N}$ at κ*).*

We shall need more than weak capturing in what follows.

Definition 3.7.7 *Let $A \subset \mathbb{R}$, let $\mathcal{M}$ be a countable premouse, let η be an uncountable cardinal of $\mathcal{M}$, and let $\tau \in \mathcal{M}^{\mathrm{Col}(\omega,\eta)}$. We say that τ* captures *A* over *$\mathcal{M}$ iff there is an iteration strategy Σ witnessing that $\mathcal{M}$ is a mouse such that whenever $\mathcal{P}$ is a simple countable iterate of $\mathcal{M}$ with iteration map*

$$i \colon \mathcal{M} \to \mathcal{P},$$

then $i(\tau) \in \mathcal{P}^{\mathrm{Col}(\omega,i(\eta))}$ weakly captures A over $\mathcal{P}$.

Some effort will be needed in order to come up with mice which move certain term relations correctly, i.e., which have terms capturing certain sets of reals.

The following lemma allows us to climb up a projective-like hierarchy, and it will be used in combination with the preceeding Lemma which provides one ingredient for obtaining capturing terms for the bottom of such a hierarchy. The proof of the following lemma is basically identical to the proof of Lemma 1.4.18

Lemma 3.7.8 *Let $\mathcal{M}$ be a countable mouse, and let $\delta < \eta \in M$ be such that $\mathcal{M} \models$ "η is a Woodin cardinal." Let $B \subset \mathbb{R} \times \mathbb{R}$. Suppose that $\tau \in \mathcal{M}^{\mathrm{Col}(\omega,\eta)}$ captures B over $\mathcal{M}$. Then there is $\sigma \in \mathcal{M}^{\mathrm{Col}(\omega,\delta)}$ capturing $\exists^{\mathbb{R}} B$ over* M.

Proof: Let us define $\sigma \in \mathcal{M}^{\mathrm{Col}(\omega,\delta)}$ as follows. We put $(p, \rho) \in \sigma$ iff $p \in \mathrm{Col}(\omega, \delta)$, ρ is a name for a real in $\mathcal{M}^{Col(\omega,\delta)} \cap (H_{\delta^+})^{\mathcal{M}}$, and

$$p \Vdash \text{“}\exists q \in \mathrm{Col}(\check{\omega}, \check{\eta})\ q \Vdash \exists y(\check{\rho}, y) \in \check{\tau}.''$$

We claim that σ captures $\exists^{\mathbb{R}} B$ over M. It is easy to see that in order to verify this, it suffices to show that σ weakly captures $\exists^{\mathbb{R}} B$ over $\mathcal{M}$.

Let $g \in V$ be $\mathrm{Col}(\omega, \delta)$–generic over $\mathcal{M}$. First let $x \in \exists^{\mathbb{R}} B \cap \mathcal{M}[g]$. We aim to show $x \in \sigma^g$. Pick y_0 such that $(x, y_0) \in B$. By Woodin's genericity theorem there is a countable iterate $\mathcal{P}$ of $\mathcal{M}[g]$ with iteration map $i \colon \mathcal{M}[g] \to \mathcal{P}$ such that y_0 is $\mathrm{Col}(\omega, i(\delta))$–generic over $\mathcal{P}$, i.e., there is some $h \in V$ which is $\mathrm{Col}(\omega, i(\delta))$–generic over $\mathcal{P}$ such that $y_0 \in \mathcal{P}[h]$. Notice that we may also write $\mathcal{P} = \mathcal{Q}[g]$, where $\mathcal{Q}$ is the iterate of $\mathcal{M}$ induced by $i \colon \mathcal{M}[g] \to \mathcal{P}$. Therefore, $\mathcal{P}[h] = \mathcal{Q}[g][h]$, where g is $\mathrm{Col}(\omega, \eta)$–generic over $\mathcal{Q}$ and h is $\mathrm{Col}(\omega, i(\delta))$–generic over $\mathcal{Q}[g]$.

As τ captures B over $\mathcal{M}$, we have that $(i(\check{\tau})^g)^h = i(\tau)^{g \oplus h} = B \cap \mathcal{Q}[g, h]$, and thus $(x, y_0) \in (i(\check{\tau})^g)^h$, i.e., $\mathcal{Q}[g, h] \models \exists y\ (x, y) \in (i(\check{\tau})^g)^h$. So $\mathcal{Q}[g] \models \exists q\ q \Vdash \exists y(\check{x}, y) \in i(\check{\tau})^g$, which implies $\mathcal{M}[g] \models \exists q\ q \Vdash \exists y(\check{x}, y) \in \check{\tau}^g$ by elementarity of i. This then gives $x \in \sigma^g$ by the construction of σ.

It is now easy to see that on the other hand if $x \in \sigma^g$, then $x \in \exists^{\mathbb{R}} B$. $\square$

Chapter 4

The inadmissible cases in $L(\mathbb{R})$

In this chapter we prove Theorem 3.6.1, the Witness Dichotomy, in the *inadmissible cases*, i.e., in the cases (1) and (2) of Lemma 3.3.2. Throughout this section, let us fix ν and g as in the hypothesis of Theorem 3.6.1. That is, ν is a cardinal, $\nu \geq 2$, g is $\mathrm{Col}(\omega, < \nu)$–generic over V, and we assume $L(\mathbb{R}^g) \models \mathsf{DC}$, where $\mathbb{R}^g = \bigcup_{\eta<\nu} \mathbb{R}^{V[g\restriction\eta]}$. Let us further fix α such that in $L(\mathbb{R}^g)$, α is as in case (1) or (2) of Lemma 3.3.2, i.e., inside $L(\mathbb{R}^g)$, α is critical, W^*_β holds true for all $\beta \leq \alpha$, and either

Case (1) $\alpha = \eta + 1$ for some critical η, or else
Case (2) α is a limit of critical ordinals and
Case (2)(a) $\mathrm{cf}(\alpha) = \omega$ or
Case (2)(b) $\mathrm{cf}(\alpha) > \omega$, but α is still inadmissible.

We also assume inductively that the Witness Dichotomy holds true for all $\beta < \alpha$. We aim to prove that there is a mouse operator $J^0 = J^0_\alpha$ such that, setting $J^{n+1}_\alpha = (J^n_\alpha)^w$ for $n < \omega$, if J^n_α is total on $(H_{\nu\cdot\aleph_1^{V[g]}})^V$ for all $n < \omega$, then $W^*_{\alpha+1}$ holds in $L(\mathbb{R}^g)$.

The cases (1) and (2)(a) are simpler and very similar to each other, and we deal with them first. We then take care of the harder case, (2)(b).

4.1 The easier inadmissible cases

Let us thus suppose that α is as in case (1) or (2)(a) of Lemma 3.3.2. Let us first assume that α is as in case (2)(a). We'll then briefly discuss how to adjust the argument which is to come to the case when α is as in case (1).

We may pick a sequence $\langle\Gamma_n|n<\omega\rangle$ of scaled Σ-pointclasses which is "cofinal" in $\Sigma_1(J_\alpha(\mathbb{R}))$ in the following sense. Let $\langle\alpha_n|n<\omega\rangle$ be cofinal in α such that each α_n begins a gap, and set $\Gamma_n=\Sigma_1(J_{\alpha_n}(\mathbb{R}^g))$ for $n<\omega$. Notice that we shall have that $C_{\Sigma_1(J_\alpha(\mathbb{R}^g))}=\bigcup_{n<\omega} C_{\Gamma_n}$.

Let $n<\omega$. As W^*_β holds for every $\beta\leq\alpha_n+1$, Theorem 3.6.1 applied to α_n yields that $J^m_{\alpha_n}$ is total on $(H_{\nu\cdot\aleph_1^{V[g]}})^V$ for each $m<\omega$. Our mouse operator $J^0=J^0_\alpha$ will then just be the "amalgamation" of all the $J^m_{\alpha_n}$.

For $n,m<\omega$, let $A_{n,m}\in H_\nu$ be the base of the cone of all $A\in H_\nu$ such that $J^m_{\alpha_n}(A)$ is defined. Let us then call $A\in H_\nu$ *suitable* iff A is transitive and self–wellordered and $\oplus_{n,m<\omega}A_{n,m}\in A$.

We may now define J^0 as follows.

Definition 4.1.1 *For any suitable A, $J^0(A)$ is the shortest initial segment $\mathcal{M}$ of $Lp(A)$ such that for some $\gamma<o(\mathcal{M})$, $\mathcal{M}|\gamma$ is a ZFC^-–model which is closed under all $J^m_{\alpha_n}$, $n,m<\omega$.*

The following is trivial, as all $J^m_{\alpha_n}$ relativise well.

Lemma 4.1.2 *J^0 relativises well.*

The following lemma finishes the proof of the Witness Dichotomy in the current case.

Lemma 4.1.3 *Suppose that J^n is total for all $n<\omega$. Then $W^*_{\alpha+1}$ holds in $L(\mathbb{R}^g)$.*

Proof. Let U be a set of reals in $J_{\alpha+1}$ and $k<\omega$. There is then a real z and an $n<\omega$ such that U is Σ_n–definable over $J_\alpha(\mathbb{R}^g)$ in the parameter z. Suppose $z=\rho^{g\restriction\mu'}$ for some $\mu'<\nu$. Let A be the base of the cone of all $B\in H_\nu$ such that $J^{k+n+2}(B)$ is defined. Set

$$\mathcal{P}=J^{k+n+2}(\langle A,\rho\rangle).$$

We show that $P[g\restriction\mu']$ is the desired witness.

Let $\delta_0<\cdots<\delta_{k+n+1}$ be the Woodin cardinals of P. Since $\mathcal{P}$ is closed under each $J^m_{\alpha_n}$, $\mathcal{P}[g\restriction\mu']$ is closed under each $C_{\Sigma_1(J_{\alpha_n}(\mathbb{R}^g))}$. By Lemma 3.7.5 we thus have in $\mathcal{P}[g\restriction\mu']$ a capturing term $\dot{W}\in\mathcal{P}[g\restriction\mu']^{\mathrm{Col}(\omega,\delta_{k+n+1})}$ for the universal $\Sigma_1(J_\alpha(\mathbb{R}^g))$–set which is obtained by "amalgamating" terms for $\Sigma_1(J_{\alpha_n}(\mathbb{R}^g))$–sets. By Lemma 1.4.18, there is then a capturing term $\tau\in\mathcal{P}[g\restriction\mu']^{\mathrm{Col}(\omega,\delta_{k+1})}$ for U.

We may now define the trees S and T required by 3.2.1 as follows. T searches for $y, \pi, \mathcal{Q}, h$ such that $\pi\colon \mathcal{Q}[g \restriction \mu'] \to P[g \restriction \mu']|\gamma$ (where γ is the largest limit ordinal of P), h is $\mathrm{Col}(\omega, \pi^{-1}(\delta_{k+1}))$–generic over $\mathcal{Q}[g \restriction \mu']$, and $y \in \pi^{-1}(\tau)^h$. S searches for $y, \pi, \mathcal{Q}, h$ such that $\pi\colon \mathcal{Q}[g \restriction \mu'] \to P[g \restriction \mu']|\gamma$, h is $\mathrm{Col}(\omega, \pi^{-1}(\delta_{k+1}))$–generic over $\mathcal{Q}[g \restriction \mu']$, and $y \notin \pi^{-1}(\tau)^h$. It is straightforward to verify that this works.

□

Case (1) is similar and uses the S–hierarchy.

4.2 The main inadmissible case

Let us now suppose that α is as in case (2)(b) of Lemma 3.3.2. I.e., inside $L(\mathbb{R}^g)$, α is a critical limit of critical ordinals, W^*_β holds true for all $\beta \leq \alpha$, $\mathrm{cf}(\alpha) > \omega$, and α is inadmissible. Since α is a limit ordinal, we have W_α inside $L(\mathbb{R}^g)$ by Lemma 3.5.4.

Let $\varphi(v_0, v_1)$ and $x \in \mathbb{R}^g$ determine the failure of admissibility, so that φ is Σ_1,

$$\forall y \in \mathbb{R}^g\ \exists \beta < \alpha\ J_\beta(\mathbb{R}^g) \models \varphi[x, y],$$

and letting $\beta(x, y)$ be the least such β,

$$\alpha = \sup\{\beta(x, y) \mid y \in \mathbb{R}^g\}.$$

Notice that since α begins a gap, $J_\alpha(\mathbb{R}^g)$ is the Σ_1 hull of its reals, so the parameter from which a failure of admissibility is defined can indeed be taken to be a real. Pick $\mu < \nu$ and $p \in g \restriction \mu$ such that $x = \tau^{g \restriction \mu}$, where τ is (or may be construed as) a $\mathrm{Col}(\omega, < \mu)$-term, and p forces in $\mathrm{Col}(\omega, < \nu)$ over V about τ all the properties of x in $V[g]$ which we have listed so far, in particular,

$$p \Vdash \forall y \in \dot{\mathbb{R}}\ \exists \beta < \check{\alpha}\ J_\beta(\dot{\mathbb{R}}) \models \varphi[\tau, y],$$

and

$$p \Vdash \forall \beta < \check{\alpha}\ \exists y \in \dot{\mathbb{R}}\ J_\beta(\dot{\mathbb{R}}) \models \neg\varphi[\tau, y].$$

(Here, $\dot{\mathbb{R}}$ is the canonical term for $\mathbb{R}^g$.)

Let us call a set A in V *suitable* if $A \in H_\nu$, A is transitive and self-wellordered, and $\tau \in A$. Our mouse operator $J^0 = J^0_\alpha$ is to be defined on all suitable A.

Let A be suitable, let $\mathcal{M}$ be an A-premouse, and $G \times H$ be $\mathcal{M}$-generic for $\mathrm{Col}(\omega, <\mu) \times \mathrm{Col}(\omega, A)$; then $\mathcal{M}[G][H]$ can be regarded as a z-premouse, where $z = z(G, H)$ is a real obtained in some simple fashion from G, H, and A, and which in turn codes G, H, and A in some simple fashion. (See [40] or [32].) Also, there is a term $\sigma = \sigma_A$ defined uniformly from A, τ in $\mathcal{M}$ such that whenever $G \times H$ is generic as above, then $\sigma^{G \times H} \in \mathbb{R}$ and

$$(\sigma^{G \times H})_0 = \tau^G,$$

and

$$\{(\sigma^{G \times H})_i \mid i > 0\} = \{\rho^{G \times H} \mid \rho \in L_1(A) \text{ and } \rho^{G \times H} \in \mathbb{R}\}.$$

Here $(w)_i$ is the i^{th} real coded into the real w, in some fixed simple way, and $L_1(A)$ is the first level of Gödel's L over A. For $n < \omega$, let φ_n^* be the Σ_1 formula

$$\varphi_n^*(v) \equiv \exists\gamma(J_\gamma(\mathbb{R}) \models \forall i \in \omega(i > 0 \Rightarrow \varphi((v)_0, (v)_i)) \wedge (\gamma + \omega n) \text{ exists }).$$

The requirement in φ_n^* that $\gamma + \omega n$ is to exists will be used in the proof that J^0 relativises well (cf. Lemma 4.2.3).

Now let ψ be the natural sentence in the language of A-premice (having therefore a name for A) such that for any A-premouse $\mathcal{M}$:

$$\mathcal{M} \models \psi$$

iff whenever $G \times H$ is $\mathcal{M}$-generic over $\mathrm{Col}(\omega, <\mu) \times \mathrm{Col}(\omega, A)$ and $p \in G$, then for any n there is a $\gamma < o(\mathcal{M})$ such that

$$\mathcal{M}[z(G, H)]|\gamma \text{ is a } \langle\varphi_n^*, \sigma_A^{G \times H}\rangle\text{- prewitness.}$$

Definition 4.2.1 *For any suitable A, $J^0(A)$ is the shortest initial segment of* $\mathrm{Lp}(A)$ *which satisfies ψ, if it exists, and is undefined otherwise.*

Lemma 4.2.2 *For any suitable A, $J^0(A)$ exists, and moreover, $J^0(A)$ is ordinal definable from A over $J_\gamma(\mathbb{R}^g)$, for some $\gamma < \alpha$.*

Proof. Since W_α holds true in $L(\mathbb{R}^g)$, there is some work to be done in going back to V, as is done in this lemma.

Fix $h \times H \in V[g]$ which is V-generic over $\mathrm{Col}(\omega, <\mu) \times \mathrm{Col}(\omega, A)$, and such that $p \in h$. For $q \leq p$ in $\mathrm{Col}(\omega, <\mu)$, let h_q be the finite variant of h such that $q \in h_q$, i.e.,

$$h_q = q \cup h \restriction (\mathrm{dom}(h) \setminus \mathrm{dom}(q))$$

(where we identify h with $\bigcup h\colon \mu\times\omega\to\mu$). So h_q is V-generic for $\mathrm{Col}(\omega,<\mu)$, and $V[h_q]=V[h]$. Hence $h_q\times H$ is V-generic for $\mathrm{Col}(\omega,<\mu)\times\mathrm{Col}(\omega,A)$, and $V[h_q,H]=V[h,H]$.

Since p forced what it did over V, we have in $V[g]$ a $\langle\varphi_n^*,\sigma_A^{h_q\times H}\rangle$-witness, for each $q\le p$ and each $n<\omega$. To see this, fix $q\le p$ and $n<\omega$. Now notice that there is some $K\in V$ which is $V[h_q,H]$-generic for $\mathrm{Col}(\omega,<\nu)$ such that $V[h_q,H,K]=V[g]$, and that $h_q\times H\times K$ may be construed as a V-generic filter for $\mathrm{Col}(\omega,<\nu)$ which contains the condition q. Hence if $i>0$, then $\exists\gamma<\alpha\ J_\gamma(\mathbb{R}^g)\models\varphi[\tau^h,(\sigma_A^{h_q,H})_i]$. Because α has uncountable cofinality, there is then in fact some $\gamma<\alpha$ such that $J_\gamma(\mathbb{R}^g)\models\varphi[\tau^h,(\sigma_A^{h_q,H})_i]$ for all $i>0$. Hence by W_α in $V[g]$, there is in $V[g]$ a $\langle\varphi_n^*,\sigma_A^{h_q\times H}\rangle$-witness whose iteration strategy for countable trees is in $J_\alpha(\mathbb{R}^g)$, call it $\mathcal{N}_q^n$. Here $\mathcal{N}_q^n$ can be regarded as a mouse over $z(h_q,H)$.

Note that $z(h_q,H)$ is Turing equivalent to $z(h_r,H)$, for $q,r\le p$. So the $\mathcal{N}_q^n$ are all essentially mice over the same real z (though the parameters τ^{h_q} and τ^{h_r} for the Σ_1 sentences witnessed by these mice have nothing to do with one another), and we may and shall assume that they are pairwise lined up. So we can, in $L(\mathbb{R}^g)$, take the union of the countably many mice over z corresponding to different q's and n's. Call this union $\mathcal{N}$.

Let $\mathcal{P}$ be the structure constructed over A from the extender sequence of $\mathcal{N}$. One can show that in $L(\mathbb{R}^g)$, $\mathcal{P}$ is an iterable mouse over A such that $\mathcal{P}[h\times H]=\mathcal{N}$. This is done by showing, by induction on η, that $\mathcal{P}|\eta$ is an iterable mouse, that $\mathcal{P}|\eta\in V$ (so that $h\times H$ is generic over $\mathcal{P}|\eta$), and that $(\mathcal{P}|\eta)[h\times H]=\mathcal{N}|\eta$. See the proof of Therorem 3.9 of [40] for a detailed version of this "inverting a generic extension" argument. (See also [32].)

$\mathcal{P}$, or an initial segment thereof, is the desired $J^0(A)$. In order to show this, it suffices to see that $\mathcal{P}\models\psi$. To verify that $\mathcal{P}\models\psi$, let $h'\times H'$ be $\mathcal{M}$-generic for $\mathrm{Col}(\omega,<\mu)\times\mathrm{Col}(\omega,A)$, and suppose that for some n there is no $\gamma<o(\mathcal{P})$ such that

$$\mathcal{P}[z(h',H')]|\gamma \text{ is a } \langle\varphi_n^*,\sigma_A^{h'\times H'}\rangle-\text{prewitness.}$$

Let $(q,r)\in h'\times H'$ force over $\mathcal{M}$ that "no initial segment of myself is a $\langle\varphi_n^*,\sigma_A\rangle$–prewitness." By the construction of $\mathcal{P}$, there is some $\gamma<o(\mathcal{P})$ such that

$$\mathcal{P}[z(h_q,H)]|\gamma \text{ is a } \langle\varphi_n^*,\sigma_A^{h_q\times H}\rangle-\text{prewitness.}$$

But then if H_r is the finite variant of H such that $r\in H_r$, then $\mathcal{P}[z(h_q,H_r)]|\gamma=$

$\mathcal{P}[z(h_q, H)]|\gamma$, and

$$\{(\sigma_A^{h_q\times H})|i>0\} = \{(\sigma_A^{h_q\times H_r})|i>0\}.$$

Therefore,

$$\mathcal{P}[z(h_q, H_r)]|\gamma \text{ is a } \langle\varphi_n^*, \sigma_A^{h_q\times H_r}\rangle - \text{prewitness},$$

contradicting what (q, r) was supposed to force. □

Lemma 4.2.3 *J_α^0 relativises well, as do all the J_α^n, $n<\omega$.*

Proof. We show that J_α^0 relativises well. We prove (1) of Definition 3.1.5 and leave (2) as an exercise to the reader.

We give an informal description of how to compute $J^0(A)$ from A and $J^0(B)$, and leave it to the reader to convert this into a formula which defines $J^0(A)$ over any transitive model N of ZFC^- with $J^0(B)\in N$. Our description will make use of generic extensions of $J^0(A)$ and $J^0(B)$, but in the end we only use the forcing relations of these models, so the description works in N.

Let $G\times H$ be $J^0(A)$–generic for $\mathrm{Col}(\omega,<\mu)\times\mathrm{Col}(\omega,A)$. For any $m<\omega$, let γ_m be the least γ such that $J^0(A)[z(G,H)]|\gamma$ is a $(\varphi_m^*, \sigma_A^{G\times H})$–witness. We have that $o(J^0(A)) = \sup_{m<\omega}\gamma_m$. Therefore, it suffices to see that we can recover $J^0(A)|\gamma_m$ from A and $J^0(B)$ uniformly in $m<\omega$.

Let us fix $m<\omega$. Let β_m be the least β such that $J_\beta(\mathbb{R}^g)\models\varphi_m^*(\sigma_A^{G\times H})$. By $\mathrm{cf}(\alpha)>\omega$, $\beta_m<\alpha$. The hypothesis $W_{\beta_m+\omega}$ (which holds true in $L(\mathbb{R}^g)$) shows that there is a $(\varphi_m^*, \sigma_A^{G\times H})$–witness with an iteration strategy in $J_{\beta_m+\omega}(\mathbb{R}^g)$, and in fact the proof of Lemma 4.2.2 shows that $J^0(A)|\gamma_m$ itself has an iteration strategy in $J_{\beta_m+\omega}(\mathbb{R}^g)$, say in $J_{\beta_m+k}(\mathbb{R}^g)$, where $k<\omega$. Hence $J^0(A)|\gamma_m$ is definable from A over $J_{\beta_m+k}(\mathbb{R}^g)$.

Now let K be such that $G\times K$ is $J^0(B)$–generic for $\mathrm{Col}(\omega,<\mu)\times\mathrm{Col}(\omega,B)$. Let us also assume that H is coded into K. If $\tilde\beta$ is least with $J_{\tilde\beta}(\mathbb{R}^g)\models\varphi_m^*(\sigma_A^{G\times H})$, then

$$\{(\sigma_A^{G\times H})_i\ :\ i>0\}\subset\{(\sigma_B^{G\times K})_i\ :\ i>0\}$$

yields that $\beta_m\le\tilde\beta$. For each $l<\omega$, $J^0(B)[z(G\times K)]$ has an initial segment which is a $(\varphi_l^*, \sigma_B^{G\times K})$–witness, and therefore $J^0(B)[z(G\times K)]$ has an initial segment which "knows" the theory of $J_{\beta_m+k}(\mathbb{R}^g)$ in the sense of the proof of Lemma 3.5.3. Using the homogeneity of the relevant forcings we finally get that $J^0(A)|\gamma_m\in J^0(B)$, and we also get a way of defining $J^0(A)|\gamma_m$ from A and $J^0(B)$.

□

The following Lemma finishes the proof of the Witness Dichotomy in the current case.

Lemma 4.2.4 *Suppose that J^n is defined for all $n < \omega$; then $W^*_{\alpha+1}$ holds in $L(\mathbb{R}^g)$.*

Proof. Let U be a set of reals in $J_{\alpha+1}(\mathbb{R}^g)$, and $k < \omega$; we seek a coarse (k, U)-Woodin mouse. Suppose that U is Σ_n-definable over $J_\alpha(\mathbb{R}^g)$ from the parameter z. As there is a lightface Σ_1 partial map of $\mathbb{R}^g$ onto $J_\alpha(\mathbb{R}^g)$, we may and shall assume z to be a real. Let $\bar{g} = g \restriction \mu'$, where $\mu \leq \mu' < \nu$, with $z = \rho^{g\restriction\mu'}$, and let

$$\mathcal{P} = J^{k+n+3}(\mathrm{TC}(\langle \tau, \rho \rangle)).$$

We show that $\mathcal{P}[\bar{g}]$ is the desired witness.

Let $\delta_0 < \cdots < \delta_{k+n+2}$ be the Woodin cardinals of $\mathcal{P}$. Let W be a universal $\Sigma_1^{J_\alpha(\mathbb{R}^g)}$ set of reals, and ψ a Σ_1 formula which defines it over $J_\alpha(\mathbb{R}^g)$. Let Σ be the canonical iteration strategy for $\mathcal{P}$, and hence for $\mathcal{P}[\bar{g}]$.

Claim. There is a term $\dot{W} \in \mathcal{P}[\bar{g}]^{\mathrm{Col}(\omega, \delta_{k+n+1})}$ which captures W, i.e., whenever

$$i \colon \mathcal{P}[\bar{g}] \to \mathcal{Q}[\bar{g}]$$

is an iteration map by Σ, and h is $\mathrm{Col}(\omega, i(\delta_{k+n+1}))$-generic over $\mathcal{Q}[\bar{g}]$, and $y \in \mathbb{R} \cap \mathcal{Q}[\bar{g}][h]$, then

$$y \in W \Leftrightarrow y \in i(\dot{W})^h.$$

Proof of the Claim. Roughly speaking, the term $\dot{W}$ asks: if we Levy collapse δ_{k+n+2} via l, and then use $J^0(\mathcal{P}[\bar{g}][h][l])|\delta_{k+n+2}$ as our oracle for the theory of the first level of $L(\mathbb{R})$ at which $\varphi(x, \sigma^l)$ is seen to be true for all terms $\sigma \in L_1(P[\bar{g}][h]|\delta_{k+n+2})$, do we see that $\psi(y)$ has been verified before that level?

In order to be more precise, set $A = \mathcal{P}|\delta_{k+n+2}$, i.e., $\mathcal{P}$ cut off at its largest Woodin cardinal. We have that $\mathcal{P} = J^0(A)$. If H is $\mathcal{P}[\bar{g}]$-generic for $\mathrm{Col}(\omega, A)$, then we may pick some H' which is $\mathcal{P}[g \restriction \mu]$-generic for $\mathrm{Col}(\omega, A)$ such that $\mathcal{P}[g \restriction \mu, H'] = \mathcal{P}[\bar{g}, H]$. (Recall that $\mu \leq \mu'$.) Hence for all $m < \omega$ there is some $\gamma < o(\mathcal{P})$ such that

$$\mathcal{P}[z(g \restriction \mu, H')]|\gamma \text{ is a } \langle \varphi^*_m, \sigma_A^{g\restriction\mu \times H'} \rangle\text{–witness.}$$

In particular, for some $k < \omega$, $\mathcal{P}[z(g \upharpoonright \mu, H')]|\gamma$ has a tree, T, such that $\mathcal{P}[z(g \upharpoonright \mu, H')]|\gamma$ thinks that "$p[T]$ is the Σ_{k+3}–theory of $J_\eta(\mathbb{R})$, where η is least such that $J_\eta(\mathbb{R}) \models \theta^k[z]$." (Here, θ^k is as in definition 3.5.1 where now φ^*_m plays the role of θ.) This theory depends both on m and H (sic!). Let us denote by $\mathsf{Th}(H, \gamma, k)$ the Σ_{k+3}–theory of $J_\eta(\mathbb{R}^{\mathcal{P}[z(g\upharpoonright\mu,H')]|\gamma})$ where η is least with $J_\eta(\mathbb{R}^{\mathcal{P}[z(g\upharpoonright\mu,H')]|\gamma}) \models \theta^k[z]$. I.e., $\mathcal{P}[z(g \upharpoonright \mu, H')]|\gamma \models$ "$p[T] = \mathsf{Th}(H, \gamma, k)$.

It is now straightforward to construct a name $\dot{W} \in \mathcal{P}[\bar{g}]^{\mathrm{Col}(\omega,\delta_{k+n+1})}$ such that whenever h is $\mathrm{Col}(\omega, \delta_{k+n+1})$-generic over $\mathcal{P}[\bar{g}]$, then for all $y \in \mathbb{R} \cap \mathcal{P}[\bar{g}, h]$ the following holds true: $y \in \dot{W}^h$ iff for any (all) l being $\mathrm{Col}(\omega, \delta_{k+n+2})$-generic over $\mathcal{P}[\bar{g}, h]$, if there exists some γ such that

$$\ulcorner\varphi[x, \sigma^l]\urcorner \in \mathsf{Th}(h \times l, \gamma, k) \text{ for all terms } \sigma \in L_1(\mathcal{P}[\bar{g}, h]|\delta_{k+n+2}),$$

then

$$\ulcorner\psi[y]\urcorner \in \mathsf{Th}(h \times l, \gamma, k).$$

It is easy to see that if $y \in \dot{W}^h$, then $y \in W$. This is because all oracles of the form $\mathsf{Th}(h \times l, \gamma, k)$ indeed give the theory of some initial segment of $L(\mathbb{R})$, restricted to parameters in $\mathbb{R} \cap P[\bar{g}, h]$. But as $\mathsf{Th}(h \times l, \gamma, k)$ gives the theory of the first level of $L(\mathbb{R})$ at which $\varphi[x, \sigma^l]$ is verified to be true for all terms $\sigma \in L_1(\mathcal{P}[\bar{g}, h]|\delta_{k+n+2})$, any such initial segment $J_\gamma(\mathbb{R})$ in question must in fact be such that $\gamma < \alpha$. Therefore, no unwanted real will be picked up by $\dot{W}^h$.

Now let $y \in W \cap \mathcal{P}[\bar{g}, h]$. Let $\gamma < \alpha$ be such that $J_\gamma(\mathbb{R}^g) \models \psi[y]$. Pick some real $u \in V$ such that $\beta(x, u) \geq \gamma$. There is an iteration

$$j : \mathcal{P}[\bar{g}, h] \to \mathcal{Q}[\bar{g}, h]$$

such that u is $\mathrm{Col}(\omega, j(\delta_{k+n+2}))$-generic over $\mathcal{Q}[\bar{g}, h]$, say $u \in \mathcal{Q}[\bar{g}, h, l]$. Let $\mathcal{Q}[z(g \upharpoonright \mu, H'')]|\gamma'$ be a $\langle\varphi^*_0, \sigma_A^{g\upharpoonright\mu,H''}\rangle$-witness, where $\mathcal{Q}[\bar{g}, h, l] = \mathcal{Q}[g \upharpoonright \mu, H'']$. As the canonical term σ for u is in $L_1(\mathcal{Q}[\bar{g}, h])$, we will have that

$$\ulcorner\varphi[x, u]\urcorner \in \mathsf{Th}(H'', \gamma', k).$$

Hence if $\mathsf{Th}(H'', \gamma', k)$ is the Σ_{k+3}–theory of $J_\xi(\mathbb{R}^g)$, restricted to real parameters in $\mathcal{Q}[\bar{g}, h, u]$, then $\xi \geq \beta(x, u) \geq \gamma$. This implies that $y \in j(\dot{W})^l$. However, this is then true for all l, and hence by pulling back we get $y \in \dot{W}^l$ whenever l is $\mathrm{Col}(\omega, \delta_{k+n+2})$-generic.

The same argument now gives the full Claim. □

Since α is inadmissible as witnessed by the real parameter x, and α begins a gap, the Σ_n theory of $J_\alpha(\mathbb{R}^g)$ can be computed from the Σ^1_n theory of $(\mathbb{R}, W, x)$. By Lemma 1.4.18, we then get a term $\dot{U}$ in $P[\bar{g}]^{\mathrm{Col}(\omega,\delta_{k+2})}$ such that if h is P-generic over $\mathrm{Col}(\omega, \delta_{k+2})$ and y is a real in $P[\bar{g}][h]$, then

$$y \in U \Leftrightarrow y \in \dot{U}^h.$$

Moreover, letting $\gamma = (\delta_0^+)^P$, and $\pi\colon \mathcal{Q}[\bar{g}] \to P[\bar{g}]|\gamma$ and $\pi(\dot{Z}) = \dot{U}$, and h is $\mathcal{Q}[\bar{g}]$ generic over $\mathrm{Col}(\omega, \pi^{-1}(\delta_{k+2}))$, then again, $\dot{Z}^h = U \cap \mathcal{Q}[\bar{g}][h]$.[1] In $P[\bar{g}]$ we can then construct the absolutely complementing trees S and T required by 3.2.1: T_y tries to build $\pi, \mathcal{Q}, h$ as above with $y \in \dot{Z}^h$, and S_y tries to build $\pi, \mathcal{Q}, h$ as above with $y \notin \dot{Z}^h$. □

[1]This fact about Skolem hulls follows from the construction. It comes down to the fact that an elementary submodel of an iterable structure is still iterable.

Chapter 5

The end of gap cases in $L(\mathbb{R})$

5.1 Scales at the end of a gap

In this chapter, we fix α and β as in (3) of Lemma 3.3.2. That is,

$$\alpha = \sup(\{\eta < \beta | \eta \text{ is critical }\}) < \beta$$

and either $[\alpha, \beta]$ is a weak Σ_1 gap, or $\beta - 1$ exists, and $[\alpha, \beta - 1]$ is a strong Σ_1 gap. Let us write β^* for β if (3)(a) holds and for $\beta - 1$ if (3)(b) holds. (We can mostly ignore the distinction between (3)(a) and (3)(b), though.) We have that $J_\alpha(\mathbb{R}^g)$ is admissible, and Σ_1- projects to $\mathbb{R}^g$. We have W^*_β, and hence W_α.

Here is a basic facts about scales which we shall need.

Theorem 5.1.1 ([35]) *Assume W^*_β, where β is critical and case 3 holds at β; then*

(1) every set of reals $A \in J_\beta(\mathbb{R}^g)$ admits a scale $\vec{\psi}$ such that each prewellorder $\leq_{\psi_i}$ belongs to $J_\beta(\mathbb{R}^g)$, and

(2) letting n be least such that $\rho_n(J_\beta(\mathbb{R}^g)) = \mathbb{R}$, and U be any boldface $\Sigma_n^{J_\beta(\mathbb{R}^g)}$ set of reals, we have $U = \bigcup_{n<\omega} U_n$, where each $U_n \in J_\beta(\mathbb{R}^g)$.

In part (1), the sequence of prewellorders may not belong to $J_\beta(\mathbb{R}^g)$. Part (2) implies that the boldface pointclass $\Sigma_n^{J_\beta(\mathbb{R}^g)}$ is in fact the class of countable unions of sets of reals in $J_\beta(\mathbb{R}^g)$, and has the scale property.

Motivated by Theorem 5.1.1, we gave the Definition 3.7.1. To recall, a self justifying system (sjs, for short) is a countable set $\mathcal{A} \subseteq P(\mathbb{R})$ which is closed under complements (in $\mathbb{R}^g$), and such that every $A \in \mathcal{A}$ admits a scale $\vec{\psi}$ such that $\leq_{\psi_i} \in \mathcal{A}$ for all $i < \omega$. So for any set of reals $A \in J_\beta(\mathbb{R}^g)$, there is a self-justifying system $\mathcal{A} \subseteq J_\beta(\mathbb{R}^g)$ such that $A \in \mathcal{A}$.

5.2 The Plan

The set coding truth at the bottom of the Levy hierarchy over $J_\beta(\mathbb{R}^g)$ which we feed into our mice will be an iteration strategy Σ for a mouse $\mathcal{N}$ with a Woodin cardinal δ. There will be a sjs $\{A_i \mid i < \omega\}$ such that $U = \bigcup_i A_{2i}$ is universal at the bottom of the Levy hierarchy over $J_\beta(\mathbb{R}^g)$, and such that for each i, there is a term $\tau_i \in \mathcal{N}$ which captures A_i, in that

$$\pi(\tau_i)^h = A_i \cap \mathcal{P}[h],$$

whenever $\pi \colon \mathcal{N} \to \mathcal{P}$ is an iteration map by Σ, and h is $\mathcal{P}$-generic for $\mathrm{Col}(\omega, \pi(\delta))$. Thus $\mathcal{N}$ is close to being a coarse $(1, U)$-Woodin mouse, as witnessed by Σ.

From $(\mathcal{N}, \Sigma)$ we shall construct *hybrid Σ-mice*, mice over $\mathcal{N}$ constructed from an extender sequence as usual, while simultaneously closing under Σ. The condensation properties of Σ will imply that these hybrid mice behave like ordinary mice. The upshot is that we have a *hybrid mouse operator* J^0. J^0 will be a (ν, A)-hmo, where A codes $\mathcal{N}$. Setting $J^{n+1} = (J^n)^w$, we can capture truth at the higher levels of the Levy hierarchy over $J_\beta(\mathbb{R}^g)$ via the $J^n(B)[g]$'s, for $B \in H_\nu$.

So either some $(J^n)^w$ fails to exist (be defined on a cone), or we have $W^*_{\beta+1}$.

5.3 Fullness-preserving iteration strategies

Definition 5.3.1 *For any self-wellordered (swo) $A \in H_\nu$, let $\mathrm{Lp}^\alpha(A)$ be the "union" of all sound A-mice $\mathcal{N}$ projecting to $\sup(A)$ such that $J_\alpha(\mathbb{R}^g) \models \mathcal{N}$ is ω_1-iterable.*

Note $\mathrm{Lp}^\alpha(A)$ is an initial segment of $\mathrm{Lp}(A)$, since the iteration strategy witnessing $\mathcal{N} \in \mathrm{Lp}^\alpha(A)$ is unique, so that its restriction to V is in V.

Definition 5.3.2 *Let $A \in H_\nu$ be swo. An A-premouse $\mathcal{N}$ is α-*suitable*, or just* suitable*, iff card $(\mathcal{N}) < \nu$ and*

(a) $\mathcal{N} \models$ *there is exactly one Woodin cardinal. We write* $\delta^{\mathcal{N}}$ *for the unique Woodin cardinal of* $\mathcal{N}$.

(b) *Letting* $\mathcal{M}_0 = \mathcal{N}|\delta^{\mathcal{N}}$, *and* $\mathcal{M}_{i+1} = \mathrm{Lp}^{\alpha}(\mathcal{M}_i)$, *we have that* $\mathcal{N} = \bigcup_{i<\omega} \mathcal{M}_i$. *That is,* $\mathcal{N}$ *is the* Lp^{α} *closure of* $\mathcal{N}|\delta^{\mathcal{N}}$, *up to its* ω^{th} *cardinal above* $\delta^{\mathcal{N}}$.

(c) *If* $\xi < \delta^{\mathcal{N}}$ *is a cardinal of* $\mathcal{N}$, *then* $\mathrm{Lp}^{\alpha}(\mathcal{N}|\xi) \models \xi$ *is not Woodin.*

We say an iteration tree $\mathcal{U}$ on a premouse $\mathcal{N}$ *lives below* η if $\mathcal{U}$ can be regarded as an iteration tree on $\mathcal{N}|\eta$. If $\mathcal{U}$ is normal, then as usual we write $\delta(\mathcal{U})$ for $\sup\{\mathrm{lh}(E^{\mathcal{U}}_{\alpha}) \mid \alpha < \mathrm{lh}(\mathcal{U})\}$, and $\mathcal{M}(\mathcal{U})$ for $\bigcup\{\mathcal{M}^{\mathcal{U}}_{\alpha}|\mathrm{lh}(E^{\mathcal{U}}_{\alpha}) \mid \alpha < \mathrm{lh}(\mathcal{U})\}$.

Definition 5.3.3 *Let* $\mathcal{U}$ *be a normal iteration tree of length* $< \nu$ *on a suitable* $\mathcal{N}$, *and suppose* $\mathcal{U}$ *lives below* $\delta^{\mathcal{N}}$; *then* $\mathcal{U}$ *is* short *iff for all limit* $\xi \leq \mathrm{lh}(\mathcal{U})$, $\mathrm{Lp}^{\alpha}(\mathcal{M}(\mathcal{U} \restriction \xi)) \models \delta(\mathcal{U} \restriction \xi)$ *is not Woodin. Otherwise, we say* $\mathcal{U}$ *is* maximal.

Just to emphasize, a non-normal iteration tree is neither short nor maximal. Similarly, a tree on $\mathcal{N}$ which cannot be regarded as a tree on $\mathcal{N}|\delta^{\mathcal{N}}$ is neither short nor maximal.

Definition 5.3.4 *Let* Σ *be a* ν*-iteration strategy on a suitable* $\mathcal{N}$; *then* Σ *is* fullness-preserving *iff whenever* $\mathcal{P}$ *is an iterate of* $\mathcal{N}$ *by* Σ, *via a tree which lives below* $\delta^{\mathcal{N}}$, *then*

(1) *if* $\mathcal{N}$*-to-*$\mathcal{P}$ *does not drop (in model), then* $\mathcal{P}$ *is suitable, and*

(2) *if* $\mathcal{N}$*-to-*$\mathcal{P}$ *drops (in model), then* $J_{\alpha}(\mathbb{R}^g) \models \mathcal{P}$ *is* ω_1*-iterable.*

It is not hard to see that in case (2) of 5.3.4, we have that for all ξ, $\mathrm{Lp}^{\alpha}(\mathcal{P}|\xi) \models \xi$ is not Woodin, and thus no initial segment of $\mathcal{P}$ is suitable.

Of course, we should really speak of α-suitability, etc., but α has been fixed.

Lemma 5.3.5 *Suppose* Σ *is a fullness-preserving iteration strategy for* $\mathcal{N}$, *and* $\mathcal{T}$ *is an iteration tree living below* $\delta^{\mathcal{N}}$, *played by* Σ, *which has a last normal component tree* $\mathcal{U}$ *having base model* $\mathcal{P}$ *and of limit length. Let* b *be the branch of* $\mathcal{U}$ *chosen by* Σ; *then*

(1) *if* $\mathcal{N}$*-to-*$\mathcal{P}$ *drops, then* $\mathcal{U}$ *is short, and* $\mathcal{Q}(b, \mathcal{U})$ *is a proper initial segment of* $\mathrm{Lp}^{\alpha}(\mathcal{M}(\mathcal{U}))$, *and*

(2) if $\mathcal{N}$-to-$\mathcal{P}$ does not drop, so that $\mathcal{P}$ is suitable, then

(a) for all $\xi < \mathrm{lh}(\mathcal{U})$, $\mathcal{U} \restriction \xi$ is short,

(b) if $\mathcal{U}$ is short, then $\mathcal{Q}(b,\mathcal{U})$ exists and is a proper initial segment of $\mathrm{Lp}^\alpha(\mathcal{M}(\mathcal{U}))$, and

(c) if $\mathcal{U}$ is maximal, then b does not drop, and $i_b^{\mathcal{U}}(\delta^{\mathcal{N}}) = \delta(\mathcal{U})$.

We shall omit the straightforward proof of this lemma.

According to this lemma, a fullness-preserving strategy is guided by $\mathcal{Q}$-structures in Lp^α, unless, for the current normal component $\mathcal{U}$, there is no such $\mathcal{Q}$-structure. That is case (2)(c) above, and then from (2)(c) we see that $\mathcal{U}$ has no normal continuation below $\delta(\mathcal{U})$. Moreover, although Lp^α cannot tell us what b is, it can identify $\mathcal{M}_b(\mathcal{U})$, since

$$\mathcal{M}_b(\mathcal{U}) = (\mathrm{Lp}^\alpha)\text{-closure of } (\mathcal{M}(\mathcal{U}))$$

up to its ω^{th} cardinal. This important insight is due to Woodin. It means that Lp^α can "track" a fullness-preserving iteration strategy, in that it can find the models of an evolving iteration tree, although it cannot always find the branches and embeddings.[1]

We wish to describe a condensation property for iteration strategies. For the notion of a finite support in an iteration tree, see [36]. Let $\mathcal{T}$ be an iteration tree on $\mathcal{N}$, and

$$\sigma \colon \beta \to \mathrm{lh}(\mathcal{T})$$

an order preserving map such that $\mathrm{ran}(\sigma)$ is support-closed. (In particular, we demand $\sigma(0) = 0$.) Then there is a unique iteration tree $\mathcal{S}$ on $\mathcal{N}$ of length β such that there are maps

$$\pi_\gamma \colon \mathcal{M}_\gamma^{\mathcal{S}} \to \mathcal{M}_{\sigma(\gamma)}^{\mathcal{T}},$$

for $\gamma < \beta$, (with π_0 being the identity and) which commute with the tree embeddings, with

$$\pi_\gamma(E_\gamma^{\mathcal{S}}) = E_{\sigma(\gamma)}^{\mathcal{T}}$$

for all $\gamma < \beta$, and $\pi_{\gamma+1}$ determined by the shift lemma. (Support-closure is just what we need to keep this process going.)

[1] For infinite stacks of normal trees, more work is needed even to find the models using only Lp^α as a guide. Using "quasi-iterations", Woodin has solved this problem. We shall not need quasi-iterations for our proof of $\mathsf{AD}^{L(\mathbb{R})}$, but they are needed in adapting Ketchersid's work.

Definition 5.3.6 *Let $\mathcal{S}$ and $\mathcal{T}$ be iteration trees related as above; then we say that $\mathcal{S}$ is a* hull *of $\mathcal{T}$, as* witnessed by *σ and the π_γ, for $\gamma < \mathrm{lh}(\mathcal{T})$.*

Definition 5.3.7 *An iteration strategy Σ* condenses well *iff whenever $\mathcal{T}$ is an iteration tree played according to Σ, and $\mathcal{S}$ is a hull of $\mathcal{T}$, then $\mathcal{S}$ is according to Σ.*

It is clear that if Σ is the unique iteration strategy[2] on $\mathcal{N}$, then Σ condenses well. More generally, if $\mathcal{N}$ is an initial segment of $\mathcal{M}$, and Γ is the unique iteration strategy for $\mathcal{M}$, and Σ is the strategy for $\mathcal{N}$ which is determined by Γ, then Σ condenses well. One can think of an iteration strategy which condenses well as the "trace" of a unique iteration strategy on a stronger mouse.

5.4 An sjs-guided iteration strategy in $V[g]$

In this and the next section, we shall prove

Theorem 5.4.1 *There is, in V, a suitable $\mathcal{N}$ and a fullness-preserving ν-iteration strategy Σ for $\mathcal{N}$ such that Σ condenses well.*

Proof. We work in $V[g]$ in this section, and obtain $\mathcal{N}$ and Σ there. In the next section, we move back to V.

Recall that $\mathrm{OD}^\gamma(z)$ is the collection of sets which are ordinal definable from z over $J_\gamma(\mathbb{R}^g)$; we write $\mathrm{OD}^{<\xi}(z)$ for $\bigcup_{\gamma<\xi} \mathrm{OD}^\gamma(z)$.

Let $\langle A_i | i \in \omega\rangle$ be a self-justifying system, with each $A_i \in J_\beta(\mathbb{R}^g)$, and A_0 the universal $\Sigma_1^{J_\alpha(\mathbb{R}^g)}$ set. Let

$$x^* = \tau^g$$

be a real such that for all i, A_i is $\mathrm{OD}^{<\beta}(x^*)$. Here τ is (essentially) a bounded subset of ν, and of course $\tau \in V$. The suitable $\mathcal{N}$ we seek will be a τ-mouse.

We need some concepts and results, due to Woodin, which are explained in more detail in [42] and [43]. First

Recall that for $\mathcal{N}$, z,κ, and A as in Lemma 3.7.4, $\tau^{\mathcal{N}}_{A,\mu}$ is the unique standard term τ such that $\tau^g = A \cap \mathcal{N}[g]$ for all $\mathrm{Col}(\omega,\kappa)$-generics g over $\mathcal{N}$. Cf. Definition 3.7.6.

[2] For some reasonable sort of iteration game.

Definition 5.4.2 *If $\mathcal{N}$ is suitable, then we write $\tau_A^{\mathcal{N}}$ for $\tau_{A,\delta}^{\mathcal{N}}$, where $\delta = \delta^{\mathcal{N}}$.*

See [42] for further explanation. Woodin proved the following key condensation result:

Theorem 5.4.3 (Term-relation condensation, Woodin) *Let $\mathcal{N}$ be a suitable premouse over $z \in HC$, and let $\mathcal{B}$ be a self-justifying family of subsets of $\mathbb{R}^g$ containing the universal $\Sigma_1^{J_\alpha(\mathbb{R}^g)}$ set, and such that each $B \in \mathcal{B}$ is $OD^{<\beta}(z)$. Suppose*

$$\pi \colon \mathcal{M} \to \mathcal{N}$$

is Σ_1-elementary and such that

$$\forall B \in \mathcal{B}\ \forall \mu \geq \delta^{\mathcal{N}}\ \ \tau_{B,\mu}^{\mathcal{N}} \in \operatorname{ran}(\pi).$$

Then

(a) $\mathcal{M}$ is suitable, and for all $B \in \mathcal{B}$,

$$\pi(\tau_{B,\bar{\mu}}^{\mathcal{M}}) = \tau_{B,\mu}^{\mathcal{N}},$$

where $\pi(\bar{\mu}) = \mu$,

(b) $\operatorname{ran}(\pi)$ is cofinal in $\delta^{\mathcal{N}}$, and

(c) if $\delta^{\mathcal{N}} \subseteq \operatorname{ran}(\pi)$, then $\pi =$ identity.

Proof. For part (a): This follows from Lemma 3.7.2.

For part (b): Let $\gamma = \sup(\operatorname{ran}(\pi) \cap \delta^{\mathcal{N}})$. Let $\psi \colon \mathcal{P} \to \mathcal{N}$ be the transitive collapse of the set of points definable over some $\mathcal{N}|\mu$, $\mu \in \operatorname{ran}(\pi)$, from the $\tau_{B,\mu}$ for ordinals $< \gamma$. Using the regularity of $\delta^{\mathcal{N}}$ in $\mathcal{N}$, we get that $\psi \restriction \gamma =$identity, and $\psi(\gamma) = \delta^{\mathcal{N}}$ as follows. Suppose that $\xi = \tau^{\mathcal{N}|\mu}(\vec{\eta}, \vec{p}) < \delta^{\mathcal{N}}$, where μ is a cardinal of $\mathcal{N}$ above $\delta^{\mathcal{N}}$, τ is a Skolem term, $\vec{\eta}$ is a finite subset of γ, and $\vec{p}$ is a finite subset of $\{\tau_{B,\bar{\mu}} | B \in \mathcal{B}, \bar{\mu} < \mu\}$. Then

$$\rho = \sup(\{\tau^{\mathcal{N}|\mu}(\vec{\eta}', \vec{p}) | \vec{\eta}' \in [\delta^{\mathcal{N}}]^{<\omega}\} \in \operatorname{ran}(\psi),$$

and $\rho < \delta^{\mathcal{N}}$ as $\delta^{\mathcal{N}}$ is regular in $\mathcal{N}$. Hence $\rho < \gamma$, so that $\xi < \gamma$ also.

From part (a), we then have that $\mathcal{P}$ is suitable. [3] But $\mathcal{P}|\gamma = \mathcal{N}|\gamma$, so we have then that $\mathcal{N}|\gamma$ is Lp^α-Woodin. The minimality condition in the suitability of $\mathcal{N}$ then implies $\gamma = \delta(\mathcal{T})$, as desired.

Part (c) follows at once from the Lp^α-fullness of $\mathcal{M}$. □

[3]The reason is essentially that Lp^α-fullness is a $\Pi_1^{J_\alpha(\mathbb{R}^g)}$ statement, true of reals coding $\mathcal{N}|\eta$ added by collapsing η, and Skolemized by the τ's.

Definition 5.4.4 *If $\mathcal{N}$ is suitable, and $\mathcal{T}$ is a maximal normal iteration tree on $\mathcal{N}$, then $\mathcal{M}(\mathcal{T})^+$ is the unique suitable $\mathcal{P}$ such that $\mathcal{M}(\mathcal{T}) = \mathcal{P}|\delta^{\mathcal{P}}$. We call a suitable $\mathcal{N}^*$ a* pseudo iterate *of $\mathcal{N}$ iff $\mathcal{N}^* = \mathcal{M}(\mathcal{T})^+$ for some maximal normal iteration tree on $\mathcal{N}$.*

Definition 5.4.5 *Let $\mathcal{N}$ be suitable z-premouse, where $z \in HC$ and codes x^*, and $A \subseteq \mathbb{R}^g$ be $OD^{<\beta}(z)$. We say $\mathcal{N}$ is* weakly A-iterable *just in case for all $n < \omega$, there is a fullness-preserving winning strategy Σ for* II *in the iteration game $\mathcal{G}(\omega, n, \omega_1)$*[4] *such that whenever*

$$i \colon \mathcal{N} \to \mathcal{P}$$

is an iteration map produced by an iteration according to Σ, then

$$i(\tau^{\mathcal{N}}_{A,\mu}) = \tau^{\mathcal{P}}_{A,i(\mu)}$$

for all cardinals $\mu \geq \delta^{\mathcal{N}}$ of $\mathcal{N}$.

We should remark that if $\mathcal{N}$ is weakly A-iterable, and Σ, Γ are iteration strategies for $\mathcal{G}(\omega, n, \omega_1)$ and $\mathcal{G}(\omega, k, \omega_1)$ witnessing this with $n \leq k$, then Σ and Γ can only disagree at some maximal normal component $\mathcal{U}$, and then their disagreement has no effect on the remainder of either game, since they agree that $\mathcal{M}(\mathcal{U})^+$ will be the base model for the next normal component. In particular, any model reached using Σ is itself weakly A-iterable.

We rely heavily on a basic result of Woodin, Theorem 5.4.8. In order to prove it we need a method for coiterating $L[E]$ constructions by coiterating the models in which the $L[E]$ constructions are performed.

Lemma 5.4.6 *Let $\mathcal{P}$, $\mathcal{Q}$ be transitive models of* ZFC *such that*

$$\kappa = \max(\mathrm{Card}(\mathcal{P}), \mathrm{Card}(\mathcal{Q})),$$

and let $x \in \mathcal{P} \cap \mathcal{Q}$ be transitive. Let $\mathcal{E} \in \mathcal{P}$ be a collection of (total) extenders from $\mathcal{P}$ with critical points above x, and let $\mathcal{F} \in \mathcal{Q}$ be a collection of (total) extenders from $\mathcal{Q}$ with critical points above x. Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are both $\kappa^+ + 1$ iterable with respect to extenders in $\mathcal{E}$ and $\mathcal{F}$ (and their images), respectively. There is then an iteration

$$\pi \colon \mathcal{P} \to \mathcal{P}^*$$

[4] The output of this game is a linear stack of n normal iteration trees, the first one being on $\mathcal{N}$.

of $\mathcal{P}$ by extenders from $\mathcal{E}$ (and the images thereof) and there is an iteration

$$\sigma\colon \mathcal{Q} \to \mathcal{Q}^*$$

of $\mathcal{Q}$ by extenders from $\mathcal{F}$ (and the images thereof) such that if

$$(\mathcal{N}_\xi^{\mathcal{P}^*}, \mathcal{M}_\xi^{\mathcal{P}^*} | \xi \leq \mathcal{P}^* \cap \mathrm{OR})$$

is the sequence of models from the $L[E,x]$ construction of $\mathcal{P}^*$ using extenders from $\pi(\mathcal{E})$ and if

$$(\mathcal{N}_\xi^{\mathcal{Q}^*}, \mathcal{M}_\xi^{\mathcal{Q}^*} | \xi \leq \mathcal{Q}^* \cap \mathrm{OR})$$

is the sequence of models from the $L[E,x]$ construction of $\mathcal{Q}^*$ using extenders from $\sigma(\mathcal{F})$, then for all $\xi \leq \mathcal{P}^* \cap \mathcal{Q}^* \cap \mathrm{OR}$, $\mathcal{N}_\xi^{\mathcal{P}^*} = \mathcal{N}_\xi^{\mathcal{Q}^*}$ and $\mathcal{M}_\xi^{\mathcal{P}^*} = \mathcal{M}_\xi^{\mathcal{Q}^*}$.

Proof. One needs to check that the obvious proof idea works. We omit further details. □

In the situation of Lemma 5.4.6, we'll say that the (sequence of models from the) $L[E,x]$ constructions of $\mathcal{P}^*$, $\mathcal{Q}^*$ are *lined up*.

Lemma 5.4.7 *Let $z \in HC^{V[g]}$. Then there is a(n) (α-)suitable z-premouse, which is ω_1 iterable with respect to short trees.*

Proof. Suppose not. We derive a contradiction via a reflection argument. As $\alpha < \beta^*$, we then have that in $J_{\beta^*}(\mathbb{R})$ there is some γ (namely, α) such that there is no γ-suitable z-premouse $\mathcal{N}$ which is ω_1 iterable with respect to short trees via an iteration strategy in $\Sigma_1^{J_\gamma(\mathbb{R})}(\{\mathcal{N}\})$. As $[\alpha, \beta^*]$ is a gap, we then also have that in $J_\alpha(\mathbb{R})$ there is some γ such that there is no γ-suitable z-premouse $\mathcal{N}$ which is ω_1 iterable with respect to short trees via an iteration strategy in $\Sigma_1^{J_\gamma(\mathbb{R})}(\{\mathcal{N}\})$. If $\gamma < \alpha$ is the least such, then $\gamma + 1$ begins a gap, as "there is no γ-suitable z-premouse $\mathcal{N}$ which is ω_1 iterable with respect to short trees via an iteration strategy in $\Sigma_1^{J_\gamma(\mathbb{R})}(\{\mathcal{N}\})$" can be formulated in a Σ_1 fashion. Therefore, the pointclasses $\Sigma_{2n+1}^{J_{\gamma+1}(\mathbb{R})}$ and $\Pi_{2n+2}^{J_{\gamma+1}(\mathbb{R})}$ have the scale property. We shall now derive a contradiction by constructing some γ-suitable z-premouse $\mathcal{N}$ which is ω_1 iterable with respect to short trees via an iteration strategy in $\Sigma_1^{J_\gamma(\mathbb{R})}(\{\mathcal{N}\})$.

Write $\Gamma = \Sigma_3^{J_{\gamma+1}(\mathbb{R})}$. Let N be a coarse $(1, U)$-Woodin mouse with $z \in N$, as given by W_α^*, where U is a universal Γ set of reals. Let us consider $W = L[E, z]^N$. Let δ be the Woodin cardinal of N, so that δ is also the Woodin cardinal of W. By the proof of the Mouse Capturing Theorem

3.4.6, there are $\eta < \zeta < \delta$ such that η is an inaccessible cardinal in N, η is a Woodin cardinal in $W|\zeta$, and $W|\zeta \notin C_\Gamma(N|\eta)$. This implies that η is a Woodin cardinal in $\mathrm{Lp}^\gamma(W|\eta)$. Let us set $W_0 = W|\eta$, $W_{n+1} = \mathrm{Lp}^\gamma(W_n)$ for $n < \omega$, and let W_ω be the "union" of the W_n, $n < \omega$. Then W_ω, which is in fact an initial segment of W, is our first candidate for $\mathcal{N}$; however, W_ω is too complicated.

Let us construct a sequence $(\mathcal{P}_n, \mathcal{T}_n \colon n < \omega)$ as follows. We set $\mathcal{P}_0 = W_\omega$. Having constructed $\mathcal{P}_n$, let $\mathcal{T}_n$ be some (possibly trivial) countable tree on $\mathcal{P}_n$ such that $\mathcal{T}_n$ has successor length and for some $\eta^* \in \mathcal{M}_\infty^{\mathcal{T}}$, η^* is a Woodin cardinal in $\mathrm{Lp}^\gamma(\mathcal{M}_\infty^{\mathcal{T}}|\eta^*)$, and let $\mathcal{P}_{n+1} = W_\omega^*$, where W_ω^* is the "union" of the W_n^*, $n < \omega$, with $W_0^* = \mathcal{M}_\infty^{\mathcal{T}}|\eta^*$ and $W_{n+1}^* = \mathrm{Lp}^\gamma(W_n^*)$ for $n < \omega$. As there is no degenerate iteration of $\mathcal{P}_0$, there must be some $n < \omega$ such that $\mathcal{P}_m = \mathcal{P}_n$ for all $m \geq n$. Let $\mathcal{N}$ be this eventual value of $\mathcal{P}_n$.

We must then have that $\mathcal{N}$ is iterable with respect to short trees via $\mathcal{Q}$-structures which are iterable in $J_\gamma(\mathbb{R})$. It is easy to see that in fact $\mathcal{N}$ is a γ-suitable z-premouse which is ω_1 iterable with respect to short trees via an iteration strategy in $\Sigma_1^{J_\gamma(\mathbb{R})}(\{\mathcal{N}\})$. □

Theorem 5.4.8 (Woodin) *Let $z \in HC^{V[g]}$, and let $A \subseteq \mathbb{R}^g$ be $OD^{<\beta}(z)$. Then there is a suitable, weakly A-iterable z-premouse.*

Proof. Let us fix $\mathcal{N}$, a suitable z-premouse which is ω_1-iterable with respect to short trees. Such an $\mathcal{N}$ exists by Lemma 5.4.7. We aim to show that there is no sequence

$$(\mathcal{N}_n, \mathcal{T}_n \mid n < \omega)$$

such that $\mathcal{N}_0 = \mathcal{N}$, and for each $n < \omega$, $\mathcal{N}_{n+1}$ is an iterate or a pseudo iterate of $\mathcal{N}_n$, as witnessed by the tree $\mathcal{T}_n$ (i.e., either $\mathcal{N}_{n+1} = \mathcal{M}_\infty^{\mathcal{T}_n}$ or else $\mathcal{N}_{n+1} = \mathcal{M}(\mathcal{T}_n)^+$), but there is no (cofinal) branch b through $\mathcal{T}_n$ such that $\mathcal{N}_{n+1} = \mathcal{M}_b^{\mathcal{T}_n}$ and

$$i_b(\tau_{A,\mu}^{\mathcal{N}_n}) = \tau_{A,i_b(\mu)}^{\mathcal{N}_{n+1}}$$

for all cardinals $\mu \geq \delta^{\mathcal{N}_n}$ of $\mathcal{N}_n$. If no such sequence exists, then there will be a finite sequence $(\mathcal{N}_n | n \leq N)$, for some $N < \omega$, such that $\mathcal{N}_0 = \mathcal{N}$, for each $n < N$, $\mathcal{N}_{n+1}$ is an iterate or a pseudo iterate of $\mathcal{N}_n$, and $\mathcal{N}_N$ is a suitable, weakly A-iterable z-premouse. The terms $\tau_{A,\mu}^{\mathcal{N}_n}$ are well-defined by Lemma 3.7.5.

Let us thus suppose such a sequence

$$(\mathcal{N}_n, \mathcal{T}_n \mid n < \omega)$$

to exist, and let us work towards a contradiction.

Let $\bar{\mathcal{P}}$ be a suitable $\mathcal{N}$-premouse. (In particular, $\mathcal{N} = \bar{\mathcal{P}}|(\delta^{\mathcal{N}})^{(+\omega)\bar{\mathcal{P}}}$.) There is a suitable $\mathcal{N}$-premouse $\mathcal{P}'$ such that $(\mathcal{N}_n, \mathcal{T}_n \mid n < \omega)$ is generic over $\mathcal{P}'$ for the extender algebra at the largest Woodin cardinal of $\mathcal{P}'$. $\mathcal{P}'$ is obtained either as an iterate or as a pseudo iterate of $\bar{\mathcal{P}}$ by iterating away extenders which do not satisfy the relevant axioms.

Now let us set $\mathcal{P}_1 = \mathcal{P}'$, and let, for $n \geq 1$, $\mathcal{P}_{n+1}$ be a suitable $\mathcal{P}_n$-premouse. Then $\mathcal{P}_n$ has $n+1$ Woodin cardinals "above z"; let us denote them by $\delta_0 < \delta_1 < \cdots < \delta_n$. We'll work with $\mathcal{P}_4$ in what follows. Let us write $\mathcal{P} = \mathcal{P}_4$.

Let $L[E]^{\mathcal{P}}$ denote the $L[E]$ (over $\emptyset$) constructed inside $\mathcal{P}|\delta_4$ by using extenders with critical point above δ_2. As $\mathcal{P}$ is Lp^{α}-closed, and by the universality of $L[E]$,

$$\mathcal{M} = L[E]^{\mathcal{P}}|(\delta_3)^{(+\omega)L[E]^{\mathcal{P}}}$$

is a suitable premouse, and thus by Lemma 3.7.4, $\tau^{\mathcal{M}}_{A,\delta_3}$ is well-defined. Let us write

$$\tau = \tau^{\mathcal{M}}_{A,\delta_3}.$$

We have that $\mathcal{P}'$ is generic over $L[E]^{\mathcal{P}}$ (and hence over $\mathcal{M}$) for the extender algebra at δ_3, and therefore $(\mathcal{N}_n, \mathcal{T}_n \mid n < \omega)$ is also generic over $\mathcal{M}$. Moreover, the sequence

$$(\tau^{\mathcal{N}_n}_{A,\mu} \mid n < \omega, \mu \geq \delta^{\mathcal{N}_n})$$

is definable over $\mathcal{M}[(\mathcal{N}_n | n < \omega)]$ from the parameters τ and $(\mathcal{N}_n | n < \omega)$. The model $\mathcal{P}[(\mathcal{N}_n, \mathcal{T}_n | n < \omega)]$ can therefore use $L[E]^{\mathcal{P}}$ and $\tau \in (L[E]^{\mathcal{P}})^{\mathrm{Col}(\omega,\delta_3)}$ to *certify* that $(\mathcal{N}_n, \mathcal{T}_n | n < \omega)$ is such that for each $n < \omega$ there is no (cofinal) branch b through $\mathcal{T}_n$ such that $\mathcal{N}_{n+1} = \mathcal{M}^{\mathcal{T}_n}_b$ and

$$i_b(\tau^{\mathcal{N}_n}_{A,\mu}) = \tau^{\mathcal{N}_{n+1}}_{A,i_b(\mu)}$$

for all cardinals $\mu \geq \delta^{\mathcal{N}_n}$ of $\mathcal{N}_n$.

In what follows, if $\mathcal{N}_\xi$ is a model from the $L[E]$ construction of $\mathcal{P}$ producing $L[E]^{\mathcal{P}}$ such that $\mathcal{N}_\xi$ has a Woodin cardinal, δ, if $\sigma \in (\mathcal{N}_\xi)^{\mathrm{Col}(\omega,\delta)}$ is a name weakly capturing a set of reals over $\mathcal{N}_\xi$, if $\mathcal{N}'$ is a premouse which is generic over $\mathcal{N}_\xi$ for the extender algebra at δ, and if $\mu < \delta$ is a cardinal of $\mathcal{N}'$, then we shall write $\tau^{\mathcal{N}'}_{\sigma,\mu}$ for the name derived from σ in the canonical fashion, i.e., $\tau^{\mathcal{N}'}_{\sigma,\mu}$ is the restriction of σ to terms for reals in ${\mathcal{N}'}^{\mathrm{Col}(\omega,\mu)}$. (In general, we need not have that $\tau^{\mathcal{N}'}_{\sigma,\mu} \in \mathcal{N}'$, but we do have that $\tau^{\mathcal{N}_n}_{\tau,\mu} = \tau^{\mathcal{N}_n}_{A,\mu} \in \mathcal{N}$ for each $n < \omega$ and $\mu \geq \delta^{\mathcal{N}_n}$.)

The model $\mathcal{P}[(\mathcal{N}_n, \mathcal{T}_n | n < \omega)]$ can also use $L[E]$ constructions to *certify* that each $\mathcal{T}_n$ is guided by Lp^{α}, and that each $\mathcal{N}_n$ is suitable. If $\mathcal{P}'$ is a premouse which is generic over $\mathcal{P}$ for the extender algebra at δ_1, then we let $L[E, \mathcal{P}']^{\mathcal{P}}$ denote the $L[E]$ over $\mathcal{P}'$ constructed inside $\mathcal{P}|\delta_4[\mathcal{P}']$ by using extenders with critical point above δ_2. Now let $\mathcal{M}^{\mathcal{T}_n}_{\alpha}$ be one of the models from $\mathcal{T}_n$, where α is a limit ordinal (> 0), or else $\alpha = 0$, $n > 0$, and $\mathcal{M}^{\mathcal{T}_n}_0 = \mathcal{N}_n$ is an iterate of $\mathcal{N}_{n-1}$. As $\mathcal{P}$ is Lp^{α}-closed, and by the universality of $L[E, \mathcal{M}(\mathcal{T}_n \restriction \alpha)]^{\mathcal{P}}$, $L[E, \mathcal{M}(\mathcal{T}_n \restriction \alpha)]^{\mathcal{P}}$ can construct the $\mathcal{Q}$-structure for $\mathcal{M}(\mathcal{T}_n \restriction \alpha)$, i.e., the least initial segment $\mathcal{Q}$ of $\mathcal{M}^{\mathcal{T}_n}_{\alpha}$ over which $\delta(\mathcal{T}_n \restriction \alpha)$ is not definably Woodin. Moreover, if $n < \omega$, then again because $\mathcal{P}$ is Lp^{α}-closed, and by the universality of $L[E, \mathcal{N}_n]^{\mathcal{P}}$, $\mathcal{N}_n$ is an initial segment of $L[E, \mathcal{N}_n]^{\mathcal{P}}$, in fact

$$\mathcal{N}_n = L[E, \mathcal{N}_n]^{\mathcal{P}} | (\delta^{\mathcal{N}_n})^{(+\omega) L[E, \mathcal{N}_n]^{\mathcal{P}}}$$

.

In what follows, if $(\mathcal{N}'_n, \mathcal{T}'_n \mid n < \omega)$ is a sequence which is $\mathrm{Col}(\omega, \delta')$-generic over $\mathcal{P}$ for some $\delta' \leq \delta_2$, where $\mathcal{N}'_0 = \mathcal{P}|\eta$ for some η, and for each $n < \omega$, $\mathcal{N}'_{n+1}$ is an iterate or a pseudo iterate of $\mathcal{N}'_n$ as witnessed by the tree $\mathcal{T}'_n$, then by "the relevant $L[E]$ contructions of $\mathcal{P}$" we shall mean

(a) the $L[E]$ construction (over $\emptyset$) constructed inside $\mathcal{P}|\delta_4$ by using extenders with critical point above δ_2, as well as

(b) the $L[E, \mathcal{P}']$ constructions (over $\mathcal{P}'$) constructed inside $\mathcal{P}|\delta_4[\mathcal{P}']$ by using extenders with critical point above δ_2, where $\mathcal{P}'$ is one of the models $\mathcal{N}'_n$ or $\mathcal{M}(\mathcal{T}'_n \restriction \lambda)$.

We have now verified

$$\exists \delta' < \delta_2 \ (*)(\delta', \delta_0, \delta_3, \tau)$$

to hold inside $\mathcal{P}$, where

$$\mathcal{P} \models (*)(\delta', \eta', \delta^*, \sigma')$$

if and only if the following statement holds true.

Inside $\mathcal{P}^{\mathrm{Col}(\omega, \delta')}$ there is a sequence $(\mathcal{N}'_n, \mathcal{T}'_n | n < \omega)$ such that

(a) $\mathcal{N}'_0 = \mathcal{P}|\eta'^{(+\omega)\mathcal{P}}$,

(b) for each $n < \omega$, $\mathcal{N}'_{n+1}$ is an iterate or a pseudo iterate of $\mathcal{N}'_n$, as witnessed by the tree $\mathcal{T}'_n$, where the $(\delta^*)^{\text{th}}$ models from the relevant $L[E]$ constructions of $\mathcal{P}$ certify the sequence $(\mathcal{N}'_n, \mathcal{T}'_n \mid n < \omega)$, but

(c) for each $n < \omega$, there is no (cofinal) branch b through $\mathcal{T}'_n$ such that $\mathcal{N}'_{n+1} = \mathcal{M}_b^{\mathcal{T}'_n}$ and

$$i_b(\tau^{\mathcal{N}'_n}_{\sigma',\mu}) = \tau^{\mathcal{N}'_{n+1}}_{\sigma',i_b(\mu)}$$

for all cardinals $\mu \geq \delta^{\mathcal{N}'_n}$ of $\mathcal{N}'_n$.

Let us from now on write (η, δ, σ) for the lexicographically least triple such that $\exists\delta'\ (*)(\delta', \eta, \delta, \sigma)$ holds true in $\mathcal{P}$.

Claim 1. $\exists\delta' \leq \delta_1\ \mathcal{P} \models (*)(\delta', \eta, \delta, \sigma)$.

Proof. Suppose not. Let us work inside $\mathcal{P}$ to pick some (sufficiently) elementary embedding

$$\pi_0 \colon \mathcal{R}_0 \to \mathcal{P},$$

where $\mathcal{R}_0$ is countable and transitive (in $\mathcal{P}$). We are going to construct a sequence $(\mathcal{R}_m, \rho_m, \pi_m \mid m < \omega)$.

Suppose $\mathcal{R}_m$, π_m to have been constructed, where

$$\pi_m \colon \mathcal{R}_m \to \mathcal{P}.$$

Let us write $\eta^m = \pi_m^{-1}(\eta)$, $\delta^m = \pi_m^{-1}(\delta)$, and $\sigma^m = \pi_m^{-1}(\sigma)$. We may force over $\mathcal{R}_m$ to add a sequence $(\mathcal{N}_n^m, \mathcal{T}_n^m \colon n < \omega) \in \mathcal{P}$ witnessing that $\exists\delta'\ (*)(\delta', \eta^m, \delta^m, \sigma^m)$ holds true in $\mathcal{R}_m$. By hypothesis, if δ' is least such that $(*)(\delta', \eta^m, \delta^m, \sigma^m)$ holds true in $\mathcal{R}_m$, then $\delta' > \pi_m^{-1}(\delta_1)$.

Let us iterate $\mathcal{R}_m$ above $\pi_m^{-1}(\delta_0)$ to make $(\mathcal{N}_n^m, \mathcal{T}_n^m \mid n < \omega)$ generic at the image of $\pi_m^{-1}(\delta_1)$. This produces a iteration

$$\rho_m \colon \mathcal{R}_m \to \mathcal{R}_{m+1}$$

such that $(\mathcal{N}_n^m, \mathcal{T}_n^m \mid n < \omega)$ is generic over $\mathcal{R}_{m+1}$ for the extender algebra at $\rho_m(\pi_m^{-1}(\delta_1)) < \rho_m(\delta')$. There is also some

$$\pi_{m+1} \colon \mathcal{R}_{m+1} \to \mathcal{P}$$

such that $\pi_{m+1} \circ \rho_m = \pi_m$.

Let us iterate all the $\mathcal{R}_m$ above their $\pi_m^{-1}(\delta_2)$ to simultaneously line up all the relevent $L[E]$ constructions (cf. Lemma 5.4.6). This produces maps

$$i_m \colon \mathcal{R}_m \to \mathcal{R}_m^*.$$

We claim that we cannot have $i_m(\delta^m) \leq i_{m+1}(\delta^{m+1})$. Well, if $i_m(\delta^m) \leq i_{m+1}(\delta^{m+1})$, then the $i_m(\delta^m)^{\text{th}}$ models from the relevant $L[E]$ constructions of $\mathcal{R}^*_{m+1}$ would certify $(\mathcal{N}^m_n, \mathcal{T}^m_n \mid 1 \leq n < \omega)$, so that in fact the ordinal $i_{m+1} \circ \pi^{-1}_{m+1}(\delta')$ (which is the least witness for

$$\exists \delta^*(*)(\delta^*, \eta^{m+1}, i_{m+1}(\delta^{m+1}), i_{m+1}(\sigma^{m+1}))$$

to hold in $\mathcal{R}^*_{m+1}$) is less than or equal to $i_{m+1} \circ \pi^{-1}_{m+1}(\delta_1)$. But $\pi^{-1}_{m+1}(\delta') > \pi^{-1}_{m+1}(\delta_1)$.

We must therefore have that $i_m(\delta^m) > i_{m+1}(\delta^{m+1})$ for every $m < \omega$. But this yields a descending chain of ordinals. Contradiction! □

Now let us work inside $\mathcal{P}$ to (again) pick some (sufficiently) elementary embedding

$$\pi_0 \colon \mathcal{R}_0 \to \mathcal{P},$$

where $\mathcal{R}_0$ is countable and transitive (in $\mathcal{P}$). Let us write $\eta_0 = \pi_0^{-1}(\eta)$, $\delta_0 = \pi_0^{-1}(\delta)$, and $\sigma_0 = \pi_0^{-1}(\sigma)$. By Claim 1, we may force over $\mathcal{R}_0$ with $\mathrm{Col}(\omega, \pi_0^{-1}(\delta_1))$ to add a sequence $(\mathcal{N}^0_n, \mathcal{T}^0_n \colon n < \omega) \in \mathcal{P}$ witnessing that $\exists \delta'\ (*)(\delta', \eta_0, \delta_0, \sigma_0)$ holds true in $\mathcal{R}_0$.

It is easy to verify that $\mathcal{T}^0_0$, which lives on $\mathcal{N}^0_0 = \mathcal{R}_0|\eta_0$, picks those (unique) branches at limit stages which can be realized back into $\mathcal{P}$, i.e., if $\lambda < \mathrm{lh}(\mathcal{T}^0_0)$ is a limit ordinal, then there is some $\pi \colon \mathcal{M}^{\mathcal{T}^0_0}_\lambda \to \mathcal{P}$ with $\pi \circ \pi^{\mathcal{T}^0_0}_{0\lambda} = \pi_0$. Therefore, there is also a cofinal branch, call it b_0, through $\mathcal{T}^0_0$ which can be realized back into $\mathcal{P}$, i.e., there is some $\pi \colon \mathcal{M}^{\mathcal{T}^0_0}_b \to \mathcal{P}$ with $\pi \circ \pi^{\mathcal{T}^0_0}_{0\lambda} = \pi_0$; let us write π_1 for this map π.

Let us now continue in this fashion. Suppose that we are given

$$\pi_m \colon \mathcal{R}_m \to \mathcal{P},$$

and let us write $\eta_m = \pi_m^{-1}(\eta)$, $\delta_m = \pi_m^{-1}(\delta)$, and $\sigma_m = \pi_m^{-1}(\sigma)$. By Claim 1, we may force over $\mathcal{R}_m$ with $\mathrm{Col}(\omega, \pi_m^{-1}(\delta_1))$ to add a sequence $(\mathcal{N}^m_n, \mathcal{T}^m_n \colon n < \omega) \in \mathcal{P}$ witnessing that $\exists \delta'\ (*)(\delta', \eta_m, \delta_m, \sigma_m)$ holds true in $\mathcal{R}_m$.

It is again easy to verify that $\mathcal{T}^m_0$, which lives on $\mathcal{N}^m_0 = \mathcal{R}_m|\eta_m$, picks those (unique) branches at limit stages which can be realized back into $\mathcal{P}$, i.e., if $\lambda < \mathrm{lh}(\mathcal{T}^m_0)$ is a limit ordinal, then there is some $\pi \colon \mathcal{M}^{\mathcal{T}^m_0}_\lambda \to \mathcal{P}$ with $\pi \circ \pi^{\mathcal{T}^m_0}_{0\lambda} = \pi_m$. Therefore, there is also a cofinal branch, call it b_m, through $\mathcal{T}^m_0$ which can be realized back into $\mathcal{P}$, i.e., there is some $\pi \colon \mathcal{M}^{\mathcal{T}^m_0}_b \to \mathcal{P}$ with $\pi \circ \pi^{\mathcal{T}^m_0}_{0\lambda} = \pi_m$; let us write π_{m+1} for this map π.

Let us now simultaneously iterate all the $\mathcal{R}_m$ above η_m to make the "double sequence"

$$(\mathcal{N}^p_n, \mathcal{T}^p_n \colon p, n < \omega)$$

generic for the extender algebra at the image of $\pi_m^{-1}(\delta_2)$. I.e., for each $m < \omega$ we produce embeddings

$$\pi_{\mathcal{R}_m,\mathcal{R}'_m} \colon \mathcal{R}_m \to \mathcal{R}'_m$$

such that the entire sequence $(\mathcal{N}_n^p, \mathcal{T}_n^p \colon p, n < \omega)$ is generic over $\mathcal{R}'_m$ for the extender algebra at $\pi_{\mathcal{R}_m,\mathcal{R}'_m}(\pi_m^{-1}(\delta_2))$.

Let us next simultaneously iterate all the $\mathcal{R}'_m$ to line up the relevant $L[E]$ constructions of $\mathcal{R}_m[(\mathcal{N}_n^p, \mathcal{T}_n^p \colon p, n < \omega)]$, namely the sequence of models from the constructions of $L[E, \mathcal{M}(\mathcal{T}_n^p)]^{\mathcal{R}'_m[(\mathcal{N}_n^p,\mathcal{T}_n^p \colon p,n<\omega)]}$ and also the sequence of models from the constructions of $L[E]^{\mathcal{R}'_m[(\mathcal{N}_n^p,\mathcal{T}_n^p \colon p,n<\omega)]}$. This produces embeddings

$$\pi_{\mathcal{R}'_m,\mathcal{R}^*_m} \colon \mathcal{R}'_m \to \mathcal{R}^*_m$$

such that the sequence of models from the constructions of

$$L[E, \mathcal{M}(\mathcal{T}_n^p)]^{\mathcal{R}^*_m[(\mathcal{N}_n^p,\mathcal{T}_n^p \colon p,n<\omega)]}$$

and also the sequence of models from the constructions of

$$L[E]^{\mathcal{R}^*_m[(\mathcal{N}_n^p,\mathcal{T}_n^p \colon p,n<\omega)]}$$

are lined up. (Cf. Lemma 5.4.6.) Let us write

$$\pi^*_m = \pi_{\mathcal{R}'_m,\mathcal{R}^*_m} \circ \pi_{\mathcal{R}_m,\mathcal{R}'_m},$$

and $\delta^*_m = \pi^*_m(\delta_m)$ and $\sigma^*_m = \pi^*_m(\sigma_m)$. So for each $m < \omega$, $(\mathcal{N}_n^m, \mathcal{T}_n^m | n < \omega)$ witnesses that $\exists \delta'(*)(\delta', \eta_m, \delta^*_m, \sigma^*_m)$ holds in $\mathcal{R}^*_m$, $(\mathcal{N}_n^m, \mathcal{T}_n^m | n < \omega)$ is generic over $\mathcal{R}^*_{m+1}$ (as well as over $\mathcal{R}^*_m$, of course), and the relevant $L[E]$ constructions are lined up.

It is not hard to see that that

$$\mathcal{R}_{m+1} | \delta^{\mathcal{N}_1^m} = \mathcal{N}_1^m | \delta^{\mathcal{N}_1^m}.$$

This is because if $\mathcal{T}_0^m$ is short, then b_m is in fact the branch chosen by $\mathcal{T}_0^m$ and $\mathcal{R}_{m+1} = \mathcal{M}_{b_m}^{\mathcal{T}_m^0} = \mathcal{M}_\infty^{\mathcal{T}_m^0} \trianglerighteq \mathcal{N}_1^m$, and if $\mathcal{T}_0^m$ is maximal, then $\delta(\mathcal{T}_0^m) = \delta^{\mathcal{N}_1^m}$ and $\mathcal{R}_{m+1} | \delta^{\mathcal{N}_1^m} = \mathcal{N}_1^m | \delta^{\mathcal{N}_1^m}$.

Claim 1. If $\eta_{m+1} > \delta^{\mathcal{N}_1^m}$, then $\delta^*_m > \delta^*_{m+1}$.

Proof. Suppose that $\eta_{m+1} > \delta^{\mathcal{N}_1^m}$, but $\delta^*_m \leq \delta^*_{m+1}$. Then the $(\delta^*_m)^{\text{th}}$ models from the $L[E, \mathcal{M}(\mathcal{T}_n^m)]$ constructions of $\mathcal{R}^*_{m+1}[(\mathcal{N}_n^m, \mathcal{T}_n^m \colon n < \omega)]$

(which are the same as the $(\delta^*_m)^{\text{th}}$ models from the $L[E,\mathcal{M}(\mathcal{T}^m_n)]$ constructions of $\mathcal{R}^*_m[(\mathcal{N}^m_n,\mathcal{T}^m_n\colon n<\omega)]$) certify $(\mathcal{N}^m_n,\mathcal{T}^m_n\colon 0<n<\omega)$, so that $\eta_{m+1}\leq\delta^{\mathcal{N}^m_1}$. Contradiction! □

Claim 2. If $\eta_{m+1}=\delta^{\mathcal{N}^m_1}$, then $\delta^*_m\geq\delta^*_{m+1}$.

Proof. Suppose that $\eta_{m+1}=\delta^{\mathcal{N}^m_1}$, but $\delta^*_m<\delta^*_{m+1}$. As in the previous proof, the $(\delta^*_m)^{\text{th}}$ models from the $L[E,\mathcal{M}(\mathcal{T}^m_n)]$ constructions of

$$\mathcal{R}^*_{m+1}[(\mathcal{N}^m_n,\mathcal{T}^m_n\colon n<\omega)]$$

certify $(\mathcal{N}^m_n,\mathcal{T}^m_n\colon 0<n<\omega)$. But this contradicts the minimality of δ^*_{m+1}. □

By Claims 1 and 2, we may pick some $m_0<\omega$ such that for all $m\geq m_0$, $\delta^*_m=\delta^*_{m_0}$. By Claim 1, $\eta^*_{m+1}=\delta^{\mathcal{N}^m_1}$ for all $m\geq m_0$.

Claim 3. $\sigma^*_m\geq\sigma^*_{m+1}$ for all $m\geq m_0$.

Proof. Let $m\geq m_0$. By $\delta^*_m=\delta^*_{m_0}$ and $\eta^*_{m+1}=\delta^{\mathcal{N}^m_1}$ we in fact get that $(\mathcal{N}^m_n,\mathcal{T}^m_n|1\leq n<\omega)$ witnesses $\exists\delta'(*)(\delta',\eta_{m+1},\delta^*_{m+1},\tau^*_m)$ (sic!) to hold in $\mathcal{R}^*_{m+1}$. But $(\mathcal{N}^{m+1}_n,\mathcal{T}^{m+1}_n|n<\omega)$ witnesses $\exists\delta'(*)(\delta',\eta_{m+1},\delta^*_{m+1},\sigma^*_{m+1})$ to hold in $\mathcal{R}^*_{m+1}$. Therefore $\sigma^*_{m+1}\leq\sigma^*_m$ by the minimality of σ^*_{m+1}. □

By Claim 3, we may hence pick some $k_0\geq m_0$ such that $\sigma^*_m=\sigma^*_{k_0}$ for all $m\geq k_0$. The next claim shows that the branches b_m for $m\geq k_0$ move the relevant terms correctly.

Claim 4. For all $m\geq k_0$ and $\mu\geq\delta^{\mathcal{N}^m_0}$, $\pi^{\mathcal{T}^m_0}_{b_m}(\tau^{\mathcal{N}^m_0}_{\sigma_m,\mu})=\tau^{\mathcal{N}^{m+1}_0}_{\sigma_m,\pi^{\mathcal{T}^m_0}_{b_m}(\mu)}$.

Proof. Writing $\mu'=\pi^{\mathcal{T}^m_0}_{b_m}(\mu)$, we have that

$$\begin{aligned}\pi^{\mathcal{T}^m_0}_{b_m}(\tau^{\mathcal{N}^m_0}_{\sigma_m,\mu}) &= \tau^{\mathcal{N}^{m+1}_0}_{\pi^{\mathcal{T}^m_0}_{b_m}(\sigma_m),\mu'}\\ &= \tau^{\mathcal{N}^{m+1}_0}_{\sigma_{m+1},\mu'}\\ &= \tau^{\mathcal{N}^{m+1}_0}_{\sigma^*_{m+1},\mu'}\\ &= \tau^{\mathcal{N}^{m+1}_0}_{\sigma^*_m,\mu'}\\ &= \tau^{\mathcal{N}^{m+1}_0}_{\sigma_m,\mu'}\,.\end{aligned}$$

□

Now let $m \geq k_0$. There is a cofinal branch through $\mathcal{T}_0^m$, namely b_m, which moves the relevant terms correctly in the sense of Claim 4. By absoluteness, there is then some such branch in $(\mathcal{R}_m)^{\mathrm{Col}(\omega,\pi_m^{-1}(\delta_1))}$. But this is a contradiction with what $\mathcal{T}_0^m$ is supposed to be a witness for. □

The reader can find a proof of 5.4.8, in the weak gap case, outlined in [44, Lemma 1.12.1].

Remark 5.4.9 It is quite easy to derive Martin's reflection theorem 3.3.4 from 5.4.8. Of course, Martin's result is a one-liner in the weak gap case, while 5.4.8 still has content there.

Remark 5.4.10 Going further in the same direction, it is easy to combine 5.4.8 with the main result of Neeman's [23] (see also [24]), and obtain thereby a proof that $J_\beta(\mathbb{R}^g) \models \mathsf{AD}$. In the weak gap case, this follows immediately from $J_\alpha(\mathbb{R}^g) \models \mathsf{AD}$, which we get from W_α^*, together with $J_\alpha(\mathbb{R}^g) \prec_1 J_\beta(\mathbb{R}^g)$, which is the weak gap case hypothesis. However, in the strong gap case, we have here a highly nontrivial "determinacy transfer" theorem known as the *Kechris-Woodin transfer theorem.* The original proof of Kechris and Woodin (cf. [12]) was purely descriptive set theoretic; no mice got involved.

Theorem 5.4.8, together with our self-justifying system, yields a fullness-preserving strategy that condenses well, as we now show.

Definition 5.4.11 *Let $\mathcal{N}$ be a suitable z-premouse, and $\mathcal{A}$ a collection of $OD^{<\beta}(z)$ sets of reals; then we say $\mathcal{N}$ is* weakly $\mathcal{A}$-iterable *iff for all finite $F \subseteq \mathcal{A}$, $\mathcal{N}$ is weakly $\oplus F$-iterable, where $\oplus F$ is the join of the sets of reals in F.*

Corollary 5.4.12 (Woodin) *Let $\mathcal{A}$ be a countable collection of $OD^{<\beta}(z)$ sets of reals, where $z \in HC^{V[g]}$ and codes x^*; then there is a suitable, weakly $\mathcal{A}$-iterable z-premouse.*

Proof. For each $F \subseteq \mathcal{A}$ finite, we have by theorem 5.4.8 a suitable, weakly $\oplus F$-iterable $\mathcal{N}_F$. Let Σ_F be a fullness-preserving strategy for II in $\mathcal{G}(\omega, 1, \omega_1)$ for $\mathcal{N}_F$. We now simultaneously coiterate all the $\mathcal{N}_F$, using Σ_F to iterate $\mathcal{N}_F$.

Claim. The coiteration ends successfully at some countable ordinal.

Proof. Let M be the Lp^α-closure of $\langle \mathcal{N}_F | F \in [\mathcal{A}]^{<\omega}\rangle$.

Subclaim. $\omega_1^M < \omega_1$.

Proof. Otherwise, define $f\colon \omega_1 \to \alpha$ by letting $f(\gamma)$ be the least ξ such that there is in $J_\xi(\mathbb{R}^g)$ an ω_1-iteration strategy for a mouse $\mathcal{P}$ over $\langle \mathcal{N}_F | F \in [\mathcal{A}]^{<\omega}\rangle$ such that $\mathcal{P}$ projects to $\langle \mathcal{N}_F | F \in [\mathcal{A}]^{<\omega}\rangle$ and $o(\mathcal{P}) \geq \gamma$. Since α is admissible, there is a $\xi < \alpha$ such that $\mathrm{ran}(f) \subset \xi$. But then there is an uncountable sequence of distinct reals definable over $J_\xi(\mathbb{R}^g)$, which cannot happen since $J_\alpha(\mathbb{R}^g) \models \mathsf{AD}$. This shows the Subclaim.

M can track the coiteration generated by the Σ_F until some maximal tree $\mathcal{U}_F$ on $\mathcal{N}_F$ is produced. (Note that coiterations always generate normal trees.) But as soon as that happens, the coiteration is over. For let $\mathcal{P}_G$ be the next model selected by Σ_G to continue $\mathcal{U}_G$, for all $G \in [\mathcal{A}]^{<\omega}$. As $\mathcal{U}_F$ is maximal, $\mathcal{P}_F = \mathcal{M}(\mathcal{U}_F)^+$ is suitable. If $\mathcal{N}_G$-to-$\mathcal{P}_G$ drops, then because Σ_G is fullness-preserving, $\mathcal{M}(\mathcal{U}_F)$ has a $\mathcal{Q}$-structure in $\mathrm{Lp}^\alpha(\mathcal{M}(\mathcal{U}_F))$[5], a contradiction. But then $\mathcal{P}_G$ is suitable, and the minimality condition in suitability easily implies $\mathcal{P}_G = \mathcal{P}_F$, for all G.

The usual regressive function argument shows the coiteration cannot be tracked in M for $\omega_1^M + 1$ steps. Thus it must terminate successfully at some stage $\leq \omega_1^M$. This proves the claim. □

The proof of the claim also shows that if $\mathcal{P}_F$ is the last model on the tree $\mathcal{U}_F$ produced in the successful coiteration by Σ_F, then no branch $\mathcal{N}_F$-to-$\mathcal{P}_F$ drops, and $\mathcal{P}_F = \mathcal{P}_G$ for all F, G. (*Some* branch doesn't drop by general coiteration theory, and then the proof of the claim gives the rest.) It is clear that the common last model $\mathcal{P}$ is suitable, and weakly $\mathcal{A}$-iterable. □

Definition 5.4.13 *Let $\mathcal{N}$ be a suitable z-premouse, where $z \in HC^{V[g]}$, let $\mathcal{A}$ be a collection of $OD^{<\beta}(z)$ subsets of $\mathbb{R}^g$, and let Σ be an ω_1-iteration strategy for $\mathcal{N}$. We say Σ is* guided by $\mathcal{A}$ *just in case Σ is fullness preserving, and whenever $\mathcal{T}$ is a countable (necessarily normal) iteration tree by Σ of limit length, and $b = \Sigma(\mathcal{T})$, then*

(a) if $\mathcal{T}$ is short, then $\mathcal{Q}(b, \mathcal{T})$ exists and $\mathcal{Q}(b, \mathcal{T}) \in \mathrm{Lp}^\alpha(\mathcal{M}(\mathcal{T}))$, and

(b) if $\mathcal{T}$ is maximal, then

$$i_b(\tau^{\mathcal{N}}_{A,\mu}) = \tau^{\mathcal{M}^{\mathcal{T}}_b}_{A,i(\mu)}$$

[5]Some initial segment of $\mathcal{P}_G$ is a $\mathcal{Q}$-structure for $\mathcal{M}(\mathcal{U}_F)$ because of the drop. This $\mathcal{Q}$-structure cannot lie beyond $\mathrm{Lp}^\alpha(\mathcal{M}(\mathcal{U}_F))$, as otherwise $\mathcal{P}_G$ would have a suitable initial segment.

for all $A \in \mathcal{A}$ and cardinals $\mu \geq \delta^{\mathcal{N}}$ of $\mathcal{N}$.

Notice that in case (b) above, b does not drop and $\mathcal{M}_b = \mathcal{M}(\mathcal{T})^+$, as Σ is fullness-preserving.

Theorem 5.4.14 (Woodin) *Let $z \in HC^{V[g]}$, and let $\mathcal{A}$ be a countable, self-justifying system of $OD^{<\beta}(z)$ sets which contains the universal $\Sigma_1^{J_\alpha(\mathbb{R}^g)}$ set. Then there is a suitable z-premouse $\mathcal{N}$, and a unique fullness-preserving ω_1-iteration strategy for $\mathcal{N}$ which is guided by $\mathcal{A}$; moreover, this strategy condenses well.*

Proof. By 5.4.12, we have a suitable z-premouse $\mathcal{N}$ which is weakly $\mathcal{A}$-iterable. Let $\mathcal{A} = \{A_k \mid k < \omega\}$, and for each $k < \omega$, let Γ_k be a fullness-preserving ω_1-iteration strategy witnessing that $\mathcal{N}$ is weakly $A_0 \oplus ... \oplus A_k$-iterable. The desired strategy Σ will be a sort of limit of the Γ_k.

So long as all Γ_k agree, Σ simply plays according to their common prescription. So suppose $\mathcal{T}$ is a normal tree of limit length which has been played according to all Γ_k, but there are k and l such that $\Gamma_k(\mathcal{T}) \neq \Gamma_l(\mathcal{T})$. Since the Γ_k are fullness-preserving and guided by Lp^α $\mathcal{Q}$-structures when these exist, $\mathcal{T}$ must be maximal, and letting

$$b_k = \Gamma_k(\mathcal{T})$$

for all k, and

$$i_k : \mathcal{N} \to \mathcal{M}(\mathcal{T})^+ = \mathcal{M}^{\mathcal{T}}_{b_k}$$

be the canonical embedding, we have that i_k moves the term relations for all A_i with $i \leq k$ correctly.

For $k < \omega$, let μ_k be the k^{th} cardinal of $\mathcal{M}(\mathcal{T})^+$ which is $\geq \delta(\mathcal{T})$, and set

$$\mathcal{M}_k = \mathcal{M}(\mathcal{T})^+ | \mu_k,$$
$$\tau_{j,k} = \tau^{\mathcal{M}(\mathcal{T})^+}_{A_j, \mu_k},$$

and

$\gamma_k = \sup\{\xi \mid \xi$ is definable over $\mathcal{M}_k$ from points of the form $\tau_{i,j}$, where $i, j \leq k.\}$.

Let $\mathcal{M} = \mathcal{M}(\mathcal{T})^+$.

Claim 1. The γ_k are cofinal in $\delta(\mathcal{T})$.

Proof. This follows at once from part (b) of Theorem 5.4.3. □

The usual uniqueness proof for good branches in iteration trees [6] yields

[6] The "zipper argument".

Claim 2. Let $k \leq l$, and let E be an extender of length $\leq \gamma_k$; then E is used in b_k if and only if E is used in b_l.

Proof. This is a simple consequence of the fact that $\mathrm{ran}(i_k) \cap \mathrm{ran}(i_l)$ is cofinal in γ_k. □

Define now $\xi \in b \Leftrightarrow \exists k \forall l \geq k(\xi \in b_l)$. By claim 2, we have

$$E \text{ is used in } b \text{ iff } E \text{ is used in } b_k, \text{ for some (all) } k \text{ such that } \mathrm{lh}(E) \leq \gamma_k.$$

Claim 3. b is cofinal in $\mathrm{lh}(\mathcal{T})$.

Proof. If not, then let $\eta = \bigcup b < \mathrm{lh}(\mathcal{T})$. Fix k such that

$$\mathrm{lh}(E_\eta^{\mathcal{T}}) < \gamma_k.$$

All extenders used in b have length $< \mathrm{lh}(E_\eta^{\mathcal{T}}$, so by claim 2,

$$b \subseteq b_l, \text{ for all } l \geq k.$$

This implies that $\eta \in b$. (If not, then b is cofinal in η, but then all b_l for $l \geq k$ are cofinal in η, so $\eta \in b_l$ for all $l \geq k$ since branches are closed.) Now let F be the extender applied to $\mathcal{M}_\eta^{\mathcal{T}}$ along the branch b_k. We have $\mathrm{crit}(F) < lh(E_\eta)$, and since F is not used in b, we must have $\mathrm{lh}(F) > \gamma_k$. But $\mathrm{ran}(i_k) \cap [\mathrm{crit}(F), \mathrm{lh}(F)] = \emptyset$, and $\mathrm{ran}(i_k)$ is cofinal in γ_k, a contradiction. □

Now set

$$T_k^{\mathcal{M}} = \mathrm{Th}^{\mathcal{M}_{k+1}}(\delta^{\mathcal{M}} \cup \{\tau_{i,j} \mid i, j < k\}),$$

and let $T_k^{\mathcal{N}}$ be defined from $\mathcal{N}$ and its capturing terms in parallel fashion. Thus we have

$$i_k(T_k^{\mathcal{N}}) = T_k^{\mathcal{M}}$$

because i_k moves the relevant term relations correctly.

Claim 4. For all k, $i_b(T_k^{\mathcal{N}}) = T_k^{\mathcal{M}}$.

Proof. Fix k. We regard $T_k^{\mathcal{N}}$ as a subset of $\delta^{\mathcal{N}}$. Since b is cofinal, it is enough to see that $i_b(T_k^{\mathcal{N}}) \cap \mathrm{lh}(E) = T_k^{\mathcal{M}} \cap \mathrm{lh}(E)$ whenever E is used in b. But fixing such an E, we can find $l \geq k$ such that E is used in b_l. It follows that $i_b(X) \cap \mathrm{lh}(E) = i_l(X) \cap \mathrm{lh}(E)$ for all $X \in \mathcal{N}$, and applying this to $X = T_k^{\mathcal{N}}$, we have the desired conclusion. □

By part (c) of 5.4.3, $\mathcal{N}$ is pointwise Σ_0-definable from ordinals $< \delta^{\mathcal{N}}$ and the $\tau_{i,j}^{\mathcal{N}}$. The parallel fact holds for $\mathcal{M}$. Thus $\mathcal{N}$ is coded by the join of the

$T_k^{\mathcal{N}}$, so that $\mathcal{M}_b^{\mathcal{T}}$ is coded by the join of the $i_b(T_k^{\mathcal{N}})$. It follows from claim 4 that $\mathcal{M}_b^{\mathcal{T}} = \mathcal{M}$ and i_b moves all the term relations correctly. Thus b satisfies all the requirements for the choice of a fullness-preserving, $\mathcal{A}$-guided iteration strategy, and we can set $\Sigma(\mathcal{T}) = b$. Since $\mathcal{T}$ was maximal, the iteration game we were playing is now over, and Σ has won.

Using the term-condensation lemma 5.4.3, it is straightforward to show that the strategy Σ we have just defined condenses well. This finishes the proof of 5.4.14. □

5.5 Back to V

We are finally ready to complete the proof of Theorem 5.4.1. Roughly speaking, 5.4.14 gives us what we want, except that it exists in $V[g]$, and depends on g. Slightly re-arranging things so as to ease notation, we may assume

$$g = h \times l,$$

where $h \times l$ is $\mathrm{Col}(\omega, \mu) \times \mathrm{Col}(\omega, < \nu)$ -generic over V, for $\mu < \nu$, and

$$x^* = \tau^h$$

is the real in $V[g]$ from which we can ordinal-define our sjs $\vec{A}$. So there is in $V[g]$ a suitable $\mathcal{N}$ over x^*, with an $\vec{A}$-guided iteration strategy. By considering all possible finite variants of h, and comparing the mice associated to each of them, we shall produce a mouse which does not depend on g. We shall then show that this mouse has the form $\mathcal{N}[h]$, where $\mathcal{N}$ is a mouse over τ in V.[7]

Let $p_0 \in h$ be a condition such that $(p_0, \emptyset)$ forces everything about τ and $V[g]$ which we have used so far. For each $p \leq p_0$ in $\mathrm{Col}(\omega, \mu)$, let h_p be given by

$$h_p = p \cup h \restriction (\omega \setminus \mathrm{dom}(p)).$$

Here we are identifying h with $\bigcup h \colon \omega \to \mu$. So h_p is V-generic, and $V[h_p] = V[h]$, for all $p \leq p_0$.

We work in $V[g]$ for a while. For $p \leq p_0$, let $\mathcal{A}_p$ be the self-justifying system of sets which are $\mathrm{OD}^{<\beta}(\tau^{h_p})$ associated to τ^{h_p}. Let

$$z_p = \langle \tau, h_p \rangle,$$

[7] The *Boolean-valued comparison* method is due to Woodin.

so that the sets in $\mathcal{A}_p$ are all $\mathrm{OD}^{<\beta}(z_p)$. Let

$$\mathcal{A} = \bigcup_{p \le p_0} \mathcal{A}_p,$$

and notice that since z_p easily computes z_q, all sets in $\mathcal{A}$ are $\mathrm{OD}^{<\beta}(z_p)$, for all p. Let $\dot{\mathcal{A}}$ be a natural $\mathrm{Col}(\omega,\mu) \times \mathrm{Col}(\omega, <\nu)$-term for $\mathcal{A}$, so that

$$\forall p \le p_0 \quad \forall q \dot{\mathcal{A}}^{h_p \times l_q} = \mathcal{A},$$

and $(p_0, \emptyset)$ forces that $\dot{A}$ is a self-justifying system containing the universal $\Sigma_1^{J_\alpha(\mathbb{R})}$ set. From now on, let's assume $p_0 = \emptyset$ to save ink. For each p, we have by 5.4.14 terms $\dot{N}_p, \dot{\Sigma}_p$ such that

$$(p, \emptyset) \quad \Vdash \quad \dot{\Sigma}_p \text{ is an } \dot{\mathcal{A}}\text{-guided, fullness-preserving strategy for the } \langle \tau, \dot{h} \rangle \text{ mouse } \dot{\mathcal{N}}_p.$$

(Here we use $\dot{h}$ to name the first coordinate of the generic pair.) Since $\dot{N}_p$ is a term for a countable transitive set, we may assume it has support bounded in ν. By increasing μ if necessary, we may assume $\dot{N}_p$ is a $\mathrm{Col}(\omega,\mu)$-term. Since $\dot{A}^{h\times l} = \dot{A}^{h_p \times l_q}$ for all p, q, we get that $\dot{\Sigma}_p^{h_p\times l} = \dot{\Sigma}^{h_p \times l_q}$ for all p, q, and therefore we have for each p a $\mathrm{Col}(\omega,\mu)$-term $\dot{\Gamma}_p$ such that

$$\dot{\Gamma}_p^{h_p} = \dot{\Sigma}_p^{h_p \times l} \restriction V[h_p],$$

for all $p \in \mathrm{Col}(\omega,\mu)$.

We work in $V[h]$ for a while. Let $\mathcal{N}_p = \dot{\mathcal{N}}^{h_p}$ and $\Gamma_p = \dot{\Gamma}^{h_p}$. Now $\mathcal{N}_p$ is a z_p-mouse, but it can also be regarded as a z_q mouse for any q, since z_p and z_q compute each other easily. It therefore makes sense to simultaneously compare all the $\mathcal{N}_p$ in $V[h]$, using the Γ_p to iterate them. We can show the comparison terminates using the argument of the claim in 5.4.12. Let

$$\mathcal{N}_\infty = \text{ common iterate of all } \mathcal{N}_p.$$

Because the Γ_p are fullness-preserving, $\mathcal{N}_p$-to-$\mathcal{N}_\infty$ does not drop for all p, and $\mathcal{N}_\infty$ can be regarded as a suitable z_p-mouse, for each p. These are different presentations so perhaps we should write $\mathcal{N}_\infty^p$, but there is a fixed extender sequence

$$\vec{E}_\infty = \dot{E}^{\mathcal{N}_\infty^p}, \text{ for all } p.$$

Moreover, $\mathcal{N}_\infty$ is weakly $\mathcal{A}$-iterable, and thus by 5.4.14 has a unique $\mathcal{A}$-guided strategy Γ which is fullness-preserving and condenses well.

Since the comparison which produced $\mathcal{N}_\infty$ depends only on the *set* of all $\dot{\mathcal{N}}^{h_p}$, and not any enumeration of this set, we have symmetric terms for $\vec{E}_\infty$ and Γ; that is $\dot{\vec{E}}_\infty$ and $\dot{\Gamma}$ such that

$$\dot{\vec{E}}_\infty{}^{h_p} = \vec{E}_\infty \wedge \dot{\Gamma}^{h_p} = \Sigma$$

for all p. It follows from the homogeneity of $\mathrm{Col}(\omega, \mu)$ that any subset of V which is definable in $V[g]$ from $\{h_p \mid p \leq p_0\}$, $\vec{E}_\infty$, Γ, and elements of V is itself in V.

In V, we can now inductively build a τ-mouse $\mathcal{N}$. We maintain

$$\mathcal{N}|\eta[h] = \mathcal{N}_\infty^\emptyset[h]|\eta,$$

by induction on η. The first few levels of $\mathcal{N}$ are just initial segments of $L(\tau)$. Given $\mathcal{N}|\eta$, we get $\mathcal{N}|\eta + 1$ by letting the next extender be

$$\dot{E}_\eta^{\mathcal{N}} = (\vec{E}_\infty)_\eta \cap \mathcal{N}|\eta.$$

Note that $\dot{E}_\eta^{\mathcal{N}}$ is in V, and can be defined from η over V uniformly in η. One can show that $\mathcal{N}$ is a τ-mouse, and $\mathcal{N}[h] = \mathcal{N}_\infty$. The proof is given in [40]. (Cf. also [32].) It relies on the fact that fine-structure is preserved, level-by-level, by small forcing. That also implies that any iteration tree $\mathcal{T}$ on $\mathcal{N}$ can be regarded as a tree $\mathcal{T}^*$ on $\mathcal{N}[h] = \mathcal{N}_\infty$, with the same drop and degree structure, and $\mathcal{M}_\xi^{\mathcal{T}^*} = \mathcal{M}_\xi^{\mathcal{T}}[h]$ for all ξ. Thus Γ induces a μ^+-iteration strategy, which we also call Γ, on $\mathcal{N}$. Moreover, $\Gamma \in V$. We leave it to the reader to check that Γ condenses well in V. This proves 5.4.1. □

5.6 Hybrid strategy-mice and operators

Let $\mathcal{N}$ and Σ be as in Theorem 5.4.1. We need to use *hybrid* mice obtained by constructing from some A coding $\mathcal{N}$, $A \in H_\nu$, adding extenders to a coherent sequence we are building, and at the same time closing the model we are building under Σ. This is parallel to the method of building K^c's in the inadmissible case by explicitly closing the model under some J-operator. (Whereas one may often argue that this explicit closure under J is unnecessary, in the present situation we have no way to argue that a pure extender model over $\mathcal{N}$ must be closed under Σ.)

Iterability for these hybrid mice includes the provision that Σ is moved correctly. (All critical points on the coherent sequence must be above the height of A, and hence iterations of a hybrid fix $\mathcal{N}$, and move Σ to a strategy for non-dropping trees on $\mathcal{N}$. Our requirement is that this strategy is Σ itself.) If this is done in a natural way, the resulting model has a fine structure. The key to the fine structure is that Σ condenses well (cf. Definition 5.3.7). Condensation for Σ is also used in the realizability proof that size μ elementary submodels of levels of $K^c_\Sigma(A)$ are countably iterable in $V[g]$.[8] We shall call such mice "Σ-hybrid mice."

We shall now first introduce hybrid premice and hybrid mice and then show how we may construe them as F-mice in the sense of Definition 1.3.8. This will enable us to have a version of Theorem 1.3.20, the K^F-Existence Dichotomy, for our hybrid mice available.

Definition 5.6.1 *A* hybrid (strategy) potential premouse *is a J-structure*

$$\mathcal{M} = J_\alpha[E, S, A]$$

over a set of ordinals A such that the following hold true.

(1) A codes a premouse in the sense of [22, Definition 3.5.1] (cf. also [33, Definition 2.4]), call it $\mathcal{N}'$.

(2) E codes a sequence $(E_\beta \mid \beta \in X_0)$ of extenders, where $X_0 \subset \alpha + 1$, $\min(X_0) > \sup(A)$. This sequence needs to satisfy [22, Definition 1.0.4] (cf. also [33, Definition 2.4]) with the understanding that if $E_\beta \neq \emptyset$, then the relevant initial segment of $\mathcal{M}$ to consider is $J_\beta[E, S, A]$ (rather than just $J_\beta[E]$).

(3) S codes fragments of an iteration (pre-)strategy for $\mathcal{N}'$ in the following way. There is some X_1 such that $S = ((\mathcal{T}_\beta, \beta, k_\beta) \colon \beta \in X_1)$, where

(a) $X_1 \cap X_0 = \emptyset$ and $X_1 \subset \alpha$,

(b) if $\beta \in X_1$, then $\mathcal{T}_\beta \in \mathcal{M}$ is an iteration tree on $\mathcal{N}'$ and $k_\beta \in \{0, 1\}$,

(c) if $\beta \in X_1$, then, setting $B_\beta = \{\gamma \in X_1 \mid \mathcal{T}_\gamma = \mathcal{T}_\beta\}$, we have that B_β is an half-open interval of ordinals, i.e., $B_\beta = [\min(B_\beta), \sup(B_\beta))$, and

$$\{\xi \mid k_{\min(B_\beta)+\xi} = 1\}$$

is a branch through $\mathcal{T}_\beta$.

[8] It would be possible to talk only about countable iterability in V. Given $\pi \colon \mathcal{M} \to \mathcal{Q}$, where $\mathcal{M}$ is countable and $\mathcal{Q}$ is a level of $K^{c,\Sigma}(A)$, iterability for $\mathcal{M}$ means that the collapse of Σ is moved to its pullback Σ^π. By condensation for Σ, this is what happens along realizable branches of trees on $\mathcal{M}$.

If $\beta \in X_1$, then we shall write λ_β for $\min(B_\beta)$ and b_β for $\{\xi \mid k_{\lambda_\beta+\xi} = 1\}$. We shall say that an iteration tree $\mathcal{T} \in \mathcal{M}$ is according to S *iff for all limit ordinals $\lambda < \mathrm{lh}(\mathcal{T})$, $[0,\lambda)_\mathcal{T} = b_\beta$ for some $\beta \in X_1$ with $\mathcal{T} = \mathcal{T}_\beta$. We shall say that an iteration tree $\mathcal{T} \in \mathcal{M}$ which is according to S is* taken care of in $\mathcal{M}$ *iff there is some $\beta \in X_1$ such that $\mathcal{T} = \mathcal{T}_\beta$ and b_β is a cofinal branch through $\mathcal{T}$.*

In what follows, we shall also write $X_0^\mathcal{M}$ for X_0, $X_1^\mathcal{M}$ for X_1, and $\mathcal{T}_\beta^\mathcal{M}$ for $\mathcal{T}_\beta$, $B_\beta^\mathcal{M}$ for B_β, and $\lambda_\beta^\mathcal{M}$ for λ_β (if $\beta \in X_1$).

Definition 5.6.2 *A hybrid (strategy) potential premouse*

$$\mathcal{M} = J_\alpha[E, S, A]$$

is called a hybrid (strategy) premouse *provided the following hold true.*

(a) Every strict initial segment of $\mathcal{M}$ is sound.

(b) If $\beta \in X_1^\mathcal{M}$, then $\mathcal{M}|\lambda_\beta^\mathcal{M} \models \mathsf{ZFC}^-$, $\lambda_\beta \leq \beta \leq \lambda_\beta^\mathcal{M} + \mathrm{lh}(\mathcal{T}_\beta^\mathcal{M})$, and $\mathcal{T}_\beta^\mathcal{M} \in \mathcal{M}|\lambda_\beta^\mathcal{M}$ is the $\mathcal{M}|\lambda_\beta^\mathcal{M}$-least iteration tree $\mathcal{T}$ on $\mathcal{N}$ which is according to S such that $\mathcal{T}$ is not taken care of in $\mathcal{M}|\beta$, and

(c) if $\lambda \leq \beta < \alpha$ are such that $\mathcal{M}|\lambda \models \mathsf{ZFC}^-$ and there is some iteration tree $\mathcal{T}$ on $\mathcal{N}$ which is according to S and such that $\beta \leq \lambda + \mathrm{lh}(\mathcal{T})$ and $\mathcal{T}$ is not taken care of in $\mathcal{M}|\beta$, then λ, $\beta \in X_1^\mathcal{M}$ and $\lambda = \lambda_\beta^\mathcal{M}$.

Let $\mathcal{M} = J_\alpha[E, S, A]$ be a hybrid premouse. A trivial induction shows that if $\beta \in X_1^\mathcal{M}$ and $b_\beta^\mathcal{M}$ is *not* cofinal through $\mathcal{T}_\beta^\mathcal{M}$, then $\sup(B_\beta^\mathcal{M}) = \alpha$. I.e., the only non-cofinal branch through an iteration tree on $\mathcal{N}$ which is provided by S can be the one coded by an end-segment of S. We need to allow this possibility to have that all initial segments of hybrid premice are again hybrid preemice and that "I'm a hybrid premouse" can be expressed by a Q-sentence.

Let $\mathcal{M} = J_\alpha[E, S, A]$ be a hybrid premouse. We say that $\mathcal{M}$ is *of type IV* iff $\sup(B_\beta^\mathcal{M}) = \alpha$ for some $\beta \in X_1^\mathcal{M}$. We say that $\mathcal{M}$ is *of type V* iff $\mathcal{M}$ is not of type IV and $\mathcal{M}$ does not have a top extender. If $\mathcal{M}$ does have a top extender then we let the meaning of "$\mathcal{M}$ is *of type* x," where $x \in \{I, II, III\}$, be determined by [22, Definition 2.0.1].

If the hybrid premouse $\mathcal{M}$ is of type IV, then we assume the language which is associated with $\mathcal{M}$ to have additional terms $\dot\lambda$ and $\dot{\mathcal{T}}$ for λ_β and $\mathcal{T}_\beta$, respectively. It is now somewhat tedious but straightforward to verify that "I'm a hybrid premouse of type x" where $x \in \{I, II, III, IV, V\}$, can in fact be expressed by a Q-sentence (cf. Exercise 5.7.1).

It is easy to show that the five tpes of hybrid premice exclude one another. In particular, if $\mathcal{M}$ has a top extender and if $\mathcal{T} \in \mathcal{M}$ is according to S, then $\mathcal{T}$ is taken care of in $\mathcal{M}$ (cf. Exercise 5.7.2).

Definition 5.6.3 *Let $\mathcal{M} = J_\alpha[E, S, A]$ be a hybrid (strategy) premouse. Let Σ be a (partial) iteration strategy for $\mathcal{N}$. Then $\mathcal{M}$ is called a* (hybrid) Σ-premouse *provided the following holds true. If $\beta \in X_1^{\mathcal{M}}$, then $b_\beta^{\mathcal{M}} = \Sigma(\mathcal{T}_\beta^{\mathcal{M}}) \cap \alpha$.*

Definition 5.6.4 *Let $\mathcal{M} = J_\alpha[E, S, A]$ be a (hybrid) Σ-premouse, where Σ be a (partial) iteration strategy for $\mathcal{N}$. Let Γ be a (partial) iteration strategy for $\mathcal{M}$. We say that* Γ moves Σ correctly *iff every Γ-iterate of $\mathcal{M}$ is a Σ-premouse.*

We say that $\mathcal{M}$ is γ-iterable iff there is some iteration strategy Γ for $\mathcal{M}$ for trees of length $< \gamma$ which moves Σ correctly.

We say that $\mathcal{M}$ is a Σ-mouse iff for every sufficiently elementary $\pi\colon \bar{\mathcal{M}} \to \mathcal{M}$, where $\bar{\mathcal{M}}$ is countable (and transitive) and $\pi \restriction \mathcal{N} \cup \{\mathcal{N}\} = \mathrm{id}$, $\bar{\mathcal{M}}$ is an $\omega_1 + 1$ iterable Σ-premouse.

The point of having an iteration strategy for a given $\mathcal{N}$ which condenses well is summarized by the following condensation result for Σ-premice.

Lemma 5.6.5 *Let $\mathcal{N}$ be countable, and let $\mathcal{M}$ be a Σ-premouse, where Σ is an iteration strategy for $\mathcal{N}$ which condenses well. Let $\pi\colon \bar{\mathcal{M}} \to \mathcal{M}$ be sufficiently elementary, where $\bar{\mathcal{M}}$ is transitive. Then $\bar{\mathcal{M}}$ is a Σ-premouse.*

Proof. This is shown by induction on $\mathcal{M} \cap \mathrm{OR}$. Let us consider the (only interesting) case where $\mathcal{M}$ is of type IV. We have that $\sup(B_\beta^{\mathcal{M}}) = \alpha = \mathcal{M} \cap \mathrm{OR}$ for some $\beta \in X_1^{\mathcal{M}}$. Due to the presence of $\dot{\lambda}$ and $\dot{\mathcal{T}}$, $\lambda_\beta^{\mathcal{M}}$, $\mathcal{T}_\beta^{\mathcal{M}} \in \mathrm{ran}(\pi)$. Write $\bar{\lambda} = \pi^{-1}(\lambda_\beta)$, $\bar{\mathcal{T}} = \pi^{-1}(\mathcal{T}_\beta)$, and $\bar{\alpha} = \pi^{-1}(\alpha)$. Exploiting the induction hypothesis, we only need to prove that

$$\{\xi \mid k_{\lambda_\beta^{\mathcal{M}} + \pi(\xi)} = 1\} = \Sigma(\bar{\mathcal{T}}) \cap \bar{\alpha}.$$

Well, by our inductive hypothesis, we get that $\bar{\mathcal{T}}$ is according to Σ. We must have that the downward closure (under the tree order of $\mathcal{T}_\beta^{\mathcal{M}}$) of

$$b_\beta^{\mathcal{M}} \cap \{\xi \mid \lambda_\beta^{\mathcal{M}} + \xi \in \mathrm{ran}(\pi) \wedge k_{\lambda_\beta^{\mathcal{M}} + \xi} = 1\},$$

call it c, is an initial segment of $b_\beta^{\mathcal{M}}$ of limit length, so that $c \subset b_\beta^{\mathcal{M}} \subset \Sigma(\mathcal{T})$ and hence

$$c = \Sigma(\mathcal{T} \restriction \sup\{\xi \mid \lambda_\beta^{\mathcal{M}} + \xi \in \mathrm{ran}(\pi)\}).$$

But then

$$\{\xi \mid k_{\lambda_\beta^{\mathcal{M}}+\pi(\xi)} = 1\} = \Sigma(\bar{\mathcal{T}}) \cap \bar{\alpha},$$

because Σ condenses well. □

It is now straightforward to define an operator $F = F^\Sigma$ in such a way that if $\mathcal{M}$ is a *sound* hybrid Σ-premouse with $\alpha = \mathcal{M} \cap \mathrm{OR}$, then $F(\mathcal{M})$ is the unique hybrid Σ-premouse $\mathcal{M}^+$ such that $\mathcal{M} \triangleleft \mathcal{M}^+$, $\mathcal{M}^+ \cap \mathrm{OR} = \alpha + \omega$, and $\alpha \notin X_0^{\mathcal{M}^+}$. $F(\mathcal{M})$ would be obtained from $\mathcal{M}$ either just as the rudimentary closure of $\mathcal{M} \cup \{\mathcal{M}\}$, or as the rudimentary closure of $\mathcal{M} \cup \{\mathcal{M}\}$ together with adding a predicate which feeds in more information about a cofinal branch through $\mathcal{T}_\lambda$, or as the rudimentary closure of $\mathcal{M} \cup \{\mathcal{M}\}$ together with adding a predicate which starts feeding in information on a cofinal branch through $\mathcal{T}_\alpha$.

Let us illustrate the use of Lemma 5.6.5. Let us suppose that $\mathcal{N}$ is a countable premouse whose OR-iterability is witnessed by an iteration strategy Σ which condenses well, and let A code $\mathcal{N}$. Setting $F = F_\Sigma$, we may thus define $K^{c,F}(A)$, which we shall also denote by $K^{c,\Sigma}(A)$. As discussed in Definition 1.3.16, the construction of this model produces an "array" of models $\mathcal{N}_\xi$ and $\mathcal{M}_\xi$, $\xi \leq \mathrm{OR}$, in much the same way as we would run a $K^c(A)$-construction, except for the following additional proviso. Suppose that $\mathcal{M}_\xi$ has been constructed, where $\beta = \mathcal{M}_\xi \cap \mathrm{OR}$. Suppose that $\mathcal{N}_{\xi+1}$ does not result from $\mathcal{M}_\xi$ by adding a top extender. Then $\mathcal{N}_{\xi+1} = F(\mathcal{M}_\xi)$.

A special case of a $K^{c,\Sigma}(A)$–construction is when no extenders at all are added to the sequence of the resulting model, which we then denote by $L^\Sigma(A)$.

Definition 5.6.6 *Let ν be an infinite cardinal, and let $A \in H_\nu$. Then a* hybrid mouse operator over A on H_ν *is a function J such that for some Q-formula ψ,*

$$J(B) = \textit{least } \mathcal{P} \trianglelefteq \mathrm{Lp}^\Sigma(B) \textit{ such that } \mathcal{P} \models \psi[A, B]$$

for all $B \in H_\nu$ such that $A \in B$. (J must be defined at all such B.) We call J a (ν, A)-hmo. The (ν, A)-hmo J is called tame *iff for every $B \in \mathrm{dom}(J)$, $J(B)$ is a tame hybrid-B-premouse.*

Definition 5.6.7 *Let $A \in HC^{V[g]}$ code $\mathcal{N}$ in some specified way. Then $P_n^\Sigma(A)^\#$ is the minimal iterable Σ-hybrid mouse over A which is active, and satisfies "there are n Woodin cardinals".*

The following essentially completes the proof of the Witness Dichotomy, Theorem 3.6.1, in the end of gap cases.

Lemma 5.6.8 *Suppose that in V: for all $n < \omega$ and all swo $B \in H_\nu$, $P_n^\Sigma(B)^\sharp$ exists and is ν-iterable. Then $W_{\beta+1}^*$ holds in $V[g]$.*

Proof. We show first that the hybrid operators we are given on H_ν^V extend to $\mathrm{HC}^{V[g]}$:

Claim 1. In $V[g]$, we have that for all $n < \omega$ and all swo $B \in \mathrm{HC}$, $P_n^\Sigma(B)^\sharp$ exists, and has an ω_1-iteration strategy in $J_{\beta+1}(\mathbb{R}^g)$.

Proof. In $V[g]$, let $J^n(B) = P_n^\Sigma(B)^\sharp$. We show by induction on n that HC is closed under J^n, and that $J^n \in J_{\beta+1}(\mathbb{R}^g)$.

For $n = 0$: Fix $B \in \mathrm{HC}^{V[g]}$. Let $g = h \times l$, where h is on $\mathrm{Col}(\omega, \mu)$, and $\mu < \nu$ is large enough that $|\mathcal{N}| \leq \mu$, that $B = \rho^h$ for some $\mathrm{Col}(\omega, \mu)$-term ρ, and for $\vec{A}$ the sjs guiding an extension of Σ to $V[g]$, we have that $\langle \tau_{A_i}^{\mathcal{N}} \mid i < \omega \rangle = \sigma^h$, for some $\mathrm{Col}(\omega, \mu)$-term σ.

In $V[g]$, we can now construct $P_0^\Sigma(B)^\sharp$, which is simply an ordinary sharp for $L^\Sigma(B)$, from $P_0^\Sigma(\langle \mathcal{N}, \sigma \rangle)^\sharp[h]$. For this, it is clearly enough to show that $L^\Sigma(B)$ is definable over $P[h] = P_0^\Sigma(\langle \mathcal{N}, \sigma, \rho \rangle)^\sharp[h]$ from $\langle \mathcal{N}, \sigma, \rho, h \rangle$. The only trouble here is that in forming $L^\Sigma(B)$, we may need to apply Σ to iteration trees which are in $P[h]$, but not in P. For that, we use

Definition 5.6.9 *Let x be countable and transitive. Then $H(x) = x \oplus \vec{A}$ is the structure $(\mathcal{R}, T)$, where*

(a) $\mathcal{R}$ is the Lp^α-closure of x through ω cardinals, call them η_i for $i < \omega$, and

(b) for all i, y, $T(i, y)$ holds iff $y = \tau_{A_i, \eta_i}^{\mathcal{R}}$.

Here we assume the sets in $\vec{A}$ are enumerated as $(A_i | i < \omega)$ so that each is repeated infinitely often.

$x \oplus \vec{A}$ is a *term relation hybrid* over x. Notice that if $\mathcal{M}$ is α–suitable and if $\vec{A}$ is a sjs, then $H(\mathcal{M}|\delta^{\mathcal{M}})$ is an amenable structure whose universe is equal to $\mathcal{M}$. We have, as an immediate consequence of Theorem 5.4.3:

Lemma 5.6.10 *Let $\pi\colon H \to_{\Sigma_1} H(x)$, where H is transitive and $x \in \mathrm{ran}(\pi)$. Then $H = H(\pi^{-1}(x))$.*

In some contexts, we can replace our strategy hybrids by models formed by adding extenders and closing the levels under $x \mapsto x \oplus \vec{A}$. Indeed, this closure operation is equivalent to closing under Σ, *granted the parameter* $\langle \mathcal{N}, \sigma, \rho, h \rangle$. This equivalence is expressed by

Lemma 5.6.11 (Strategy-sjs equivalence) *Let P, h be as above, and let $H(x) = x \oplus \vec{A}$ for all x; then*

(a) $P[h]$ is closed under the function H, and H is definable over $P[h]$ from Σ and $\langle \mathcal{N}, \sigma, \rho, h \rangle$,

(b) $\Sigma \restriction P[h]$ is definable over $P[h]$ from $H \restriction P[h]$ and $\langle \mathcal{N}, \sigma, \rho, h \rangle$.

Proof. (a) Notice that $\mathcal{N}$ is the universe of $H(\mathcal{N}|\delta^{\mathcal{N}})$, and that by the construction of Σ, if

$$\pi\colon \mathcal{N} \to \mathcal{M}$$

is obtained by a simple iteration of $\mathcal{N}$ according to Σ, then $\mathcal{M}$ is the universe of $H(\mathcal{M}|\delta^{\mathcal{M}})$ and $\pi(\tau^{\mathcal{N}}_{A_i,\eta_i^{\mathcal{N}}}) = \tau^{\mathcal{M}}_{A_i,\eta_i^{\mathcal{M}}}$ for every $i < \omega$.

Now let $y \in P[h]$, say $y = \tau^h$. Working in P, we may find some

$$\pi\colon \mathcal{N} \to \mathcal{M},$$

which is obtained by a simple iteration of $\mathcal{N}$ according to Σ, such that τ is generic over $\mathcal{M}$ at $\delta^{\mathcal{M}}$.

We claim that we may easily read off $H(\mathcal{M}|\delta^{\mathcal{M}}[\tau, h])$ from $H(\mathcal{M}|\delta^{\mathcal{M}})[\tau, h]$. For one thing, notice that $\tau^{\mathcal{M}}_{A_i,\eta_i^{\mathcal{M}}}$ still captures A_i over $\mathcal{M}[\tau, h]$, just because if k is $\mathrm{Col}(\omega, \delta^{\mathcal{M}})$-generic over $\mathcal{M}$, then $\tau \times h \times k$ may be construed as being $\mathrm{Col}(\omega, \delta^{\mathcal{M}})$-generic over $\mathcal{M}$ as well. Also, $\mathcal{M}[\tau, h]$ is Lp^{α}-closed above $\delta^{\mathcal{M}}$. To see this, suppose that for some $\eta_i^{\mathcal{M}}$ there is an initial segment $\mathcal{S}$ of $Lp^{\alpha}(\mathcal{M}|\eta_i^{\mathcal{M}}[\tau, h])$ which is not in $\mathcal{M}[\tau, h]$. If $\mathcal{P}$ is the result of performing the $\mathcal{P}$-construction over $\mathcal{M}|\eta_i^{\mathcal{M}}$ inside $\mathcal{S}$ (cf. [32]), then $\mathcal{P} \in Lp^{\alpha}(\mathcal{M}|\eta_i^{\mathcal{M}})$, and therefore $\mathcal{P} \triangleleft \mathcal{M}$. But this gives $\mathcal{P}[\tau, h] = \mathcal{S} \in \mathcal{M}[\tau, h]$.

We now get $H(y)$ as the transitive collapse of

$$\mathrm{Hull}_1^{H(\mathcal{M}|\delta^{\mathcal{M}}[\tau,h])}(y \cup \{y\}).$$

(b) Inside $P[h]$, $\Sigma \restriction P[h]$ is defined as follows. Let $\mathcal{T}$ be a tree on $\mathcal{N}$ of limit length. Then $\Sigma(\mathcal{T}) = b$ iff either there is an initial segment of

$H(\mathcal{M}(\mathcal{T}))$ which kills the Woodinness of $\delta(\mathcal{T})$ and if $\mathcal{Q}$ is the least such then $\mathcal{Q} \lhd \mathcal{M}_b^{\mathcal{T}}$, or else $H(\mathcal{M}(\mathcal{T})) \models \delta(\mathcal{T})$ is a Woodin cardinal and b is such that $\mathcal{M}_b^{\mathcal{T}}$ is the universe of $H(\mathcal{M}(\mathcal{T}))$ and the branch given by b moves the terms correctly. There can be at most one such b, so we need to see that if $\mathcal{T}$ is according to Σ, then there is such a b.

Let $\mathcal{T}$ be a counterexample. Working inside $P[h]$, pick

$$\sigma\colon \bar{P}[\bar{h}] \to P[h],$$

where $\bar{P}[\bar{h}]$ is countable and transitive and $\{\mathcal{T}, H(\mathcal{M}(\mathcal{T}))\} \subset \mathrm{ran}(\sigma)$. Let $\bar{\mathcal{T}} = \sigma^{-1}(\mathcal{T})$. By Lemma 5.6.10, $\bar{\mathcal{T}}$ is according to Σ. In $V[g]$, $\Sigma(\bar{T})$ is hence well-defined. By absoluteness, there is hence in $\bar{P}[\bar{h}]$ a branch b through $\bar{\mathcal{T}}$ such that either there is a (least) initial segment $\mathcal{Q}$ of $\pi^{-1}(H(\mathcal{M}(\mathcal{T}))) = H(\mathcal{M}(\bar{\mathcal{T}}))$ which kills the Woodinness of $\delta(\bar{\mathcal{T}})$ and $\mathcal{Q} \lhd \mathcal{M}_b^{\bar{\mathcal{T}}}$, or else $H(\mathcal{M}(\bar{\mathcal{T}})) \models \delta(\bar{\mathcal{T}})$ is a Woodin cardinal and b is such that $\mathcal{M}_b^{\bar{\mathcal{T}}}$ is the universe of $H(\mathcal{M}(\bar{\mathcal{T}}))$. By uniqueness, $b = \Sigma(\bar{\mathcal{T}}) \in \bar{P}[\bar{h}]$. Hence by elementarity, $\Sigma(\mathcal{T})$ is defined in $\mathcal{P}[h]$ after all. Contradiction! □

Clearly, 5.6.11 completes the proof that $\mathrm{HC}^{V[g]}$ is closed under J^0. It is easy then to see that $J^0 \in J_{\beta+1}(\mathbb{R}^g)$. This finishes the case $n = 0$.

Now let $n = k+1$. The argument above easily adapts to show that $\mathrm{HC}^{V[g]}$ is closed under J^n. But then, each $J^n(B)$ has an ω_1 iteration strategy which is $\Delta^1_5(J^k)$, as J^k provides the necessary $\mathcal{Q}$-structures. This implies that J^n itself is $\Delta^1_8(J^k)$, and so in $J_{\beta+1}(\mathbb{R}^g)$.

This proves Claim 1. □

Now let $U \subseteq \mathbb{R}^g$ be in $J_{\beta+1}(\mathbb{R}^g)$ and $k < \omega$; we seek a coarse (k, U)-Woodin mouse. Let U be $\Sigma_n^{J_\beta(\mathbb{R}^g)}$ in the real parameter z. Let us also take z so that it codes $\langle \mathcal{N}, \sigma, \rho, h\rangle$, where these are as above. Our desired witness will be

$$P = P^{\Sigma}_{k+n}(z)^{\sharp}.$$

By Claim 1, P has a unique ω_1-iteration strategy in $J_{\beta+1}(\mathbb{R}^g)$. Let Γ be this strategy. Let $\langle A_i \mid i < \omega\rangle$ be our self-justifying system of sets which are $\mathrm{OD}^{<\beta}(z)$. If j is least such that $\rho_j^{J_\beta(\mathbb{R}^g)} = \mathbb{R}^g$, then Σ_j-truth at β is coded in a simple way into

$$W = \oplus_{i<\omega} A_i.$$

Claim 2. For any $\xi \in P$, there is a term $\dot{W} \in P$ relative to $\mathrm{Col}(\omega, \xi)$ such that whenever $i\colon P \to \mathcal{Q}$ is an iteration map by Γ (constructed in $V[g]$), and

l is $\mathcal{Q}$]-generic over $\mathrm{Col}(\omega, i(\xi))$, then

$$\dot{W}^l = W \cap \mathcal{Q}[l].$$

Proof. Basically, $\dot{W}$ asks what the $\tau^{\mathcal{N}}_{A_i}$ are moved to in the iteration of $\mathcal{N}$ which makes $P|\mu^+$ generic over the extender algebra of the iterate. This iteration is done inside $\mathcal{P}$, using what it knows of Σ. □

Claim 3. Let δ be the k^{th} Woodin cardinal of P; then for any $\Sigma_n^{J_\beta(\mathbb{R}^g)}(z)$ set Y, there is a term $\dot{Y} \in P$ relative to $\mathrm{Col}(\omega, \delta)$ such that whenever $i\colon P \to \mathcal{Q}$ is an iteration map by Γ (constructed in $V[g]$), and l is $\mathcal{Q}$-generic over $\mathrm{Col}(\omega, i(\delta))$, then

$$\dot{Y}^l = Y \cap \mathcal{Q}[l].$$

Proof. $\dot{Y}$ is constructed from the term $\dot{W}$ given by claim 2, applied at the $k+n^{th}$ Woodin of $\mathcal{P}$. The n-Woodins above δ are used to answer the relevant n-real-quantifier statements. □

We can now see that P is the desired coarse witness. The trees in $\mathcal{P}$ which are moved appropriately by Γ are obtained just as in the inadmissible case. □

5.7 Exercises.

Exercise 5.7.1 *Show that "I'm a hybrid premouse of type x" where $x \in \{I, II, III, IV, V\}$," can be expressed by a Q-sentence (cf. Exercise 5.7.1).*

Exercise 5.7.2 *Show that if $\mathcal{M}$ has a top extender and if $\mathcal{T} \in \mathcal{N}$ is according to S, then $\mathcal{T}$ is taken care of in $\mathcal{N}$.*

Exercise 5.7.3 *Let $\mathcal{M}$ be a hybrid Σ-premouse of type IV such that $b_{\dot{\lambda}^{\mathcal{M}}} = \Sigma(\dot{T}^{\mathcal{M}})$. Let $\mathrm{cf}(\mathrm{lh}(\mathcal{T}_{\dot{\lambda}^{\mathcal{M}}}))$ be measurable in $\mathcal{M}$ as witnessed by $E^{\mathcal{M}}_\gamma$. Then $\mathcal{M}^* = \mathrm{Ult}_0(\mathcal{M}; E^{\mathcal{M}}_\gamma)$ is also of type IV, but $b_{\dot{\lambda}^{\mathcal{M}^*}}$ is a strict initial segment of $\Sigma(\dot{T}^{\mathcal{M}})$.*

Exercise 5.7.4 *Let $\mathcal{M}$ be a hybrid Σ-premouse. Let $\beta \in X^{\mathcal{M}}_1$, and suppose that in $\mathcal{M}|\lambda^{\mathcal{M}}_\beta$, $\mathrm{cf}(\mathrm{lh}(\mathcal{T}^{\mathcal{M}}_\beta))$ is measurable (as being witnessed by an extender from the $\mathcal{M}$-sequence). Then $\Sigma(\mathcal{T}^{\mathcal{M}}_\beta) \in \mathcal{M}|\lambda^{\mathcal{M}}_\beta$.*

Chapter 6

Applications

6.1 $\mathsf{AD}^{L(\mathbb{R})}$ from a homogeneous ideal

In Chapter 2, §7 we showed that CH plus the existence of a homogeneous, presaturated ideal on ω_1 implies PD. In this section we shall join that argument to the Witness Dichotomy, and thereby derive $\mathsf{AD}^{L(\mathbb{R})}$ from the same hypothesis. We prove

Theorem 6.1.1 *Assume* CH*, and suppose there is a homogeneous poset* $\mathbb{P}$ *such that whenever* G *is* V*-generic over* $\mathbb{P}$*, then*

$$V[G] \models \exists j\colon V \to M(\mathrm{crit}(j) = \omega_1^V \wedge M^\omega \subseteq M).$$

Then for all α*,* W_α^* *holds in* V*.*

Remark 6.1.2 We are not requiring that $\mathbb{P}$ be given by an ideal on ω_1, so it is easy to obtain a model of the hypothesis by forcing. Namely, let and $j\colon V \to N$ be elementary, with $\kappa = \mathrm{crit}(j)$. Suppose $j(\kappa)$ is inaccessible in V, and $V_{j(\kappa)} \subseteq N$. Thus κ is a bit more than superstrong. Let H be $\mathrm{Col}(\omega, <\kappa)$-generic over V. Then the hypothesis of 6.1.1 holds in $V[H]$, with the homogeneous poset $\mathbb{P} = \mathrm{Col}(\omega, < j(\kappa))$ providing the witness.

We do not know whether the hypothesis of 6.1.1 can be shown consistent using less than a superstrong.

Proof of 6.1.1: Let $j\colon V \to M$, with $\mathrm{crit}(j) = \omega_1^V$ and $M^\omega \subseteq M$, be the embedding in $V[G]$ which is given by our hypothesis. Let $\nu = j(\omega_1^V)$, so that ν is a regular cardinal in V, and $\nu = \omega_1$ in $V[G]$.

It is clear from the proof of WD that it holds for $L(\mathbb{R})^V$ as well. This is where we shall use it. Suppose that W^*_α fails to hold in V for some α. Let us fix an swo A and a (ω_1, A)-hmo J such that J^w does not exist. We shall use K^J, and in particular the argument of Chapter 2, §7 to obtain a contradiction.

We shall begin with the case that J is an ordinary (ω_1, A)-mo, and then indicate the elaborations needed in the hybrid case.

Let $B \in H_{\omega_1}$ be swo'd, and in the cone over A, and such that $M_1^J(B)$ does not exist. Following Chapter 2, §7, we then have that $(K^{c,J}(B))^N$ does not reach $M_1^J(B)$, where N is the background universe $L^J(\mathbb{R})$. A preliminary argument gives indicernibles for $L^J(\mathbb{R})$, and hence enough of a measurable cardinal to get the theory of $K^J(B)$ going. We then use this theory to obtain a contradiction.

It is important for the argument that $\mathbb{R} \subseteq N$, since the countable-in-$V[G]$ fragments of E_j should be in $j(N)$, where they might be added to $(K^J(B))^{j(N)}$. It is also important that $j(N)$ be ordinal definable in $V[G]$, so that $(K^J(B))^{j(N)} \in V$ by the homogeneity of $\mathbb{P}$. We therefore cannot afford to put a wellorder of $\mathbb{R}$ into N.

Unfortunately, mouse operators like J only operate on self-wellordered sets, and not for example on $\mathbb{R}$, so we must take a little care as to what $L^J(\mathbb{R})$ is to be. We should probably modify the notion of "cone over A" so as to include non-swo's. For now, let us just simply note that the notion of an a-premouse makes sense for any transitive set a, self-wellordered or not. The reader should see [40] for a discussion of the elementary properties of such premice, in the representative special case that $a = \mathrm{HC}$. The main thing is that if $\mathcal{M}$ is an a-premouse with top extender E, and $f\colon (a \times \xi) \to E_b$ with $\xi < \mathrm{crit}(E)$ and $f \in \mathcal{M}$, then $\bigcap \mathrm{ran}(f) \in E_b$. This implies that i_E is the identity on $a \cup \{a\}$, and that we have Los' theorem for Σ_n ultrapowers, whenever $\mathrm{crit}(E) < \rho_n^{\mathcal{M}} =$ least ρ such that there is a new $\Sigma_n^{\mathcal{M}}$ subset of $a \times \rho$. These properties imply that if g is $\mathrm{Col}(\omega, a)$-generic over $\mathcal{M}$, then $\mathcal{M}[g]$ can be regarded as an ordinary premouse over the swo $\langle a, g\rangle$. This last fact summarizes what it is to be an a-premouse: you become an ordinary premouse when a wellorder of a is added generically.

We extend the lower part notation $\mathrm{Lp}(b)$ to arbitrary transitive b in the obvious way.

Definition 6.1.3 *Let $C \in H_\mu$; then a* extended mouse operator over C on

H_μ is a function H such that for some Q-formula ψ,

$$H(b) = \textit{least } \mathcal{P} \trianglelefteq \mathrm{Lp}(b) \textit{ such that } \mathcal{P} \models \psi[C, a]$$

for all transitive $b \in H_\nu$ such that $C \in b$. (H must be defined at all such b.) We call H a (μ, C)-emo.

Lemma 6.1.4 *There is a $C \in HC$, and an (ω_1, C)-emo H, such that for all $a \in \mathrm{dom}(H)$ and all g which are $\mathrm{Col}(\omega, a)$-generic over $H(a)$, $H(a)[g] = J(\langle a, g\rangle)$.*

One can show that ther operators J which arise from WD satisfy lemma 6.1.4 , but one can also just derive 6.1.4 abstractly from the Turing invariance of J. We defer further detail on the proof of 6.1.4. Fix H and C as given there.

The next lemma takes what is a key step in any local core model induction, by extending the domain of our operator H.

Lemma 6.1.5 (Extension Lemma) *There is a unique (ν, C)-emo H^* such that $H \subseteq H^*$.*

Proof. Uniqueness is a simple Lowenheim-Skolem argument, based on the fact that any such H^* has condensation.

For existence, we use j, taking H^* to be simply $j(H) \restriction V$. We must show this works.

Let ψ be the sentence which determines H over the parameter C. We need to see that whenever $b \in H_\nu$ is transitive, with $C \in b$, then there is a countably iterable b-premouse $\mathcal{P}$ such that $\mathcal{P} \models \psi[C, b]$. Fix such a b. We have that $b \in \mathrm{HC}^M = \mathrm{HC}^{V[G]}$. Working in M, we obtain a countably iterable minimal b-premouse $\mathcal{P}$ such that $\mathcal{P} \models \psi[C, b]$. We need only show that $\mathcal{P}$ is in V, and is countably iterable there. For this, we need to look more closely at our J.

Remark 6.1.6 At this point, we are using not just the statement of WD, but its proof.

Case 1. J is the diagonal operator at the bottom of the hierarchy in the inadmissible, uncountable cofinality case (2)(b) of the proof of WD.

Proof. Let β be our inadmissible of uncountable cofinality, with W^*_β holding in V. Inspecting the construction of J, we see that for each $Z \in \text{dom}(J)$,

$$J_\beta(\mathbb{R}) \models J(Z) \text{ is } \omega_1\text{-iterable.}$$

It follows that the same is true with H replacing J. (Iterations of $H(a)$ reduce to iterations of $H(a)[g] = J(\langle a, g\rangle)$.) Thus in M,

$$J_{j(\beta)}(\mathbb{R}^M) \models H(b) \text{ is } \omega_1\text{-iterable.}$$

But $J_{j(\beta)}(\mathbb{R}^M) = J_{j(\beta)}(\mathbb{R}^{V[G]})$ is a model of AD. It follows that $H(b)$ is ordinal definable in $V[G]$ from b. Thus $H(b)$ is in V.

Similarly, if Σ is the unique ω_1-iteration strategy of $J_{j(\beta)}(\mathbb{R}^M)$ for $H(b)$, then $\Sigma \restriction V$ is in V. In V, it is a ν-iteration strategy for $H(b)$, and thus certainly witnesses countable iterability.

Notice that we have shown in this case that $j(H)$ is definable over $J_{j(\beta)}(\mathbb{R}^M)$. Our desired extension of H is just $j(H) \restriction V$.

Case 2. $J = I^w$, for some (ω_1, A)-mo I.

Proof. In this case, we have $H = S^w$, for some S. By induction, we may assume that $j(S)$ is definable over $J_{j(\beta)}(\mathbb{R}^M)$. This enables us to define $H(b)$ over $J_{j(\beta)}(\mathbb{R}^M)$, as the unique model of the appropriate theory which is ω_1-iterable via the $\mathcal{Q}$-structures provided by $j(S)$. (If H^1 and H^2 are two such structures, we can compare them in $j(S)(\langle H^1, H^2\rangle)$, ending at worst by stepping outside when we reach stage ω_1 in this model.) We can also define an ω_1-strategy for $H(b)$ from $j(S)$. Again, this gives $H(b)$ and $\Sigma \restriction V$ in V. Again, we have $H^* = j(H) \restriction V$.

This completes the proof of 6.1.5 in case (2)(b). In cases (1) or (2)(a), the operator at the bottom of the $J_\beta(\mathbb{R})$ hierarchy is a countable join $\oplus_n I_n$ of operators belonging to $J_\beta(\mathbb{R})$. The I_n can be extended as in 6.1.5 to $j(I_n) \restriction V$, and hence $\oplus_n I_n$ extends to $j(\oplus_n I_n)$. This handles our Case 1 above, and the Case 2 is done in the same way as above.

Since we are ignoring the gap case of WD for now, this completes our proof of 6.1.5. □

To save notation, let us now write H for the operator H^* given by 6.1.5.

Now let

$$N = L^H_\nu(\mathbb{R})$$

be the model $\mathcal{M}_\nu$ obtained by starting with $\mathcal{M}_0 = (\text{HC}, \in)$, and letting $\mathcal{M}_{\alpha+1} = H(\mathcal{M}_\alpha)$, with unions taken at limit ordinals.

Lemma 6.1.7 *$N^\sharp$ exists.*

Pick Ω an indiscernible of N. Inside $N^{\mathrm{Col}(\omega_1,\mathbb{R})}$, we form $K^{c,H}(B)$ up to Ω. (Either explicitly close its levels under H, or argue that they must be closed above the largest Woodin by universality.)

Since $M_1^H(B)$ does not exist, we have that

$$K = (K^H(B))^N$$

exists.

Lemma 6.1.8 *1. $j(K) \in V$.*

2. ω_1^V is inaccessible in K.

Proof. Part (1) follows because $j(K)$ is definable over $j(N)$, and $j(N) = L^H(\mathbb{R}^{V[G]})$ is definable over $V[G]$.

Part (2) follows as otherwise ω_1^V is collapsed in $j(K)$. But $j(K) \in V$. □

By part (2) of the lemma, each fragment $E_j \restriction \alpha \cap j(K)$ of the extender of j, for $\alpha < \nu$, is coded by a real in $V[G]$. Hence these fragments are in $j(N)$. If they are in $j(K)$, then ω_1^V is Shelah in $j(K)$. A standard argument shows that they are indeed in $j(K)$ granted the next lemma.

Lemma 6.1.9 *Let $\omega_1^V < \alpha < \nu$, with α a cardinal of $j(K)$. Then in $j(N)$, the phalanx $(j(K), Ult(j(K), E_j \restriction \alpha), \alpha)$ is $j(\Omega) + 1$- iterable.*

Proof. Because $j(N)$ is $j(H)$-closed, it is enough to show the phalanx is countably iterable in $j(N)$. Working in $j(N)$, let $(\mathcal{P}, \mathcal{Q}, \alpha)$ be a countable phalanx embedding by (π, σ) into $(j(K), \mathrm{Ult}(j(K), E_j \restriction \alpha), \alpha)$. We have $(\mathcal{P}, \mathcal{Q}, \alpha)$ embeds into $(j(K), j(j(K)), \alpha)$ by some (π, τ) then.

Remark 6.1.10 Here is a crucial point at which $j(K) \in V$ is used.

Pulling back to V, we have for G-a.e. ξ, $(\mathcal{P}_\xi, \mathcal{Q}_\xi, \alpha_\xi)$ embeds by some (π_ξ, τ_ξ) into $(K, j(K), \alpha_\xi)$. But this implies that $(\mathcal{P}_\xi, \mathcal{Q}_\xi, \alpha_\xi))$ embeds into (K, K, α_ξ) by some (π_ξ, ψ_ξ), for G-a.e. ξ. That in turn means that $(\mathcal{P}, \mathcal{Q}, \alpha)$ embeds into $(j(K), j(K), \alpha)$ in $j(N)$, and so is countably iterable there.

□

This finishes our proof of 6.1.1 in the non-gap case. In the gap case, in which our J is a hybrid mouse operator, the main thing we have to see is how to extend the ω_1-iteration strategy Σ for a suitable mouse $\mathcal{N}$ which we got in V, and went into the definition of J, to a ν-iteration strategy.

But this is easy; our desired extension is just $j(\Sigma) \restriction V$. Σ was guided in V by an sjs $\vec{A}$. Moreover, $\vec{A}$ is $\mathrm{OD}^{\beta}(x)$ in V, for some β. It follows that $j(\Sigma)$ is $\mathrm{OD}^{j(\beta)}(x)$ in M. Since $\mathbb{R}^M = \mathbb{R}^{V[G]}$, $j(\Sigma)$ is $\mathrm{OD}^{j(\beta)}(x)$ in $V[G]$. By homogeneity, $j(\Sigma) \restriction V$ is in V.

Remark 6.1.11 For this homogeneous ideal plus CH argument, we could avoid the strategy hybrids, and work instead with the older *term-relation hybrids*. The reason is that the extension argument for J takes place in the very same universe in which we are trying to prove W^*_α for all α. The meaning of our sjs $\vec{A}$ is tied to some real x, but this x is in the domain of the embedding j we use to do the extension. Strategy hybrids do seem necessary in getting strength from a failure of $\square$ at a singular.

Question. Does $\mathsf{AD}^{L(\mathbb{R})}$ follow from the existence of a homogeneous, presaturated ideal on ω_1?

6.2 The strength of AD

In this section, we aim to show how the core model induction may be used to get strength from the hypothesis that AD holds (in $L(\mathbb{R})$). More specifically, we shall produce the following result.

Theorem 6.2.1 *Suppose that $V = L(\mathbb{R}) \models AD$. There is then a generic extension of V in which there is a fine structural inner model $L[E]$ with infinitely many Woodin cardinals (cofinal in ω_1^V) and in which $\mathbb{R}$ (the reals of V) is the set reals of a symmetric collapse over $L[E]$ of the supremum of the Woodin cardinals of $L[E]$ to ω.*

The only consequence of AD which we shall need in order to produce this theorem is given by the following classical result.

Lemma 6.2.2 (AD; Kechris) *Let $S \subset OR$. For an S–cone of reals x we have*

$$L[S,x] \models OD_S\text{–determinacy.}$$

In particular, $\omega_1^{L[S,x]}$ is measurable in $HOD_S^{L[S,x]}$.

Proof. Let us first assume that there is no S–cone of reals x such that in $L[S, x]$ all OD_S–sets of reals are determined. Define $x \mapsto A_x$ by letting A_x be the least $OD_S^{L[S,x]}$–set of reals which is not determined. I.e., if $\mathcal{G}_{A_x}$ is the usual game (in which I, II alternate playing integers) with payoff A_x, then $\mathcal{G}_{A_x}$ is not determined in $L[S, x]$. Notice that A_x only depends on the S–constructibility degree of x. Also, by hypothesis, A_x is defined for an S–cone $\mathcal{C}$ of x.

Let $\mathcal{G}$ be the game in which I, II alternate playing integers so that I produces the reals x, a, II produces the reals y, b, and I wins iff $a \oplus b \in A_{x \oplus y}$. Let us suppose that I has a winning strategy, τ, in $\mathcal{G}$. Let $\tau \in L[S, z]$, where z is in $\mathcal{C}$. Let τ^* be a strategy for I in $\mathcal{G}_{A_z}$ so that if II produces the real b, and if τ calls for I to produce the reals a, x in a play of $\mathcal{G}$ in which II plays b, $z \oplus b$, then τ^* calls for I to produce the real a. Then for every $b \in L[S, z]$, if $a = \tau^*(b)$, in fact if a, $x = \tau(b, z \oplus b)$, then

$$a \oplus b \in A_{x \oplus (z \oplus b)} = A_z.$$

So τ^* is a winning strategy for I in the game $\mathcal{G}_{A_z}$ played in $L[S, z]$. Contradiction! We may argue similarily if II has a winning strategy in $\mathcal{G}$.

Now let $L[S, x] \models OD_S$–determinacy. Working inside $L[S, x]$, we may then define a filter μ on $\omega_1^{L[S,x]}$ as follows.

For reals x, let $|x| = sup\{||y||\colon y \equiv_T x \wedge y \in WO\}$. Let $S = \{|x|\colon x \in \mathbb{R}\}$. Let $\pi\colon \omega_1 \to S$ be the order isomorphism. Now if $A \subset \omega_1$, then we put $A \in \mu$ iff

$$\{x\colon |x| \in \pi'' A\}$$

contains an S–cone of reals. It is easy to verify that $\mu \cap HOD_S$ witnesses that ω_1 is measurable in HOD_S. □

Let us assume that $V = L(\mathbb{R}) \models$ AD for the rest of this section. Fix $T = T_1^2$, a tree obtained from the scale property of Σ_1^2. So for any real x, the model $L[T, x]$ is Σ_1^2–*correct*, i.e., if $A \subset \mathbb{R}$ is a nonempty $\Sigma_1^2(z)$ set, where $z \in \mathbb{R} \cap L[T, x]$, then $A \cap L[T, x] \neq \emptyset$. Moreover, with $\delta = \boldsymbol{\delta}_1^2$ we have

$$L_\delta[T, x] = V_\delta^{HOD_x} \models \omega_1^V \text{ is measurable},$$

so that ω_1^V is measurable in $L[T, x]$, which can be seen to imply that

$$HOD_{T,\vec{Q}}^{L[T,x]} \models \omega_1^V \text{ is measurable}$$

for $\vec{Q} \in L[T,x]$, as $L[T,x]$ is a size $< \omega_1$ forcing extension of $HOD_{T,\vec{Q}}^{L[T,x]}$. We may thus try to isolate versions of K of height ω_1^V inside various models of the form $HOD_{T,\vec{Q}}^{L[T,x]}$. We shall write $\Omega = \omega_1^V$.

Lemma 6.2.3 *Let $P \in HC$ be transitive, and suppose that*

$$W_x = (K^c(P))^{HOD_{T,P}^{L[T,x]}},$$

constructed with height Ω, exists for a cone of x. Then there is a cone of x such that W_x cannot be $\Omega + 1$ iterable above P inside $HOD_{T,P}^{L[T,x]}$.

Proof: Suppose otherwise. By 6.2.2, there is then a $T \oplus P$–cone $\mathcal{C}$ so that for all $x \in \mathcal{C}$ we have $L[T,x] = L[T,P,x]$,

$$\omega_1^{L[T,x]} \text{ is measurable in } HOD_{T,P}^{L[T,x]},$$

and we may isolate

$$K_x = (K(P))^{HOD_{T,P}^{L[T,x]}}.$$

Let us fix an $x \in \mathcal{C}$, and let us write K for K_x. By "cheapo" covering and the fact that $L[T,x]$ is a size $< \Omega$ forcing extension of its $HOD_{T,P}$, we may pick some $\lambda < \Omega$ s.t.

$$\lambda^{+K} = \lambda^{+L[T,x]}.$$

Let $g \in V$ be a $Col(\omega, \lambda)$–generic over $L[T,x]$, and let $y \in V$ be a real coding (g,x). Thus

$$\omega_1^{L[T,y]} = \lambda^{+L[T,x]} = \lambda^{+K}.$$

As we also have $y \in \mathcal{C}$,

$$\omega_1^{L[T,y]} \text{ is measurable in } HOD_{T,P}^{L[T,y]}.$$

We hence get a contradiction if we can show:

Claim. $K \in HOD_{T,P}^{L[T,x]}$.

Proof: K is still fully iterable inside $L[T,y]$ by [37, Theorem 2.18]. This means that K is the core model above P of $L[T,y]$ in the sense of [37, 5.17]; i.e., from the point of view of $L[T,y]$, it is the common transitive collapse of $Def(W', S)$ for any W', S s.t. W' is $\Omega + 1$ iterable and Ω is S–thick. But

this characterization clearly establishes $K \in HOD_{T,P}^{L[T,y]}$. □ (Claim)
□

Using [37, Corollary 2.11, Theorem 2.8], we now immediately get:

Corollary 6.2.4 *In the situation of 6.2.3, there is a $T \oplus P$–cone of x such that for each x from that cone, there is $Q \rhd P$ together with $\delta \in Q \backslash P$ such that*

$$Q \in HOD_{T,P}^{L[T,x]}$$
$$HOD_{T,P}^{L[T,x]} \models Q \text{ is excellent, and}$$
$$Q \models \delta \text{ is Woodin.}$$

For our purposes, in 2.3 and in the following, we may let "Q is excellent" mean that Q is excellent in the sense of [37] *and* Q has a largest cardinal, denoted by $\delta = \delta(Q)$, such that $Q \models \delta$ is Woodin.

Lemma 6.2.5 *Let Q be excellent in $HOD_{T,P}^{L[T,x]}$, and suppose that*

$$OD_Q \cap \mathcal{P}(\delta(Q)) \subset Q.$$

Then

$$(K^c(Q))^{HOD_{T,Q}^{L[T,y]}} \textit{ exists}$$

for a cone of y.

[N.b.: "$K^c(Q)$ exists" is supposed to imply $Q \cap OR = \delta(Q)^{+K^c(Q)}$, c.f. [37, §1].]

Proof: Deny. Then let $\mathcal{C}$ be a cone such that for all $y \in \mathcal{C}$,

$$(K^c(Q))^{HOD_{T,Q}^{L[T,y]}} \text{ does not exist.}$$

Consider $y \in \mathcal{C}$. As $K^c(Q)$ does not exist in $HOD_{T,Q}^{L[T,y]}$, there is a least $\mathcal{N}_\xi$ from the $K^c(Q)$–construction (inside $HOD_{T,Q}^{L[T,y]}$) with $\rho_\omega(\mathcal{N}_\xi) \leq \delta = \delta(Q)$. But then if $A \in (\Sigma_\omega(\mathcal{N}_\xi) \cap \mathcal{P}(\delta)) \backslash Q$, we have that $A \in OD_{T,Q}^{L[T,y]} \cap \mathcal{P}(\delta)$.

We may thus define $f : \mathcal{C} \to \mathcal{P}(\delta)$ by letting $f([y]_T)$ be the $<_{HOD_{T,Q}^{L[T,y]}}$–least $X \in (OD_{T,Q}^{L[T,y]} \cap \mathcal{P}(\delta)) \backslash OD_Q$. We have $f \in OD_Q$ (notice $T \in OD$), and f is constant on a cone. Setting $A =$ the $f([y]_T)$ for a cone of y's, we then get $A \in OD_Q$. Contradiction! □ (2.4)

Definition 6.2.6 *Let M be a premouse with largest cardinal $\alpha \in M$. Then M is called* full *if for all $N \rhd \mathcal{J}_\alpha^M$ s.t. $\mathcal{J}_\alpha^M$ is a cutpoint in N and N is $\Omega+1$ iterable above α do we have that $\mathcal{J}_{\alpha^{+N}}^N \trianglelefteq M$.*

We shall need the following key consequence of the Mouse Set Theorem 3.4.7.

Lemma 6.2.7 *Let M be full with largest cardinal α. Then $OD_M \cap \mathcal{P}(\alpha) \subset M$.*

In the light of the Mouse Set Theorem 3.4.7, Lemma 6.2.7 is a trivial consequence of $\forall\alpha\ W_\alpha^*$. We shall defer the proof of $\forall\alpha\ W_\alpha^*$ to the end of this section, after the proof of Lemma 6.2.14.

Lemma 6.2.8 *Let $P \in HC$ be such that for a cone $\mathcal{C}$, if $x \in \mathcal{C}$, then $P \in L[T,x]$ and $K^c(P)^{HOD_{T,P}^{L[T,x]}}$ exists. Let $\mathcal{C}' \subset \mathcal{C}$ be given by 6.2.3. Pick $x \in \mathcal{C}'$, and let $Q = Q_x \rhd P$ be as in 6.2.4. Then Q is full.*

Proof: Suppose not. Write $\delta = \delta(Q)$. Let $N \rhd \mathcal{J}_\delta^Q$ be s.t. $\mathcal{J}_\delta^Q$ is a cutpoint in N, N is $\Omega+1$ iterable, and $(\Sigma_\omega(N) \cap \mathcal{P}(\delta)) \backslash Q \neq \emptyset$. By Σ_1^2–correctness of $L[T,x]$, there is one such N in $L[T,x]$, and the least one such is in fact in $HOD_{T,P}^{L[T,x]}$ (recall that $Q \in HOD_{T,P}^{L[T,x]}$).

Let Σ be N's (unique) Ω–iteration strategy. By uniqueness, we have that $\Sigma \operatorname{Res} HOD_{T,P}^{L[T,x]} \in HOD_{T,P}^{L[T,x]}$, so that N is iterable inside $HOD_{T,P}^{L[T,x]}$. But this gives a contradiction with the universality of $K^c(Q)$ inside $HOD_{T,P}^{L[T,x]}$. □ (2.7)

We have therefore established the following.

Corollary 6.2.9 *Let $P \in HC$ be such that*

$$K^c(P)^{HOD_{T,P}^{L[T,x]}}$$

exists for a cone of x. There is then a full $Q \rhd P$ such that Q's largest cardinal is Woodin in Q and

$$K^c(Q)^{HOD_{T,Q}^{L[T,y]}}$$

exists for a cone of y.

Definition 6.2.10 *Let $k < \omega$. We set*

$$([x_0]_T, \ldots, [x_{k-1}]_T) \in A_k$$

if there exists a sequence

$$(Q_i \colon i \in \{-1\} \cup k)$$

and some x_k *such that* $Q_{-1} = \emptyset$, x_0 *is a base for the cone from 6.2.2 (with* $S = T$*), and for all integers* $j < k$, Q_j *is the* $<_{HOD^{L[T,x_j]}_{T,Q_{j-1}}}$ *–least* Q *such that* $Q \triangleright Q_{j-1}$, Q*'s largest cardinal is Woodin in* Q*, and*

$$K^c(Q)^{HOD^{L[T,x]}_{T,Q_{j-1}}}$$

exists for all x *in the cone above* x_{j+1}.

Notice that if $([x_0], \dots, [x_{k-1}]_T) \in A_k$, then there is a unique "Q-sequence" $(Q_i \colon i < k)$ witnessing this.

We let μ_T denote Martin's measure on the T–degrees.

Definition 6.2.11 $(p, U) \in \mathbb{P}$ *iff* U *is a subtree of* $\bigcup_k A_k$ *with stem* p *and for all* $q \in U$ *with* $q \supset p$ *we have that*

$$\{r \in D_T : q^\frown r \in U\} \in \mu_T.$$

$(p', U') \leq_{\mathbb{P}} (p, U)$ *iff* $p' \supset p$, *and* $U' \subset U$.

As any element of A_k comes with its unique "Q-sequence" $(Q_0, \dots, Q_{k-1})$ of Q's, forcing with $\mathbb{P}$ will produce an infinite sequence $\vec{Q} = (Q_0, Q_1, \dots)$ of Q's, to which we'll refer as the "Q–sequence" corresponding to the generic filter.

Lemma 6.2.12 *Let* G *be* $\mathbb{P}$*–generic over* V*, and let* $\vec{Q}$ *be the corresponding* Q*–sequence. Then* $\mathcal{P}(\delta(Q_k)) \cap L[\vec{Q}] \subset Q_k$ *for all* $k < \omega$.

This immediately gives:

Corollary 6.2.13 *If* G *and* $\vec{Q}$ *are as in 6.2.12 then*

$$L[\vec{Q}] \models \text{ there are } \omega \text{ many Woodin cardinals.}$$

Proof of 6.2.12. Let $k < \omega$. By 6.2.7, 6.2.8, and the definition of $\mathbb{P}$, in order to show that $\mathcal{P}(\delta(Q_k)) \cap L[\vec{Q}] \subset Q_k$ it is enough to verify that $\mathcal{P}(\delta(Q_k)) \cap L[\vec{Q}] \subset OD_{Q_j}$ for some $j \geq k$.

Let $\delta = \delta(Q_k)$, and let $X \in \mathcal{P}(\delta) \cap L[\vec{Q}]$. Let $\dot{X}$ be a name for X; we may in fact assume $\dot{X}$ is OD. [X is ordinal definable from $\vec{Q}$, which in turn is definable from the generic filter G. We therefore have a name for X which is ordinal definable from a name for G, i.e, a name for X which is just ordinal definable.] Well, by the Prikry lemma there is some $(p, U) \in G$ deciding all $\check{\alpha} \in \dot{X}$ for $\alpha < \delta$.

Claim. $\alpha \in X$ iff $\exists q \in A_{\text{dom}(p)} \exists W (q$ gives $(Q_0, \ldots, Q_{\text{dom}(p)-1})$ and $(q, W) \Vdash \check{\alpha} \in \dot{X})$.

Proof: "$\Rightarrow$": trivial.
"$\Leftarrow$": Notice $(p, U \cap W)$ and $(q, U \cap W)$ are both conditions, and we may find $\mathbb{P}$–generics G' and G'' both giving the same Q–sequence and s.t. $(p, U \cap W) \in G'$ and $(q, U \cap W) \in G''$. But then

$$\dot{X}^{G'} = \dot{X}^{G''},$$

as this interpretation only depends on the Q–sequence, and hence

$$(p, U) \Vdash \check{a} \in \dot{X} \Leftrightarrow (q, W) \Vdash \check{a} \in \dot{X}.$$

□ (Claim)

But this shows $X \in OD_{Q_{\text{dom}(p)-1}}$, and thus the lemma. □ (2.10)

Lemma 6.2.14 *There is Q as in 6.2.4 s.t. moreover, setting $W = (K^c(Q))^{HOD^{L[T,P]}_{T,P}}$ the real x is $\mathbb{P}^W_{\delta(Q)}$–generic over W.*

Corollary 6.2.15 *If in 6.2.10 we replace "6.2.4" by "6.2.14" then still*

$$L[\vec{Q}] \models \text{ there are } \omega \text{ many Woodins},$$

but also there is G^* being $Col(\omega, \Omega)$–generic over $L[\vec{Q}]$ s.t.

$$\mathbb{R}^V = \bigcup_i \mathbb{R}^{L[\vec{Q}][G^* \text{ Res } \delta(Q_i)]},$$

i.e., $V = L(\mathbb{R})$ is a derived model of $L[\vec{Q}]$.

Proof of 6.2.14. Set $W = (K^c(P))^{HOD^{L[T,P]}_{T,x}}$.

Case 1. $W \models$ there is a Woodin $> P \cap OR$.

Let δ be such a Woodin. Then $\delta \in Q \backslash P$, and x is $\mathbb{P}^W_{\delta(Q)}$–generic over W, which easily follows from the fact that we require extenders with critical point κ to have certificates when being put onto the K^c–sequence.

Case 2. $W \models$ there is no Woodin $> P \cap OR$.

In this case we have to go a bit deeper into [37].

Inside $HOD^{L[T,P]}_{T,x}$, let Σ be the following strategy for the good player in the iteration game on $K^c(P)^{HOD^{L[T,P]}_{T,x}}$ above P; if $\mathcal{T}$ has limit length then pick a cofinal branch coming with a weakly iterable Q–structure, i.e., pick b s.t. there is $\mathcal{M}(\mathcal{T}) \trianglelefteq Q \trianglelefteq \mathcal{M}^{\mathcal{T}}_b$ bith all collapses of countable substructures of Q being $\omega_1 + 1$ iterable above $\delta(\mathcal{T})$. Standard arguments show that in fact there is nothing to pick, i.e., there is always only at most one such branch.

By 6.2.3, however, Σ cannot be an iteration strategy for $K^c(P)^{HOD^{L[T,P]}_{T,x}}$ above P (inside $HOD^{L[T,P]}_{T,x}$). A few more standard arguments then show that there is an iteration tree $\mathcal{T} \in HOD^{L[T,P]}_{T,x}$ on $W, \mathcal{T}$ being above P, s.t. $\mathcal{T}$ was formed by following $\Sigma, \mathcal{T}$ has limit length, and there is no weakly iterable Q–structure for $\mathcal{M}(\mathcal{T})$. This of course implies that

$$K^c(\mathcal{M}(\mathcal{T})) \models \delta(\mathcal{T}) \text{ is Woodin},$$

as initial segments of $K^c(\mathcal{L}(\mathcal{T}))$ are $\omega_1 + 1$ iterable.

Set $\mathcal{M} = \mathcal{M}(\mathcal{T})$. Working inside $L[T, x]$, we now define a simple iteration tree $\mathcal{U}$ on $\mathcal{M}$ as follows. (A) At successor steps, hit the least extender ($>$ the largest Woodin below, if there is one) s.t. there is a real which doesn't satisfy the associated axiom. (B) At limit stages we pick the (unique!) cofinal branch coming with a weakly iterable Q–structure.

Notice that $\mathcal{U} \in HOD^{L[T,P]}_{T,x}$. Let's work inside $HOD^{L[T,P]}_{T,x}$.

Case 2a. $\mathcal{M}^{\mathcal{U}}_\alpha$ exists, but there is no extender as in (A).

Let $\delta' = \mathcal{M}^{\mathcal{U}}_\alpha \cap OR$. In this case, to prove 6.2.14 it clearly suffices to verify the

Claim. $K^c(\mathcal{M}^{\mathcal{U}}_\alpha)$ exists and $\models \delta'$ is Woodin.

Proof: Let Θ be large enough, and pick an elementary $\pi : N \to V_\Theta$ with N countable and transitive and all sets of current interest are in $ran(\pi)$.

In V (which is $HOD_{T,x}^{L[T,P]}$ for the moment!) there is a maximal branch b thru $\overline{\mathcal{T}}$ together with a realization map $\sigma : C_k(\mathcal{N}_\xi)$, with $C_k(\mathcal{N}_\xi)$ being from the $K^c(P)$–construction. Let $\overline{Q} \trianglelefteq \mathcal{M}_b^{\overline{\mathcal{T}}}$ be the Q–structure for $\mathcal{M}(\overline{\mathcal{T}} \operatorname{Res} \sup(b))$ provided by $\mathcal{M}_b^{\overline{\mathcal{T}}}$.

By the usual argument of comparing $\overline{Q}$ with the Q–structure provided by $\mathcal{M}_{\sup(b)}^{\overline{\mathcal{T}}}$ is $\sup(b) < lh(\overline{\mathcal{T}})$ we in fact get that b is cofinal thru $\overline{\mathcal{T}}$.

We may now view $\overline{\mathcal{U}}$ as a tree acting on $\overline{Q}$ (instead of just on $\mathcal{M}(\overline{\mathcal{T}})$, so that we get a map

$$\tilde{\pi}\overline{Q} \to \tilde{Q}$$

extending the iteration map

$$\pi_{0\overline{\alpha}}^{\overline{U}} : \mathcal{M}(\overline{\mathcal{T}}) \to \mathcal{M}_{\overline{\alpha}}^{\overline{\mathcal{U}}},$$

together with a realization $\sigma' : \tilde{Q} \to C_k(\mathcal{N}_\xi)$ with $\sigma \operatorname{Res} \overline{Q} = \sigma' \circ \tilde{\pi}$.

Now let $\mathcal{N}_\eta$ be the least model from the $K^c(\mathcal{M}_\alpha^{\mathcal{U}})$–construction with the property that $\rho_\omega(\mathcal{N}_\eta) < \delta'$ or δ' is not definably Woodin over $\mathcal{N}_\eta$. Then, as usual $\tilde{Q} = \pi^{-1}(\mathcal{N}_\eta)$, so that $\tilde{Q} \in N$. But then

$$\overline{Q} \simeq h^{\tilde{Q}}(ran(\pi_{0\overline{\alpha}}^{\overline{\mathcal{U}}}) \cup \{p\}), \text{ some } p,$$

and $\tilde{\pi}$ is the inverse of the transitive collapse. Hence both $\overline{Q}$ and $\tilde{\pi}$ are elements of N.

We have shown that $\tilde{\pi} : \overline{Q} \to \tilde{Q}$ exists in N. Moreover $\overline{Q}$ is weakly iterable in N (as $\mathcal{N}_\eta$ is weakly iterable in V). This implies $\overline{Q}$ is weakly iterable in N. By elementarity, then, $\pi(\overline{Q})$ is weakly iterable in V, so that $\mathcal{M}(\mathcal{T})$ admits a weakly iterable Q–structure.

We have reached a contradiction! □ (Claim)

Case 2b. V (which is still $HOD_{T,x}^{L[T,P]}$ here) doesn't see a Q–structure for the common part model.

We then have

$$W' = K^c(\mathcal{M}(\mathcal{T})) \models \delta(\mathcal{T}) \text{ is a Woodin cardinal.}$$

But then by the construction of $\mathcal{U}$, the extenders of W' witnessing Woodinness of $\delta(\mathcal{T})$ in W' all satisfy the desired axiom. □

As discussed after the statement of Lemma 6.2.7, we are now left with having to prove that $\forall\alpha\, W_\alpha^*$.

Let us fix α, and suppose W^*_α to hold, where α is critical. We need to see that $W^*_{\alpha+1}$ holds. By the Witness Dichotomy this means that we need to see that for all $n < \omega$, J^n_α is total on $\mathbb{R}$.

Suppose that $\mathbb{R}$ is closed under J^n_α. We need to see that $\mathbb{R}$ is closed under J^{n+1}_α.

Let us first consider the inadmissible case. Let us fix $a \in \mathbb{R}$. By Lemma 6.2.3 above, there is a cone of $b \in \mathbb{R}$ such that

$$W_a(b) = (K^c(b))^{HOD^{L[T,b]}_{T,a}}$$

cannot be $\Omega + 1$ iterable inside $HOD^{L[T,b]}_{T,a}$. We may therefore define

$$b \mapsto J^{n+1}_\alpha(a)^b,$$

for a cone of b, where $J^{n+1}_\alpha(a)^b$ is a version of $J^{n+1}_\alpha(a)$ from the point of view of $HOD^{L[T,b]}_{T,a}$. In particular, $J^{n+1}_\alpha(a)$ from the point of view of $HOD^{L[T,b]}_{T,a}$ is $\omega_1 + 1$ iterable from the point of view of $HOD^{L[T,b]}_{T,a}$.

We'll have to use AD again (or rather the fact that Martin's measure on the Turing degrees $\mathcal{D}$ is a σ–complete ultrafilter) in order to get the true J^{n+1}_α. Let us consider $f : \mathcal{D} \to \mathbb{R}$ given by

$$[b] \mapsto \text{ a canonical real code for } J^{n+1}_\alpha(a)^b.$$ [1]

For each $n < \omega$, the set $\{b \in \mathbb{R} : n \in f([b])\}$ either contains a cone or is disjoint from a cone. Let $n \in \mathcal{P}$ iff for a cone of b, $n \in f([b])$. Then $f([b]) = \mathcal{P}$ on a cone of b.

It is straightforward to see that $\mathcal{P}$ is in fact ω_1 iterable (in V): if $\mathcal{T}$ is a countable tree on $\mathcal{P}$ of limit length, then the good branch through $\mathcal{T}$ is the one picked by the strategies of $HOD^{L[T,b]}_{T,a}$ for a cone of b. We have therefore found our desired J^{n+1}_α.

Let us now consider the end of gap case. Let $\mathcal{N}$ be suitable such that there is an ω_1 iteration strategy Σ for $\mathcal{N}$ which is witnesses that $\mathcal{N}$ is $\mathcal{A}$-iterable and condenses well. (Here, $\mathcal{A}$ is a sjs consisting of $\mathrm{OR}^{<\alpha}_z$ sets of reals, some $z \in \mathbb{R}$.)

We have the following analogon of Lemma 6.2.3.

[1] The fact that $\rho_\omega(J^{n+1}_\alpha(a)^b) = \omega$ means that $J^{n+1}_\alpha(a)^b$ comes with a canonical real code for itself.

Lemma 6.2.16 *Let $a \in \mathbb{R}$, and suppose that*

$$W_b = (K^{c,\Sigma}(a, \mathcal{N}))^{HOD^{L[T,\Sigma,b]}_{T,\Sigma,a}},$$

constructed with height Ω, exists for a cone of b. Then there is a cone of b such that W_b cannot be $\Omega + 1$ iterableinside $HOD^{L[T,\Sigma,x]}_{T,\Sigma,P}$.

Using this lemma, the rest is exactly as in the inadmissible case.

Chapter 7

A model of AD plus $\Theta_0 < \Theta$

7.1 The set-up

The core model induction will now take us beyond $L(\mathbb{R})$. Under AD, for any α, the sets of reals in $J_\alpha(\mathbb{R})$ comprise an initial segment of the Wadge hierarchy. It will be the Wadge hierarchy and the associated Solovay sequence which guides the induction beyond $L(\mathbb{R})$.

Let us recall the relevant definition.

Definition 7.1.1 *For A, $B \subset \mathbb{R}$, we say that A is* Wadge reducible to B*, in short: $A \leq_w B$, iff there is a continuous $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$, $x \in A$ iff $f(x) \in B$.*

Under AD, Wagde's Lemma says that for all A, $B \subset \mathbb{R}$, either $A \leq_w B$ or $B \leq_w \mathbb{R}\setminus A$. Identifying A with $\mathbb{R}\setminus A$, AD yields that $\leq_w$ is a prewellorder on the sets of reals. The *Wagde rank* $|A|_w$ of $A \subset \mathbb{R}$ is the rank of A according to this prewellorder.

If AD holds, then we shall write $P_\alpha(\mathbb{R})$ for the set of sets of reals of Wadge rank less than α.

Definition 7.1.2 *Suppose that AD holds. The* Solovay sequence *of ordinals θ_i, $i \geq 0$, is defined as follows.*

1. *$\theta_0 = \sup\{\ |A|_w\ :\ A$ is OD_x for some $x \in \mathbb{R}\ \}$.*

2. *If θ_i is defined and if there is some A with $|A|_w = \theta_i$, then θ_{i+1} is defined as $\theta_{i+1} = \sup\{\ |A|_w\ :\ A$ is $\mathsf{OD}_{x,A}$ for some $x \in \mathbb{R}\ \}$, where A*

is arbitrary such that $|A|_w = \theta_i$ (the value of θ_{i+1} is independent from the particular choice of A).

3. *If λ is a limit ordinal and if θ_i is defined for every $i < \lambda$, then θ_λ is defined as* $\sup\{ \theta_i : i < \lambda \}$.

4. *If $i \geq 0$ is least such that θ_{i+1} is undefined, then we write $\theta = \theta_i$.*

In $L(\mathbb{R})$, for every set A whatsoever there is some $x \in \mathbb{R}$ such that A is OD_x. Therefore, if $L(\mathbb{R}) \models \mathsf{AD}$, then $L(\mathbb{R}) \models \theta_0 = \theta$. On the other hand if $\mathsf{AD} + \theta_0 = \theta$ holds, then for every $A \subset \mathbb{R}$ there is some $x \in \mathbb{R}$ such that A is OD_x: this is easily seen by first picking an OD_y-set B, some $y \in \mathbb{R}$, of the same Wadge rank as A and then observing that A is computable from B using some real.

Our goal will now be to show that, granted CH plus the existence of certain kind of homogenous ideal on ω_1, there is a nontame mouse (and a bit more). More precisely, let HI be the conjunction of:

1. CH,
2. There is a homogeneous poset $\mathbb{P}$ such that whenever G is $\mathbb{P}$-generic, then in $V[G]$ there is an elementary embedding $j \colon V \to M$, where M is transitive, such that
   (a) $\mathrm{crit}(j) = \omega_1^V$ and M is closed under ω-sequences in $V[G]$,
   (b) $j \restriction \mathrm{OR}$ is amenable to V, i.e., $\forall\alpha\ (j \restriction \alpha \in V)$, and
   (c) for any countable set X of ordinals, and any OD^{M_0} set of ordinals A, there is a transitive set R such that $X, A \in R$, and $R \models \mathsf{ZFC}+$ "ω_1^V is a measurable cardinal."

The model M_0 which is mentioned in item (c) will be defined in the next section as the "maximal model of $\mathsf{AD}^+ + \theta_0 = \theta$." Item (c) will only be used in the proof of Lemma 7.9.7. We suspect that it can be avoided entirely, but we do not know how to do so.

In what follows, we shall fix some G which is $\mathbb{P}$-generic over V, and we'll write $j \colon V \to M$ for the induced generic embedding. We have that $\mathrm{crit}(j) = \omega_1^V$, ${}^\omega M \subset M$ in $V[G]$, and $\forall\alpha\ (j \restriction \alpha \in V)$ by HI.

Our goal is to show

Theorem 7.1.3 (Ketchersid [13]) *If HI holds, then there is an inner model of $\mathsf{AD}^+ + \theta_0 < \theta$ containing all real and ordinals, and consequently, there is a nontame mouse.*

In the presence of AD^+, $\theta_0 < \theta$ is equivalent to "all Π^2_1 sets are Suslin". We shall therefore assume throughout our argument that there is no inner model $\mathsf{AD}+\mathsf{DC}+$ "all sets are Suslin" (i.e. no inner model of $\mathsf{AD}_{\mathbb{R}}+\mathsf{DC}$) having all the reals and ordinals. We shall use the following two consequences, Theorems 7.1.4 and 7.1.5, of this fairly quickly.

Theorem 7.1.4 (Woodin) *Suppose N and M are transitive models of AD^+ containing all reals and ordinals such that neither $P(\mathbb{R}) \cap N \subseteq M$ nor $P(\mathbb{R}) \cap M \subseteq N$ holds; then $L(P(\mathbb{R}) \cap M \cap N) \models \mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$.*

Theorem 7.1.4 can also be phrased as saying that if there is no transitive model of $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$ containing all the reals and ordinals, and if N and M are transitive models of AD^+ containing all reals and ordinals, then either $\mathcal{P}(\mathbb{R}) \cap N = P_{\theta^N}(\mathbb{R})^M$ or else $\mathcal{P}(\mathbb{R}) \cap M = P_{\theta^M}(\mathbb{R})^N$.

Our second consequence of the nonexistence of a model of $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$ is that it implies the "Mouse Set Conjecture."

Theorem 7.1.5 (Woodin) *Assume $\mathsf{AD}^+ +$ there is no inner model having all reals and ordinals of $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$; then the Mouse Set Conjecture (MSC) holds.*

7.2 A maximal model of $\theta_0 = \theta$.

We start with a key definition.

Definition 7.2.1 $\Gamma_0 = \{A \subset \mathbb{R} : \ L(A, \mathbb{R}) \models \mathsf{AD} \ + \ \theta_0 = \theta\}$.

Lemma 7.2.2 $\Gamma_0 \neq \emptyset$.

Proof. We have shown that $L(\mathbb{R}) \models \mathsf{AD}$. As $L(\mathbb{R}) \models \theta_0 = \theta$, we have the lemma. □

Our goal is to produce a set B of reals such that $L(B, \mathbb{R}) \models \mathsf{AD} + \theta_0 < \theta$.

We first need to see that every set of reals in $L(\Gamma_0, \mathbb{R})$ is captured by an $\mathbb{R}$-mouse, so as to have a neat characterization of $L(\Gamma_0, \mathbb{R})$.

Definition 7.2.3 $K(\mathbb{R}) = L(\bigcup\{\mathcal{M} \colon \mathcal{M}$ *is an* $\mathbb{R}$-*mouse*[1] *with* $\rho_\omega(\mathbb{R}) = \mathbb{R}\})$.

[1]i.e., collapses of countable substructures are $\omega_1 + 1$-iterable

Lemma 7.2.4 $\Gamma_0 \subset K(\mathbb{R})$. *Moreover, if* $A \in \Gamma_0$, *then* $A \in K(\mathbb{R})$ *is witnessed by an* $\mathbb{R}$*-premouse* $\mathcal{N}$ *such that the* ω_1*-iteration strategies for collapses of countable substructures of* $\mathcal{N}$ *are in* $L(A, \mathbb{R})$.

Proof. Fix $A \in \Gamma_0$. By CH, there is a generic bijection $b\colon \omega \to \mathbb{R}^V$ in $V[G]$. Let $A_b \in V[G]$ be the real coding A relative to b, i.e., thinking of A_b as a subset of ω, $n \in A_b$ iff $b(n) \in A$. We have that $L(A, \mathbb{R}) \models \mathsf{AD} + \theta_0 = \theta$, so that A is $\mathsf{OD}_z^{L(A,\mathbb{R})}$ for some $z \in \mathbb{R}$, and then

$$A = j(A) \cap \mathbb{R}^V \in \mathsf{OD}_{z,\mathbb{R}^V}^{L(j(A),\mathbb{R}^{V[G]})}.$$

Therefore,

$$A_b \in \mathsf{OD}_b^{L(j(A),\mathbb{R}^{V[G]})}.$$

As $L(j(A), \mathbb{R}^{V[G]}) \models \mathsf{AD} + \theta_0 = \theta$, by the Mouse Set Theorem 7.1.5, $L(j(A), \mathbb{R}^{V[G]})$ contains a b-mouse $\mathcal{M}$ with $A_b \in \mathcal{M}$ and some Σ such that

$$L(j(A), \mathbb{R}^{V[G]}) \models \Sigma \text{ is an } \omega_1\text{−iteration strategy for } \mathcal{M}.$$

There is then some $\mathbb{R}^V$-premouse $\mathcal{N} \in L(j(A), \mathbb{R}^{V[G]})$ such that $A \in \mathcal{N}$ and

$$L(j(A), \mathbb{R}^{V[G]}) \models \Sigma \text{ is an } \omega_1\text{−iteration strategy for } \mathcal{N}.$$

We may and shall assume that $\rho_\omega(\mathcal{N}) = \mathbb{R}$

Clearly, $\mathcal{N}$ is now $\mathsf{OD}^{L(j(A),\mathbb{R}^{V[G]})}$, where $L(j(A), \mathbb{R}^{V[G]}) \models \mathsf{AD}^+$. Let us write $\beta = \theta^{L(j(A),\mathbb{R}^{V[G]})}$. By Theorem 7.1.4, applied inside $V[G]$, $L(j(A), \mathbb{R}^{V[G]})$ is equal to $L(P_\beta)(\mathbb{R})$, as computed in *any* inner model of $V[G]$ of AD^+ which contains all the reals and whose θ is at least β. This readily gives that $\mathcal{N}$ is $\mathsf{OD}_{\mathbb{R}^V}$ in $V[G]$. Hence by the homogeneity of $\mathbb{P}$, $\mathcal{N} \in V$.

If $\pi\colon \bar{\mathcal{N}} \to \mathcal{N}$ is in V and countable there, then $\Sigma \restriction V \in V$ witnesses that $\bar{\mathcal{N}}$ is $< j(\omega_1^V)$-, and hence $\omega_1 + 1$-iterable in V. Also, by elementarity, $L(A, \mathbb{R}) \models$ "$\bar{\mathcal{N}}$ is ω_1-iterable" (and this is witnessed by $\Sigma \restriction \mathsf{HC}^V$). Therefore, $A \in K(\mathbb{R})$, and this is witnessed by an $\mathbb{R}$-premouse $\mathcal{N}$ such that the ω_1-iteration strategies for collapses of countable substructures of $\mathcal{N}$ are in $L(A, \mathbb{R})$. □

Definition 7.2.5 $M_0 = L(\Gamma_0, \mathbb{R})$.

For any $A \subset \mathbb{R}$, the statement "$A \in \Gamma_0$" is absolute between inner models which contain all the reals and A. Therefore, Lemma 7.2.4 gives that in fact $\Gamma_0 \in K(\mathbb{R})$ and hence $M_0 \subset K(\mathbb{R})$.

Lemma 7.2.6 *Both $j(\Gamma_0)$ and $j(M_0)$ are $\mathsf{OD}^{V[G]}$.*

PROOF. It certainly suffices to see that $j(\Gamma_0)$ is $\mathsf{OD}^{V[G]}$. But by Theorem 7.1.4, there is some β such that $A \in j(\Gamma_0) = \Gamma_0^M$ iff $A \in \Gamma_0^{V[G]}$ and $A \in P_\beta(\mathbb{R})^{L(A,\mathbb{R})}$. Notice that whereas we might have $\Gamma_0^{V[G]} \setminus \Gamma_0^M \neq \emptyset$, we must have that $\Gamma_0^{j(M)}$ is a Wadge initial segment of $\Gamma_0^{V[G]}$ by Theorem 7.1.4 applied inside $V[G]$. □

Theorem 7.2.7 $M_0 \models \mathsf{AD}$.

PROOF. By a core model induction, guided by [40]. This uses Lemmas 7.2.4 and 7.2.6. □

The core model induction which is used to prove Theorem 7.2.7 is completely parallel to the the core model induction through $L(\mathbb{R})$ which we did earlier in this book. We do need to get $j(K) \in V$ at each step, where K is the core model of some appropriate local universe P. As in the earlier cases, $j(K) \in V$ comes from $j(P) \in \mathsf{OD}^{V[G]}$. In order to see that we seem to need that our putative $\mathbb{R}$-mouse beyond Γ_0 is countably iterable "over" Γ_0 (as in the conclusion of Lemma 7.2.4).

The same remark applies to the proof of Lemma 7.2.9 below.

Also, we don't know how to show $K(\mathbb{R}) \models \mathsf{AD}$, because of this problem of showing that $j(P) \in \mathsf{OD}^{V[G]}$ for the relevant local universes P.

Corollary 7.2.8 $\Gamma_0 = M_0 \cap \mathcal{P}(\mathbb{R})$.

PROOF. If $A \in (M_0 \cap \mathcal{P}(\mathbb{R})) \setminus \Gamma_0$, then by Theorem 7.2.7 we have that $L(\mathbb{R}, A) \models \mathsf{AD} + \theta_0 < \theta$. □

Lemma 7.2.9 *If S is an $\mathbb{R}$-premouse with $\rho_\omega(S) = \mathbb{R}$ such that every countable $\bar{S}$ which embeds into S has an ω_1-iteration strategy in M_0, then $S \in M_0$.*

PROOF. We first show that $L(S) \models \mathsf{AD}$ by a core model induction, guided by [40] and [44]. Notice that certainly $S \in \mathsf{OD}$, hence $j(S) \in \mathsf{OD}^M$. But this implies that $j(S) \in \mathsf{OD}^{V[G]}$, as in $V[G]$, for some α, $j(S)$ is the unique sound $\mathbb{R}^{V[G]}$-premouse $\mathcal{S}'$ of height α with $\rho_\omega(\mathcal{S}') = \mathbb{R}^{V[G]}$ such that for all countable $\bar{\mathcal{S}}$ which embed into $\mathcal{S}'$ there is some $B \in \Gamma_0^{V[G]}$ with $L(B, \mathbb{R}^{V[G]}) \models$ "$\bar{\mathcal{S}}$ is ω_1-iterable." This is because if $\bar{\mathcal{S}}$ and $\bar{\mathcal{S}}^*$ are countable premice and $B \in \Gamma_0^{V[G]}$ and $C \in \Gamma_0^{V[G]}$ are such that $L(B, \mathbb{R}^{V[G]}) \models$ "$\bar{\mathcal{S}}$ is ω_1-iterable" and $L(C, \mathbb{R}^{V[G]}) \models$ "$\bar{\mathcal{S}}^*$ is ω_1-iterable," then $L(B, C, \mathbb{R}^{V[G]}) \models \mathsf{AD}$ and $\bar{\mathcal{S}}$ and $\bar{\mathcal{S}}^*$

can be compared in $L(B, C, \mathbb{R}^{V[G]})$. The fact that $j(S) \in \mathsf{OD}^{V[G]}$ is crucial in the verification of $L(S) \models \mathsf{AD}$.

But we must also have that $L(S) \models \theta_0 = \theta$, because otherwise we have that $L(S) \models \mathsf{AD} + \theta_0 < \theta$. Therefore, the master code for S is in Γ_0, and hence $S \in M_0$. □

Here is another useful characterization of M_0, or rather of its sets of reals.

Lemma 7.2.10 $\mathcal{P}(\mathbb{R}) \cap M_0 = \mathcal{P}(\mathbb{R}) \cap \mathsf{HOD}^{j(M_0)}_{\mathbb{R}^V}$.

PROOF. "$\subset$": Let $A \in \mathcal{P}(\mathbb{R}) \cap M_0 = \Gamma_0$. By Lemma 7.2.4, there is some countably iterable $\mathbb{R}$-premouse $\mathcal{N}$ with $\rho_\omega(\mathcal{N}) = \mathbb{R}$ such that $A \in \mathcal{N}$ and if $\bar{\mathcal{N}} \in V$ is countable in V and embeds into $\mathcal{N}$, then $L(A, \mathbb{R}) \models$ "$\bar{\mathcal{N}}$ is ω_1^V-iterable." By Lemma 7.2.9, we then have that $\mathcal{N} \in M_0$. But then $M_0 \models$ "$\mathcal{N}$ is countably iterable," hence $j(M_0) \models$ "$j(\mathcal{N})$ is countably iterable." This implies that $j(M_0) \models$ "$\mathcal{N}$ is countably iterable." But then $A \in \mathsf{OD}^{j(M_0)}_{\mathbb{R}^V}$.

"$\supset$": Let $A \subset \mathbb{R}$, $A \in \mathsf{OD}^{j(M_0)}_{\mathbb{R}^V}$. By the Mouse Set Theorem 7.1.5, there is some $\mathbb{R}$-premouse $\mathcal{N}$ such that $A \in \mathcal{N}$ and $j(M_0) \models$ "$\mathcal{N}$ is $\omega_1^{V[G]}$-iterable.". We have that $\mathcal{N} \in V$ by Lemma 7.2.6 and the homogeneity of $\mathbb{P}$. Also, if $\bar{\mathcal{N}} \in V$ is countable in V and $\bar{\mathcal{N}}$ embeds into $\mathcal{N}$ (equivalently in V or in $j(M_0)$), then $j(M_0) \models$ "$\bar{\mathcal{N}}$ is ω_1-iterable," and hence $M_0 \models$ "$\bar{\mathcal{N}}$ is ω_1-iterable." We may now use Lemma 7.2.9 to conclude that $A \in M_0$. □

Lemma 7.2.11 *$M_0 \models$ "$\theta_0 = \theta$."*

PROOF. Let A be a set of reals in M_0. There is then an $\mathbb{R}$-premouse $\mathcal{N}$ with $A \in \mathcal{N}$ as in Lemma 7.2.4. But then $\mathcal{N} \in M_0$ by Lemma 7.2.9. We get that $\mathcal{N}$ is OD^{M_0} and A is $\mathsf{OD}^{M_0}_z$ for some real z. It follows that $M_0 \models$ "$\theta_0 = \theta$." □

We have shown that the union of all models $L(A, \mathbb{R})$ of $\mathsf{AD} + \theta_0 = \theta$ is itself a model of (ZF plus) $\mathsf{AD} + \theta_0 = \theta$ (and it is equal to M_0).

Here is something for the record.

Lemma 7.2.12 *(a) Let γ be the Wadge ordinal of Γ_0; then $j(\Gamma_0)$ and $j(M_0)$ are definable in $V[G]$ from $j(\gamma)$, uniformly in G. Thus if G, H are $\mathbb{P}$-generics such that $V[G] = V[H]$, then $j_G(\Gamma_0) = j_H(\Gamma_0)$ and $j_G(M_0) = j_H(M_0)$.*

(b) If x is HOD^{M_0}, then $j(x) \in V$, and $j(x)$ is independent of G.

(c) $HOD^{j(M_0)}_{\mathbb{R}^V}$ *is contained in* V*, and is independent of* G*.*

Definition 7.2.13 $\Gamma = (\Sigma^2_1)^{M_0}$.

We now go for a Γ-suitable premouse N. In fact, we want more, cf. Theorem 7.4.1 below. One starting point for this is $\mathsf{HOD}^{j(M_0)}_{\mathbb{R}^V}$. Notice that $\mathsf{HOD}^{j(M_0)}_{\mathbb{R}^V}$ is amenable to V by Lemma 7.2.6.

7.3 The HOD of M_0 up to its θ.

Definition 7.3.1 $\mathsf{H}_0 = \mathsf{HOD}^{M_0}|\theta^{M_0}$.[2]

Theorem 7.3.2 (Woodin) *Assume* AD *plus the mouse set conjecture* MSC[3]*. Then* $\mathsf{HOD}|\theta_0$ *is a direct limit of premice; in particular, it is itself a premouse.*

Theorem 7.3.3 (Woodin) *Assume* AD^+*. Let* $A \subset \theta_0$*; then there is some* $\kappa < \theta_0$ *such that* κ *is* A*-reflecting*[4] *in* θ_0 *via measures, i.e., for all* $\gamma < \theta_0$*, there is a measure* μ *on* κ *such that* $i_\mu(\kappa) > \gamma$ *and* $i_\mu(A) \cap \gamma = A \cap \gamma$*.*

Because under AD all measures are OD (by a result of Kunen), this immediately gives:

Corollary 7.3.4 (Woodin) *Assume* AD^+*. If* $S \subset \mathrm{OR}$*, then* $\mathsf{HOD}_S \models$ *"*θ_0 *is a Woodin cardinal."*

Lemma 7.3.5 *In* $j(M_0)$*,* H_0 *is full in the following sense: if* $X \subset \theta_0^{M_0}$ *is bounded,* $X \in \mathsf{OD}^{j(M_0)}_{H_0}$*, then* $X \in H_0$*.*

PROOF. Deny. Then by MSC, in $j(M_0)$ there is some ω_1-iterable $\mathcal{Q} \trianglerighteq \mathsf{H}_0$ such that $\rho_\omega(\mathcal{Q}) < \theta^{M_0}$. We have that $H_0 \in \mathsf{HOD}^{j(M_0)}_{\mathbb{R}^V}$, because $M_0 = L(\mathcal{P}(\mathbb{R}) \cap M_0)$ (by Corollary 7.2.8) $= L(\mathcal{P}(\mathbb{R}) \cap \mathsf{HOD}^{j(M_0)}_{\mathbb{R}^V})$ (by Lemma 7.2.10). But $\mathcal{Q}$ is $\mathsf{OD}^{j(M_0)}_{\mathsf{H}_0}$, so that $\mathcal{Q} \in \mathsf{HOD}^{j(M_0)}_{\mathbb{R}^V}$.

Now θ^{M_0} is regular in $\mathsf{HOD}^{j(M_0)}_{\mathbb{R}^V}$. This is because $\mathsf{HOD}^{j(M_0)}_{\mathbb{R}^V} = L(S, \mathbb{R}^V)$ for some set S of ordinals, $L(S, \mathbb{R}^V) \models$ "θ is regular," and $\theta^{L(S,\mathbb{R}^V)} = \theta^{M_0}$

[2] I.e., $\mathsf{H}_0 = V^{\mathsf{HOD}^{M_0}}_{\theta^{M_0}}$

[3] i.e., the conclusion of Theorem 7.1.5

[4] i.e., A-strong

(by Lemma 7.2.10). We may thus pick some $\pi\colon \bar{\mathcal{Q}} \to \mathcal{Q}$ with critical point $\pi^{-1}(\theta^{M_0})$, call it $\bar{\theta}$, such that $\bar{\theta}$ is a cardinal of H_0, π, $\bar{\mathcal{Q}} \in \mathsf{HOD}^{j(M_0)}_{\mathbb{R}^V}$, and $\rho_\omega(\mathcal{Q}) < \bar{\theta}$. Via the Coding Lemma, $\bar{\mathcal{Q}}$ is then coded by a set of reals in $\mathsf{HOD}^{j(M_0)}_{\mathbb{R}^V}$ (i.e., in M_0, cf. Lemma 7.2.10), so that $\bar{\mathcal{Q}} \in M_0$.

We claim that $M_0 \models$ "$\bar{\mathcal{Q}}$ is countably iterable." Well, if $\sigma\colon \mathcal{P} \to \bar{\mathcal{Q}}$, where $\mathcal{P} \in V$ is countable there, then in $j(M_0)$ there is some $\sigma^*\colon \mathcal{P} \to \mathcal{Q}$ (by the existence of $\pi \circ \sigma$), so that $j(M_0) \models$ "$\mathcal{P}$ is countably iterable." Pulling this back via j gives $M_0 \models$ "$\mathcal{P}$ is countably iterable."

But now $\bar{\mathcal{Q}} \in \mathsf{HOD}^{M_0}$, and we have a contradiction. □

Definition 7.3.6 *Let H_0^+ be the $C^{j(M_0)}_{\Sigma^2_1}$-closure of H_0 up thru ω cardinals.*

Lemma 7.3.7 $\mathsf{H}_0^+ \models$ *"θ^{M_0} is a Woodin cardinal."*

Proof. Let $A \subset \theta^{M_0}$, $A \in \mathsf{OD}^{j(M_0)}_{H_0}$. It suffices to see that θ^{M_0} is Woodin with respect to A, as witnessed by extenders in H_0.

We have that $A \in \mathsf{HOD}^{j(M_0)}_{\mathbb{R}^V}$, and also $A \cap \gamma \in \mathsf{H}_0$ for all $\gamma < \theta^{M_0}$. Let us pick some $\kappa < \theta^{M_0}$ such that κ is A-reflecting as witnessed by measures in $\mathsf{HOD}^{j(M_0)}_{\mathbb{R}^V}$ (cf. Theorem 7.3.3). Each such measure is in M_0 (cf. Lemma 7.2.10), and it is OD^{M_0} by Kunen. Moreover, $(i_\mu \restriction \mathcal{P}(\kappa))^{M_0} = (i_\mu \restriction \mathcal{P}(\kappa))^{\mathsf{HOD}^{j(M_0)}_{\mathbb{R}^V}}$, so that $i_\mu \restriction (\mathcal{P}(\kappa) \cap H_0) \in \mathsf{H}_0 = \mathsf{HOD}^{M_0}|\theta^{M_0}$ for each such measure. It thus easily follows that $\mathsf{H}_0^+ \models$ "κ is A-reflecting in θ^{M_0}."
□

Using j, we may now pull back the true statement $j(M_0) \models$ "there is a Σ^2_1-suitable countable premouse."

We now have:

Theorem 7.3.8 *H_0^+ is suitable in $j(M_0)$.*

7.4 A model of AD plus $\theta_0 < \theta$.

We need to prove more, namely the following.

Theorem 7.4.1 *There is a Γ-suitable countable premouse N together with a Γ-fullness-preserving iteration strategy Σ which condenses well.*

The proof of this result will be presented later. In order to prove Theorem 7.4.1, we need the following information about θ^{M_0}.

Theorem 7.4.2 *(a)* $\theta^{M_0} < j(\omega_1^V)$,

(b) $j"\theta^{M_0}$ *is cofinal in* $j(\theta^{M_0})$, *and*

(c) In V, $\mathrm{cof}(\theta^{M_0}) = \omega$.

Proof. In $j(M_0)$, there are only countably many $\mathsf{OD}_{\mathbb{R}^V}$ subsets of $\mathbb{R}^V$. With the help of Lemma 7.2.10, it thus follows that $\theta^{M_0} < j(\omega_1^V)$.

We now show that $\mathrm{cf}(\theta^{M_0}) = \omega$ (which implies that $j''\theta^{M_0}$ is cofinal in $j(\theta^{M_0})$).

Suppose that $\mathrm{cf}(\theta^{M_0}) > \omega$. Then $j''\theta^{M_0}$ is not cofinal in $j(\theta^{M_0})$. Let $\gamma = \sup(j''\theta^{M_0}) < j(\theta^{M_0})$. Then γ is a (limit of) cardinal(s) in $j(\mathsf{H}_0)$, and we have some initial segment $\mathcal{Q}$ of $j(\mathsf{H}_0)$ which defines a counterexample to the Woodinness of γ.[5] Let $\mathcal{Q}$ the least such premouse.

We have $\mathcal{Q} \in \mathsf{HOD}^{j(M_0)}$, and so $L[\mathcal{Q}, \mathsf{H}_0]|\theta^{M_0} = \mathsf{H}_0$ by Lemma 7.3.5. Moreover, $L[\mathcal{Q}, \mathsf{H}_0] \models$ "θ^{M_0} is a Woodin cardinal," by Lemma 7.3.7.

We get that $\mathcal{Q} \in V$, and the map $j \restriction L[\mathcal{Q}, \mathsf{H}_0] \colon L[\mathcal{Q}, \mathsf{H}_0] \to L[j(\mathcal{Q}), j(\mathsf{H}_0)]$ factors as $j \restriction L[\mathcal{Q}, \mathsf{H}_0] = k \circ i$, where $i \colon L[\mathcal{Q}, \mathsf{H}_0] \to \mathrm{Ult}(L[\mathcal{Q}, \mathsf{H}_0]; E_j \restriction \gamma)$, $\mathrm{crit}(k) \geq \gamma$, and $\sup i''\theta^{M_0} = \sup j''\theta^{M_0}$ (because $E_j|\gamma$ captures this much of j). But then

$$i(\theta^{M_0}) = \sup i''\theta^{M_0} = \gamma,$$

because θ^{M_0} is regular in $L[Q, \mathsf{H}_0]$. Since $k(i(\theta^{M_0})) = \theta^{j(M_0)}$,

$$\mathrm{crit}(k) = i(\theta^{M_0}) = \gamma.$$

Write $\mathrm{Ult} = \mathrm{Ult}(L[\mathcal{Q}, \mathsf{H}_0]; E_j \restriction \gamma)$. Let g be $\mathrm{Col}(\omega, \gamma)$-generic over Ult. In $\mathrm{Ult}[g]$, there is a tree T searching for some premouse S which defines a counterexample to the Woodinness of γ, $S \trianglerighteq i(\mathsf{H}_0)$, such that there is an embedding $S \to i(\mathcal{Q})$. There is a unique branch thru T, namely the one which gives $\mathcal{Q}$. Therefore, $\mathcal{Q} \in \mathrm{Ult}$, which is a contradiction, because $\mathrm{Ult} \models$ "$i(\theta^{M_0})$ is a Woodin cardinal." □

Let N, Σ be a premouse and an ω_1-iteration strategy as in the statement of Theorem 7.4.1. The following Lemmas show how we are going to use Σ to produce our model of $\mathsf{AD} + \theta_0 < \theta$.

Lemma 7.4.3 $\Sigma \notin M_0$.

[5]This follows from a result of Woodin extending Theorem 7.3.4. There is no Woodin cardinal $< \theta_0$ in HOD_S.

PROOF. Suppose that $\Sigma \in M_0$. We have that $M_0 \models \mathsf{AD}$. Let us work in M_0 for the rest of this proof.

Consider the relation $R(z, y)$ iff $z \notin \mathsf{OD}_y$. Let $n \in \mathbb{R}$ code $\mathcal{N}$. We may define a uniformizing function F for R as follows.

Let $y \in \mathbb{R}$. We may then work inside $L[n, y, \Sigma]$ to make y generic over a Σ-iterate $\mathcal{M}$ of $\mathcal{N}$. By the properties of $\mathcal{N}$, Σ we have that if δ is the Woodin cardinal of $\mathcal{M}$, then $\mathcal{P}(\mathcal{M}|\delta) \cap \mathsf{OD}_{M|\delta} \subset M$. Hence if T is the tree for a universal Σ^2_1 set which is derived from a Σ^2_1 scale on it, then $\mathcal{P}(\mathcal{M}|\delta) \cap L[T, \mathcal{M}] \subset \mathcal{M}$. This implies that $\mathcal{P}(\mathcal{M}[y]|\delta) \cap L[T, \mathcal{M}, y] \subset \mathcal{M}[y]$, and hence $\mathcal{P}(\mathcal{M}[y]|\delta) \cap \mathsf{OD}_{M|\delta[y]} \subset M[y]$. Therefore, $\mathcal{M}[y]$ contains every real which is OD_y.

But now $L[n, y, \Sigma]$ contains a real z which codes $\mathcal{P}[y]$. We must then have $z \notin \mathsf{OD}_y$. Moreover, we may let $F(y) = z$, where $(z, \mathcal{M})$ is least in $L[n, y, \Sigma]$ such that y is generic over the Σ-iterate $\mathcal{M}$ of $\mathcal{N}$ and z enumerates $\mathcal{M}[y]$.

F is obviously OD_x for some real x. Then $F(x) \in \mathsf{OD}_x$ as well. But of course we should have $F(x) \notin \mathsf{OD}_x$, as F uniformizes R. Contradiction! $\square$

Theorem 7.4.4 $L(\Sigma, \mathbb{R}) \models \mathsf{AD}$.

PROOF. By a core model induction, using the fact that Σ condenses well. $\square$

Theorem 7.4.5 $L(\Sigma, \mathbb{R}) \models \mathsf{AD} \ + \ \theta_0 < \theta$.

PROOF. Otherwise $\Sigma \in K(\mathbb{R})$ by Lemma 7.2.4. But this contradicts Lemma 7.4.3. $\square$

7.5 The Plan

Our goal is to show that there is a model of $\mathsf{AD}^+ + V = L(P(\mathbb{R}))$ properly including M_0. Suppose N were such a model. We have $\theta^{M_0} = \theta_0^N < \theta^N$ because M_0 was maximal. This also gives $(\Sigma^2_1)^{M_0} = (\Sigma^2_1)^N$. By results of Woodin, there is in N a sjs $\vec{A}$ containing the universal $(\Sigma^2_1)^{M_0}$ set, with each $A_i \in M_0$. (The sequence of A_i cannot be in M_0 since the universal Π^2_1 set is not Suslin in M_0.)

The sequence $\vec{A}$ is a Wadge minimal set of reals not in M_0, so it seems clear that our first step towards N should be to construct an sjs $\vec{A}$ containing the universal $(\Sigma^2_1)^{M_0}$ set, with each A_i in M_0. This is actually all we need to do, since then a core model induction like our $L(\mathbb{R})$ one will give $L(\vec{A}, \mathbb{R}) \models$ AD, so that $N = L(\vec{A}, \mathbb{R})$ will do.

As we have seen, such an sjs $\vec{A}$ determines fullness-preserving, $\vec{A}$-guided strategy Σ for a suitable $(\Sigma_1^2)^{M_0}$-Woodin $\mathcal{N}$, and $\vec{A}$ is in turn determined by $\mathcal{N}$ and Σ. (See Theorem 5.4.14 and Lemma 5.6.11.) We shall obtain $\vec{A}$ by making our way to $\mathcal{N}$ and Σ. Letting $j\colon V \to M \subseteq V[G]$ be as in HI, we proceed as follows:

1. We use j to show that $\mathrm{cof}(\theta^{M_0}) = \omega$. Thus $\mathrm{cof}(\theta^{j(M_0)}) = \omega$ in M, and we can pick a countable family $\mathcal{B}$ of sets of reals Wadge cofinal in $j(M_0)$.

2. Let $H_0 = \mathrm{HOD}^{M_0}|\theta^{M_0}$. We show that in M, H_0 is $(\Sigma_1^2)^{j(M_0)}$-full, and has a fullness-preserving iteration strategy Σ guided by $\mathcal{B}$.

3. We show that, when restricted to some Σ-iterate of H_0, the strategy Σ condenses well.

4. Pulling back to V, we have a $(\Sigma_1^2)^{M_0}$ -suitable $\mathcal{N}$ with an iteration strategy guided by some countable $\mathcal{A}$ which condenses well. We use this to get our sjs $\vec{A}$.

Woodin's theory of approximations to sjs-guided iteration strategies plays a heavy role in steps (2) and (3).

7.6 HOD^{M_0} as viewed in $j(M_0)$

We shall show that HOD^{M_0} yields a mouse which is suitable from the point of view of $j(M_0)$.

Let us define suitability again. (See Chapter 4, §5,6.) Assume AD^+ and MSC for a bit. Let $\Gamma = \Sigma_1^2$. By MSC, we have that for any countable transitive a and $b \subseteq a$, $b \in \mathrm{OD}(a)$ iff $b \in C_\Gamma(a)$ iff b is in some ω_1-iterable a-mouse. Let $\mathrm{Lp}(a)$ be the union of all ω_1-iterable a-mice projecting to a. We let a^+ be the a-mouse given by

Definition 7.6.1 *For any countable, transitive a, let $a^+ = \bigcup_{i<\omega}(\mathcal{M}_i)$, where $\mathcal{M}_0 = a$, and $\mathcal{M}_{i+1} = \mathrm{Lp}(\mathcal{M}_i)$.*

As before, a premouse $\mathcal{N}$ is *suitable* (with respect to Γ) just in case $\mathcal{N}$ is countable, and

(a) $\mathcal{N} \models$ there is exactly one Woodin cardinal, called $\delta^{\mathcal{N}}$,

(b) $\mathcal{N} = (\mathcal{N}|\delta^{\mathcal{N}})^+$, and

(c) If $\xi < \delta^{\mathcal{N}}$ is a cardinal of $\mathcal{N}$, then $\mathrm{Lp}(\mathcal{N}|\xi) \models \xi$ is not Woodin.

Theorem 7.3.8 gives at once

Corollary 7.6.2 *Let $Q \in HOD^{j(M_0)}$; then*

(a) $L[Q, H_0]|\theta^{M_0} = H_0$,

(b) $L[Q, H_0] \models \theta^{M_0}$ is Woodin.

And from this, we get some important confirmation that our plan of constructing an sjs containing the universal $(\Sigma_1^2)^{M_0}$ set and consisting only of sets in M_0 is plausible.

7.7 HOD below θ_0

We have already used

Theorem 7.7.1 (Woodin) *Assume* AD^+ *and* MSC*; then $HOD|\theta_0$ is a premouse.*

We shall need the proof of 7.7.1, as well as its statement. The main ideas are exposited in [43] and [41][§8], but the full proof has never been written up, and so we shall do that here. Let us assume AD^+ plus MSC for the remainder of this section.

7.7.1 Quasi-iterability

We need some material from earlier chapters. See Chapter 4, §5,6.

If $\mathcal{N}$ is suitable and A is an OD set of reals, then for any cardinal μ of $\mathcal{N}$, $\tau^{\mathcal{N}}_{A,\mu}$ is the unique standard $\mathrm{Col}(\omega,\mu)$-term which captures A over $\mathcal{N}$. Notice that if $\mu < \nu$, then $\tau^{\mathcal{N}}_{A,\mu}$ is easily definable over $\mathcal{N}$ from $\tau^{cN}_{A,\nu}$. Let $\vec{A} = \langle A_0, ..., A_{k-1}\rangle$ be a sequence of OD sets of reals, and let ν_k be the k-th cardinal of $\mathcal{N}$ which is $\geq \delta^{\mathcal{N}}$; then

$$\gamma^{\mathcal{N}}_{\vec{A}} = \sup(\{\xi \mid \xi \text{ is definable over } (\mathcal{N}|\nu_{k+2}, \tau^{\mathcal{N}}_{A_0,\nu_k}, ..., \tau^{\mathcal{N}}_{A_k,\nu_k})\}).$$

Let also

$$H^{\mathcal{N}}_{\vec{A}} = \mathrm{Hull}^{\mathcal{N}|\nu_{k+2}}(\gamma^{\mathcal{N}}_{\vec{A}} \cup \{\tau^{\mathcal{N}}_{A_0,\nu_k}, ..., \tau^{\mathcal{N}}_{A_k,\nu_k}\}),$$

where we take the full elementary hull, but without transitively collapsing. Using the regularity of $\delta^{\mathcal{N}}$ in $\mathcal{N}$, we have that

$$H^{\mathcal{N}}_{\vec{A}} \cap \delta^{\mathcal{N}} = \gamma^{\mathcal{N}}_{\vec{A}},$$

so that $\gamma^{\mathcal{N}}_{\vec{A}}$ is the image of $\delta^{\mathcal{N}}$ in the collapse of $H^{\mathcal{N}}_{\vec{A}}$.

From the proof of 5.4.14, we have

Lemma 7.7.2 *Let $\mathcal{T}$ be a maximal tree on a suitable $\mathcal{N}$, with cofinal branches b and c such that $i_b(\delta^{\mathcal{N}}) = i_c(\delta^{\mathcal{N}}) = \delta(\mathcal{T})$. Let $\vec{A} = \langle A_0, ..., A_k\rangle$ be a sequence of OD sets of reals, and suppose*

$$i_b(\tau^{\mathcal{N}}_{A_i,\nu}) = i_c(\tau^{\mathcal{N}}_{A_i,\nu})$$

for all $i \le k$, where ν is the k-th cardinal of $\mathcal{N}$ above $\delta^{\mathcal{N}}$. Then $i_b \restriction H^{\mathcal{N}}_{\vec{A}} = i_c \restriction H^{\mathcal{N}}_{\vec{A}}$.

Definition 7.7.3 *Let $i\colon H^{\mathcal{N}}_{\vec{A}} \to \mathcal{M}$, where $\mathcal{N}$ and $\mathcal{M}$ are suitable, and A is an OD set of reals. We say that i is A-*correct *iff $i(\tau^{\mathcal{N}}_{A,\nu}) = \tau^{\mathcal{M}}_{A,i(\nu)}$. In the case that $\mathcal{M} = \mathcal{M}^{\mathcal{T}}_b = \mathcal{M}(\mathcal{T})^+$ and $i = i^{\mathcal{T}}_b$, we say also that b is A-correct. Finally, correctness with respect to a set or sequence of OD sets of reals means correctness with respect to each member of the set or sequence.*

Corollary 7.7.4 *Let $\vec{A}$ be a sequence of OD sets of reals, and let $\mathcal{T}$ be a maximal tree on a suitable $\mathcal{N}$. Suppose b and c are $\vec{A}$-correct branches of $\mathcal{T}$; then $i_b \restriction H^{\mathcal{N}}_{\vec{A}} = i_c \restriction H^{\mathcal{N}}_{\vec{A}}$.*

Definition 7.7.5 *An iteration tree $\mathcal{T}$ on premouse $\mathcal{N}$ is* Lp-guided *(or Σ^2_1-guided, or OD-guided) just in case for all limit $\lambda < \mathrm{lh}(\mathcal{T})$, $\mathcal{Q}([0,\lambda]_T)$ exists, and $\mathcal{Q}([0,\lambda]) \trianglelefteq \mathrm{Lp}(\mathcal{M}(\mathcal{T}))$.*

So an Lp-guided tree on $\mathcal{N}$ is according to all iteration strategies for $\mathcal{N}$. We have seen that if Σ is an ω_1-strategy for a suitable $\mathcal{N}$, and $\mathcal{T}$ is a normal, Lp-guided tree on $\mathcal{N}$ of limit length, and $b = \Sigma(\mathcal{T})$, then either

(a) $\mathcal{T}$ is short, and $\mathcal{T}^\frown b$ is the unique Lp-guided extension of $\mathcal{T}$, or

(b) $\mathcal{T}$ is maximal, and $\mathcal{M}^{\mathcal{T}}_b = \mathcal{M}(\mathcal{T})^+$.

In case (a) we can define b from $\mathcal{T}$, without referring to Σ, but in case (b) we may not be able to do that. However, in both cases we can define $\mathcal{M}_b^{\mathcal{T}}$ from $\mathcal{T}$, without referring to Σ. This leads to

Definition 7.7.6 *A countable sequence $\langle \mathcal{N}_\alpha \mid \alpha < \beta \rangle$ is* pre-suitable *iff whenever $\alpha + 1 < \beta$, then*

(1) $\mathcal{N}_\alpha$ and $\mathcal{N}_{\alpha+1}$ are suitable, and

(2) there is a normal, Lp*-guided tree $\mathcal{T}$ on $\mathcal{N}_\alpha$ such that either*

(a) $\mathcal{T}$ is short, and $\mathcal{N}_{\alpha+1}$ is the last model of $\mathcal{T}$, or

(b) $\mathcal{T}$ is maximal, and $\mathcal{N}_{\alpha+1} = \mathcal{M}(\mathcal{T})^+$.

Notice that $\mathcal{N}_\alpha$ and $\mathcal{N}_{\alpha+1}$ uniquely determine $\mathcal{T}$. There is nothing in this definition, however, which connects $\mathcal{N}_\lambda$, for λ a limit, to the earlier $\mathcal{N}_\alpha$. We would like some way defining the direct limit of $\langle \mathcal{N}_\alpha \mid \alpha < \lambda \rangle$, given that $\langle \mathcal{N}_\alpha \mid \alpha < \lambda \rangle$ is played according to some iteration strategy for $\mathcal{N}_0$, without fully knowing the branches of maximal trees. For this, we use the approximation lemma 7.7.4.

Let $\langle \mathcal{N}_\alpha \mid \alpha < \beta \rangle$ be pre-suitable. We shall define by induction on $\gamma \leq \beta$:

(1) $\langle \mathcal{N}_\alpha \mid \alpha < \gamma \rangle$ is *suitable*,

(2) for $\xi < \gamma$ and $\vec{A}$ a finite sequence of OD sets of reals: $[\xi, \gamma)$ is $\vec{A}$-*good*,

(3) if $[\xi, \gamma)$ is $\vec{A}$-good, and $\xi \leq \eta < \gamma$, the $\vec{A}$-*guided embedding* $\pi^{\vec{A}}_{\xi,\eta} \colon H^{\mathcal{N}_\xi}_{\vec{A}} \to H^{\mathcal{N}_\eta}_{\vec{A}}$,

(4) for γ a limit: the *quasi–limit* $\operatorname{qlim}_{\alpha<\gamma} \mathcal{N}_\alpha$.

Let $\mathcal{T}_\alpha$ be the unique normal tree leading from $\mathcal{N}_\alpha$ to $\mathcal{N}_{\alpha+1}$, as in pre-suitability.

To begin with, we say that $\langle \mathcal{N}_\alpha \mid \alpha < \gamma \rangle$ is suitable just in case for all limit $\lambda < \gamma$, $\mathcal{N}_\lambda = \operatorname{qlim}_{\alpha<\lambda} \mathcal{N}_\alpha$. Concepts (2)-(4) will only be defined when $\langle \mathcal{N}_\alpha \mid \alpha < \gamma \rangle$ is suitable, so we assume that. It will follow from suitability that our embeddings commute appropriately, in that whenever $\xi < \nu < \mu < \gamma$ and $[\xi, \mu+1)$ is $\vec{A}$-good, then $\pi^{\vec{A}}_{\xi,\mu} = \pi^{\vec{A}}_{\nu,\mu} \circ \pi^{\vec{A}}_{\xi,\nu}$. So we assume that too.

Now let γ be a limit ordinal. We say $[\xi, \gamma)$ is $\vec{A}$-good just in case $[\xi, \eta)$ is $\vec{A}$-good for all $\eta < \gamma$. We say that the quasi-limit $\operatorname{qlim}_{\alpha<\gamma} \mathcal{N}_\alpha$ *exists* iff for every finite sequence $\vec{A}$ of OD sets of reals, there is a $\xi < \gamma$ such that $[\xi, \gamma)$ is $\vec{A}$-good. In this case, we have a direct limit system $\mathcal{F}$ whose indices are pairs $(\xi, \vec{A})$ such that $[\xi, \gamma)$ is $\vec{A}$-good, with the directed ordering $(\xi, \vec{A}) \le (\nu, \vec{B})$ iff $\xi \le \nu$ and $\vec{A}$ is a subsequence of $\vec{B}$, with $H_{\vec{A}}^{\mathcal{N}_\xi}$ being the structure indexed in $\mathcal{F}$ by $(\xi, \vec{A})$, and the map from $H_{\vec{A}}^{\mathcal{N}_\xi}$ to $H_{\vec{B}}^{\mathcal{N}_\nu}$ being $\pi_{\xi,\nu}^{\vec{A}}$. (Note here $H_{\vec{A}}^{\mathcal{N}_\nu} \subseteq H_{\vec{B}}^{\mathcal{N}_\nu}$.) Writing $\mathcal{F} = \mathcal{F}(\langle \mathcal{N}_\alpha \mid \alpha < \gamma \rangle)$, we set

$$\mathrm{qlim}_{\alpha<\gamma} \mathcal{N}_\alpha = \mathrm{dirlim}(\mathcal{F}(\langle \mathcal{N}_\alpha \mid \alpha < \gamma \rangle)).$$

This completes the definitions of (2)-(4) in the case that γ is a limit ordinal.

Next, suppose $\gamma = \eta + 1$, where η is a limit ordinal. We declare that $[\xi, \gamma)$ is $\vec{A}$-good just in case $\xi = \eta$, or $[\xi, \eta)$ is $\vec{A}$-good, and letting $\sigma \colon H_{\vec{A}}^{\mathcal{N}_\xi} \to \mathcal{N}_\eta$ be the map given by $\mathcal{F}(\langle \mathcal{N}_\alpha \mid \alpha < \eta \rangle)$ (whose limit is $\mathcal{N}_\eta$ by suitability), we have that σ is $\vec{A}$-correct. In the latter case, we set $\pi_{\xi,\eta}^{\vec{A}} = \sigma$. This completes the definitions (2)-(4) in the case that γ is the successor of a limit ordinal.

Finally, suppose $\gamma = \eta + 1$, where $\eta = \nu + 1$. We say $[\xi, \gamma)$ is $\vec{A}$-good just in case $\xi = \eta$, or $[\xi, \eta)$ is $\vec{A}$-good, and either

(a) $\mathcal{T}_\nu$ is short, and its canonical embedding $i \colon \mathcal{N}_\nu \to \mathcal{N}_\eta$ is $\vec{A}$-correct, or

(b) $\mathcal{T}_\nu$ is maximal, and has a cofinal, $\vec{A}$-correct branch.

In both cases, we have a canonical embedding $\pi_{\nu,\eta}^{\vec{A}} \colon H_{\vec{A}}^{\mathcal{N}_\nu} \to H_{\vec{A}}^{\mathcal{N}_\eta}$. If $\xi < \nu$, we define $\pi_{\xi,\eta}^{\vec{A}} = \pi_{\nu,\eta}^{\vec{A}} \circ \pi_{\xi,\nu}^{\vec{A}}$. This is all we need to do in the current case. This completes our inductive definitions of (1)-(4).

Definition 7.7.7 *$\mathcal{N}$ is* quasi–iterable *just in case $\mathcal{N}$ is suitable, and every suitable $\langle \mathcal{N}_\alpha \mid \alpha < \gamma \rangle$ of limit length, with $\mathcal{N} = \mathcal{N}_0$, has a suitable quasi-limit. We call the models on any suitable sequence beginning with $\mathcal{N}$* quasi-iterates *of $\mathcal{N}$.*

Definition 7.7.8 *Let $\vec{A}$ be a sequence of OD sets of reals. We say $\mathcal{N}$ is* $\vec{A}$-quasi-iterable *iff $\mathcal{N}$ is quasi-iterable, and whenever $\langle \mathcal{N}_\alpha \mid \alpha < \gamma \rangle$ is a suitable sequence with $\mathcal{N}_0 = \mathcal{N}$, then $[0, \gamma)$ is $\vec{A}$-good. We call the models on any such suitable sequence* $\vec{A}$-quasi-iterates *of $\mathcal{N}$.*

Remark 7.7.9 It is easy to see that any tail end of a suitable sequence is suitable, and any tail end of an $\vec{A}$-good sequence is $\vec{A}$-good. Thus any $\vec{A}$-quasi-iterate of an $\vec{A}$-quasi-iterable mouse is itself $\vec{A}$-quasi-iterable.

We need to strengthen $\vec{A}$-quasi-iterability by adding a Dodd-Jensen property.

Definition 7.7.10 *Let $\vec{A}$ be a sequence of OD sets of reals. We say $\mathcal{N}$ is* strongly $\vec{A}$-quasi-iterable *iff $\mathcal{N}$ is $\vec{A}$-quasi-iterable, and whenever $\mathcal{P}$ is a quasi-iterate of $\mathcal{N}$, and $\langle \mathcal{Q}_\alpha \mid \alpha \leq \beta \rangle$ and $\langle \mathcal{S}_\alpha \mid \alpha \leq \xi \rangle$ are suitable with $\mathcal{Q}_0 = \mathcal{S}_0 = \mathcal{P}$ and $\mathcal{Q}_\beta = \mathcal{S}_\xi = \mathcal{R}$, then the $\vec{A}$-iteration maps from $H^{\mathcal{P}}_{\vec{A}}$ to $\mathcal{R}$ of the two systems are the same.*

The following is central:

Theorem 7.7.11 (Woodin) *Assume AD^+ plus MSC, and let $\vec{A}$ be a sequence of OD sets of reals; then there is a strongly $\vec{A}$-quasi-iterable mouse.*

Remark 7.7.12 Although we do not need it in this section, notice that from 7.7.4 we immediately get: if $\mathcal{A}$ is a collection of OD sets of reals, and $\mathcal{T}$ is a maximal tree on a suitable $\mathcal{N}$ such that $\delta(\mathcal{N}) = \sup(\{\gamma^{\mathcal{N}}_A \mid A \in \mathcal{A}\})$, then $\mathcal{T}$ has at most one cofinal, $\mathcal{A}$-correct branch. This was part of the proof of Theorem 5.4.14. If b is an $\mathcal{A}$-correct branch of $\mathcal{T}$, and $\delta(\mathcal{T}) = \sup(\{\gamma^{\mathcal{N}}_A \mid A \in \mathcal{A}\})$,, then we can apply 7.7.4 to maximal trees on $\mathcal{M}^{\mathcal{T}}_b$, and continue iterating.

We shall use this to show that if $\mathcal{B}$ is a Wadge-cofinal countable collection of OD^{M_0} sets of reals, and $\mathcal{A} = j``\mathcal{B}$, then $\mathcal{A}$ guides a fullness preserving strategy on H^+_0 in this way.

7.7.2 HOD$|\theta_0$ as a direct limit of mice

We now define a direct limit system which will give us HOD$|\theta_0$. It is just the same system we used in the definition of $\mathrm{qlim}_{\alpha<\gamma}\mathcal{N}_\alpha$, but with $\{\mathcal{N}_\alpha \mid \alpha < \gamma\}$ replaced by the family of all quasi-iterable mice. More precisely, let

$$I = \{(\mathcal{N}, \vec{A}) \mid \mathcal{N} \text{ is strongly } \vec{A}\text{-quasi-iterable}\},$$

and order I by

$$(\mathcal{N}, \vec{A}) \leq_I (\mathcal{M}, \vec{B}) \Leftrightarrow \vec{A} \subseteq \vec{B} \text{ and } \mathcal{M} \text{ is a quasi-iterate of } \mathcal{N}.$$

Lemma 7.7.13 *I is directed under the order $\leq_I$.*

Proof. Let $(\mathcal{M}, \vec{A}), (\mathcal{N}, \vec{B}) \in I$. Pick $\mathcal{P}$ such that $(\mathcal{P}, \vec{A}^\frown\vec{B}) \in I$. We now coiterate $\mathcal{M}, \mathcal{N}, \mathcal{P}$, working in the universe $L[T, \mathcal{M}, \mathcal{N}, \mathcal{P}]$, where T is the tree of a Σ^2_1 scale on a universal Σ^2_1 set. As we have argued before, either the comparison succeeds in this universe, or at stage $\omega_1^{L[T,\mathcal{M},\mathcal{N},\mathcal{P}]}$ it produces maximal iteration trees on $\mathcal{M}, \mathcal{N}, \mathcal{P}$. (Quasi-iterability for $\mathcal{M}, \mathcal{N}, \mathcal{P}$ is enough to carry out that argument.) In either case, we get a common quasi-iterate $\mathcal{R}$ of $\mathcal{M}, \mathcal{N}, \mathcal{P}$. Because $\mathcal{R}$ is a quasi-iterate of $\mathcal{P}$, we have that $\mathcal{R}$ is strongly $\vec{A}^\frown\vec{B}$ quasi-iterable. This implies that $(\mathcal{R}, \vec{A}^\frown\vec{B})$ is the desired upper bound in I for $(\mathcal{M}, \vec{A})$ and $(\mathcal{N}, \vec{B})$. □

We regard $(\mathcal{N}, \vec{A}) \in I$ as an index of the structure $H^{\mathcal{N}}_{\vec{A}}$, and the maps

$$\pi_{(\mathcal{N},\vec{A}),(\mathcal{M},\vec{B})} : H^{\mathcal{N}}_{\vec{A}} \to H^{\mathcal{M}}_{\vec{B}}$$

of our system are those given by strong $\vec{A}$-quasi-iterability. ("Strong" implies the maps are unique, that they commute appropriately, and that the system is OD.)

We use $\mathcal{F}$ to denote this direct limit system. Let

$$M_\infty = \text{dirlim}\mathcal{F}.$$

The system is not quite countably directed, but we do have

Lemma 7.7.14 *Let $(\mathcal{M}_i, \vec{A}_i) \in I$ for all $i < \omega$; then there is a suitable $\mathcal{N}$ such that for all i, $(\mathcal{M}_i, \vec{A}_i) \leq_I (\mathcal{N}, \vec{A}_i)$.*

The proof uses a simultaneous comparison of the $\mathcal{M}_i$, like the proof of directedness. It follows at once that M_∞ is well-founded.

We use δ_∞ for the common image of the Woodin cardinal $\delta^{\mathcal{N}}$ of the various $(\mathcal{N}, \vec{A}) \in I$.

Lemma 7.7.15 $\delta_\infty = \theta_0$.

Proof. That $\delta_\infty \leq \theta_0$ is easy to see: given $(\mathcal{N}, \vec{A})$ in I, one can show that there is a prewellorder of $\mathbb{R}$ of order type $\pi_{(\mathcal{N},\vec{A}),\infty}(\gamma^{\mathcal{N}}_{\vec{A}})$ which is ordinal definable from $(\mathcal{N}, \vec{A})$, and hence from $\mathcal{N}$. This prewellorder is just the natural one given by $\mathcal{F}$, which is definable because the comparison process (including the choice of "sufficiently much" of the last branches of the comparison trees) is appropriately definable.

So suppose $\delta_\infty < \theta_0$. We reflect this to some Wadge level where we have a sjs, and then use that to get a contradiction. This sort of argument is used often in Woodin's work.

Let $\langle\alpha,\beta\rangle$ be lexicographically least such that $L_\beta(P_\alpha(\mathbb{R})) \models \mathsf{ZF}^- + \delta_\infty < \theta_0$. Let $S = L_\beta(P_\alpha(\mathbb{R}))$. By MSC, we have that S is an initial segment of $K(\mathbb{R})$, and it is easy to see from our definition of it that it ends a weak gap in $K(\mathbb{R})$. We therefore have an sjs $\vec{A}$ containing the universal $(\Sigma^2_1)^S$ set, and such that each A_i is OD^S.

Remark 7.7.16 [40] only gives a real x such that each A_i is $\mathrm{OD}^S(x)$. Some additional work is needed to avoid the real parameter.

We showed earlier (see Chapter 4) that there is a $(\Sigma^2_1)^S$-suitable $\mathcal{N}$ with a fullness-preserving iteration strategy Σ which is guided by $\vec{A}$. Σ has the Dodd-Jensen property, by the usual argument, and thus we can let P_∞ be the direct limit of all the non-dropping Σ-iterates of $\mathcal{N}$, under the maps given by comparison. It is easy to see that P_∞ embeds into the direct limit of $\mathcal{F}^S$. (In fact, it's not hard to show they are equal.) Letting $\pi\colon \mathcal{N} \to P_\infty$ be the natural map, and $\tau_i = \pi(\tau^{\mathcal{N}}_{A_i,\delta^{\mathcal{N}}}$, and $H_i =$ $\mathrm{Hull}^{P_\infty|\nu_i}(\delta_\infty \cup \{\tau_k \mid k < i\})$, we have that H_i is coded by a subset G_i of δ_∞. By the Coding Lemma, we have then that $\langle G_i \mid i < \omega\rangle$, and the sequence of natural embeddings between the G_i, are in S. Thus $o(P_\infty) < \theta^S_0$, and $P_\infty \in S$. We have also that $\langle\tau_i \mid i \in \omega\rangle \in S$. But $\vec{A}$ was an sjs, so from $\langle\tau_i \mid i \in \omega\rangle$ we can define a tree T (in the sense of descriptive set theory) whose projection is the universal $(\Pi^2_1)^S$ set of reals. Since T is on some ordinal $< \theta^S_0$, we have a contradiction. □

Lemma 7.7.17 *$M_\infty|\delta_\infty = HOD|\theta_0$.*

Proof. Clearly, $M_\infty|\delta_\infty \subseteq \mathrm{HOD}|\theta_0$.

Let A be a bounded, ordinal definable subset of θ_0. We must show $A \in M_\infty$. Let $(\mathcal{N},\vec{B}) \in I$ and $\alpha < \gamma^{\mathcal{N}}_{\vec{B}}$ be such that $A \subseteq \pi_{(\mathcal{N},\vec{B}),\infty}(\alpha)$. We define

$$
\begin{aligned}
C(x) \quad \Leftrightarrow \quad & x \text{ is a real coding some } (P,\xi) \\
& \text{such that } \exists(\mathcal{N},\vec{B}) \in I\ (P = \mathcal{N}|(\delta^{\mathcal{N}}) \wedge \\
& \xi < \gamma^{\mathcal{N}}_{vb} \wedge \pi_{(\mathcal{N},\vec{B}),\infty}(\xi) \in A).
\end{aligned}
$$

(Note that $\mathcal{N}$ is determined by P, if it exists, by $\mathcal{N} = P^+$.) Clearly, C is an OD set of reals. Let

$$\tau_\infty = \text{ common value of } \pi_{(\mathcal{Q},C),\infty}(\tau^{\mathcal{Q}}_{C,\delta^{\mathcal{Q}}}),$$

for all strongly C-quasi-iterable $\mathcal{Q}$.

We claim that for all β,

$$\beta \in A \Leftrightarrow M_\infty \models (\mathrm{Col}(\omega, \delta_\infty) \Vdash \dot{x}_{M_\infty} \in \tau_\infty),$$

where for any appropriate $\mathcal{R}$, $\dot{x}_{\mathcal{R}}$ is the canonical collapse term for a real coding $\mathcal{R}|\delta^{\mathcal{R}}$. This gives $A \in M_\infty$, as desired. To see this claim, fix $\beta < \alpha$, and fix $(\mathcal{Q}, \vec{B}) \in I$ and ξ such that $\pi_{(\mathcal{Q},\vec{B}),\infty}(\xi) = \beta$. We may assume that $(\mathcal{Q}, \vec{B}^\frown C) \in I$. Letting $\pi = pi_{(\mathcal{Q},\vec{B}^\frown C),\infty}$, we then have

$$\begin{aligned} \beta = \pi(\xi) \in A \quad &\Leftrightarrow \quad H^{\mathcal{Q}}_{\vec{B}^\frown C} \models (\mathrm{Col}(\omega, \delta^{\mathcal{Q}}) \Vdash \dot{x}_{\mathcal{Q}} \in \tau^{\mathcal{Q}}_{C,\delta^{\mathcal{Q}}}) \\ &\Leftrightarrow \quad M_\infty \models (\mathrm{Col}(\omega, \delta_\infty) \Vdash \dot{x}_{M_\infty} \in \tau_\infty), \end{aligned}$$

by the elementarity of π.

This finishes the proof of Lemma 7.7.17, and hence of Theorem 7.7.1.

□

In fact, we do not need to consider the full system $\mathcal{F}$ in order to obtain $\mathrm{HOD}|\theta_0$. It is enough to take any collection $\mathcal{A}$ of OD sets of reals which is Wadge cofinal (i.e. $\forall B \in P(\mathbb{R}) \cap \mathrm{OD} \exists A \in \mathcal{A}(B \leq_w A)$), and consider only the subsystem of $\mathcal{F}$ corresponding to indices $(\mathcal{N}, \vec{A}) \in I$ with $\vec{A} \in \mathcal{A}^{<\omega}$. This follows from

Lemma 7.7.18 *Assume* AD^+ *and* MSC. *Let $\mathcal{A}$ be a Wadge-cofinal collection of OD sets of reals, and let $x \in HOD|\theta_0$. Then there is a suitable $\mathcal{N}$ and an $\vec{A} \in \mathcal{A}^{<\omega}$ such that $x \in \mathrm{ran}(\pi_{(\mathcal{N},\vec{A}),\infty})$.*

Proof. We may assume x is an ordinal. Let $(\mathcal{N}, B) \in I$ and $\xi < \gamma^{\mathcal{N}}_B$ be such that $\pi_{(\mathcal{N},B),\infty}(\xi) = x$. Let $A \in \mathcal{A}$ be such that $B \leq_w A$ via the Wadge reduction z. Using genericity iteration, we can find $(\mathcal{M}, \langle B, A\rangle) \in I$ such that $(\mathcal{N}, B) \leq_I (\mathcal{M}, \langle B, A\rangle)$, and z is generic over $\mathcal{M}$ for the extender algebra at $\delta^{\mathcal{M}}$. It is an easy exercise to show, using the $\delta^{\mathcal{M}}$-chain condition for the extender algebra, that then $\gamma^{\mathcal{M}}_B \leq \gamma^{\mathcal{M}}_A$. But then $\pi_{(\mathcal{N},B),(\mathcal{M},\langle B,A\rangle)}(\xi) < \gamma^{\mathcal{M}}_A$, so $x \in \mathrm{ran}(\pi_{(\mathcal{M},A),\infty})$, as desired. □

7.8 A fullness preserving strategy for H_0^+

We return to our particular situation. By the results of the last section, H_0 is the direct limit of $\mathcal{F}^{M_0}$, restricted to θ^{M_0}. We have shown already that

H_0 is Σ^2_1-full in $j(M_0)$, and that in fact H_0^+ is suitable there. We have not yet shown that H_0^+ is the full direct limit of $\mathcal{F}^{M_0}$, but that will follow from what we are about to do.

Definition 7.8.1 *(a) $\mathcal{P}_\infty$ is the direct limit of the system $\mathcal{F}^{M_0}$.*

(b) $\mathcal{O}$ is the collection of finite sequences of ordinal definable sets of reals.

(c) Working in $j(M_0)$, let $\langle(\mathcal{N}_i, \vec{A}_i) \mid i < \omega\rangle$ be a cofinal, linearly ordered sequence from $(I, \leq_I)^{M_0}$. Then

$$\mathcal{P}^*_\infty = \text{qlim}_{i<\omega} \mathcal{N}_i,$$

where the quasi-limit is computed in $j(M_0)$.

Concerning part (c), notice that $\mathcal{N}_i = j(\mathcal{N}_i)$ is suitable, and in fact strongly $j(\vec{A}_i)$-quasi-iterable, in $j(M_0)$. It follows that $\mathcal{P}^*_\infty$ exists, and that in $j(M_0)$, $\mathcal{P}^*_\infty$ is suitable, and in fact strongly $j(\vec{A})$-quasi-iterable, for all $\vec{A} \in \mathcal{O}^{M_0}$.

There is a natural embedding σ from $\mathcal{P}_\infty$ into $\mathcal{P}^*_\infty$: given $x \in \mathcal{P}_\infty$, we can find an i such that $x = \pi_{(\mathcal{N}_i, \vec{A}_i),\infty}(\bar{x})$ for some $\bar{x}$, and we then set

$$\sigma(x) = \pi^{j(\vec{A}_i)}_{i,\infty}(\bar{x}).$$

The range of σ is just the direct limit of the subsystem of $\mathcal{F}(\langle\mathcal{N}_i \mid i < \omega\rangle)^{j(M_0)}$ corresponding to sets in $j``\mathcal{O}^{M_0}$.

Definition 7.8.2 *Let $\mathcal{W}$ be suitable, and $\mathcal{A}$ a collection of OD sets of reals; then*

(a) $H^{\mathcal{W}}_{\mathcal{A}} = \bigcup_{\vec{B} \in \mathcal{A}^{<\omega}} H^{\mathcal{W}}_{\vec{B}}$,

(b) $\mathcal{S}$ is an $\mathcal{A}$-quasi-iterate of $\mathcal{W}$ iff for all $\vec{B} \in \mathcal{A}^{<\omega}$, $\mathcal{S}$ is a $\vec{B}$-quasi-iterate of $\mathcal{W}$,

(c) if $\mathcal{S}$ is an $\mathcal{A}$-quasi-iterate of $\mathcal{W}$, then $\pi^{\mathcal{A}}_{\mathcal{W},\mathcal{S}} \colon H^{\mathcal{W}}_{\mathcal{A}} \to H^{\mathcal{S}}_{\mathcal{A}}$ is the union of the quasi-iteration maps $\pi^{\vec{B}}_{\mathcal{W},\mathcal{S}} \colon H^{\mathcal{W}}_{\vec{B}} \to H^{\mathcal{S}}_{\vec{B}}$,

(d) $\mathcal{W}$ is $\mathcal{A}$-quasi-iterable iff $\mathcal{W}$ is $\vec{B}$-quasi-iterable, for all $\vec{B} \in \mathcal{A}^{<\omega}$,

(e) if $\mathcal{W}$ is $\mathcal{A}$-quasi-iterable, then $\pi^{\mathcal{A}}_{\mathcal{W},\infty} \colon H^{\mathcal{W}}_{\mathcal{A}} \to M_\infty$ is the natural map into the direct limit of the system $\mathcal{F}$.

We want to use these notions in $j(M_0)$, with $\mathcal{A} = j``\mathcal{O}^{M_0}$. Although $\mathcal{A} \notin j(M_0)$ in this case, $\mathcal{A} \subseteq j(M_0)$, and this is enough to make sense of $\mathcal{A}$-quasi-iterability "in" $j(M_0)$.

Lemma 7.8.3 *In $j(M_0)$, $\mathcal{P}^*_\infty$ and all of its quasi-iterates are $j``\mathcal{O}^{M_0}$-quasi-iterable.*

Proof. We observed this for $\mathcal{P}^*_\infty$ above. It passes to quasi-iterates trivially. □

Lemma 7.8.4 *Let $\mathcal{A}$ be any Wadge cofinal collection of OD^{M_0} sets of reals; then* $\mathrm{ran}(\sigma) = H_{\mathcal{A}}^{\mathcal{P}^*_\infty}$.

Proof. Clear. □

We shall show that σ is onto, so that $\mathcal{P}_\infty = \mathcal{P}^*_\infty$. To begin with,

Lemma 7.8.5 $\sigma \restriction (\delta_\infty + 1)$ *is the identity.*

Proof. We get that $\sigma \restriction \delta_\infty$ is the identity from the elementary properties of the direct limit systems in question, We leave the easy proof to the reader.

Suppose $\delta_\infty < \sigma(\delta_\infty)$. It follows that $H_0 = \mathcal{P}^*_\infty | \eta$ where $\eta = \delta_\infty < \delta^{P^*_\infty} = \sigma(\delta_\infty)$. But $\mathcal{P}^*_\infty$ is suitable in $j(M_0)$, so η is not Woodin in $\mathcal{P}^*_\infty$, so $(H_0^+)^{j(M_0)} \models \delta_\infty$ is not Woodin. This contradicts 7.3.8. □

The following extends our fullness and suitability results for H_0.

Lemma 7.8.6 *Let $k\colon H_0 \to S$ and $i\colon S \to j(H_0)$ be such that $j = i \circ k$, with S countable in $V[G]$. Then in $j(M_0)$:*

(a) S is Σ^2_1-full, and

(b) S^+ is suitable.

Proof. Working in $j(M_0)$, let T be the tree of a Σ^2_1 scale on the set of reals coding ω_1-iterable relativised mice. We have that $T \in \mathrm{HOD}^{j(M_0)}$. As in the proof of Theorem 7.4.2, we can extend k and i so as to obtain

$$k^*\colon L[T, H_0] \to \mathrm{Ult}(L[T, H_0], E_k)$$

and

$$i^*\colon \mathrm{Ult}(L[T, H_0], E_k) \to L[j(T), j(H_0)]$$

such that $j = i^* \circ k^*$. (k^* is just the canonical embedding into the ultrapower, and i^* is given by: $i^*(k^*(f)(a)) = j(f)(i(a))$, for all $a \in [o(S)]^{<\omega}$. Note that since j maps $o(H_0)$ cofinally into $o(j(H_0))$, k maps $o(H_0)$ cofinally into $o(S)$.) We have that $S = k^*(H_0)$. Also, H_0 is a rank intial segment of $L[T, H_0]$, and $o(H_0) = \theta^{M_0}$ is Woodin in $L[T, H_0]$, so S has these properties in $\mathrm{Ult}(L[T, H_0], E_k)$.

Now an absoluteness argument like that we had before shows that indeed S is Σ^2_1-full, and S^+ is suitable, in $j(M_0)$. For if not, there is a countable minimal mouse $\mathcal{Q} \in j(M_0)$ witnessing that, and there is a tree in $\mathrm{Ult}(L[T, H_0], E_k)$ whose only branch is $\mathcal{Q}$. The key is that we can use the tree $k^*(T)$ to certify the iterability of the mice our tree produces. Since T embeds into $k^*(T)$, $\mathcal{Q} \in p[k^*(T)]$. On the other hand, $k^*(T)$ embeds into $j(T)$, and so any $\mathcal{S} \in p[k^*(T)]$ is ω_1-iterable in $j(M_0)$. [Proof: We have then $\mathcal{S} \in p[j(T)]$. So it is enough to show that in M, every $\mathcal{S} \in p[j(T)]$ is ω_1-iterable via a strategy in $j(M_0)$. Pulling back, it is enough to show that in V, every $\mathcal{S} \in p[T]$ is ω_1-iterable via a strategy in M_0. But fix $\mathcal{S} \in p[T]$ with $\mathcal{S} \in V$; then $j(M_0) \models \mathcal{S} \in p[T]$, so $j(M_0) \models \mathcal{S}$ is ω_1-iterable. But $j(\mathcal{S}) = \mathcal{S}$, so $M_0 \models \mathcal{S}$ is ω_1-iterable, as desired.]

It follows that $\mathcal{Q} \in \mathrm{Ult}(L[T, H_0], E_k)$, a contradiction. □

Lemma 7.8.7 *Let $\mathcal{A}$ be Wadge cofinal in the OD^{M_0} sets of reals, and let W be a quasi-iterate of $\mathcal{P}^*_\infty$ in the sense of $j(M_0)$; then $\delta^W = \sup\{\gamma^W_{\vec{A},o(W)} \mid \vec{A} \in (j``\mathcal{A})^{<\omega}\}$. That is, $(\delta^{\mathcal{W}} + 1) \subseteq H^{\mathcal{W}}_{j``\mathcal{A}}$.*

Proof. The key is

Claim. $j \restriction \mathcal{P}_\infty = \pi^{j``\mathcal{A}}_{(\mathcal{P}^*_\infty, \infty)} \circ \sigma$.

Proof. Let $x \in \mathcal{P}_\infty$. By 7.7.18, we have $x = \pi^{M_0}_{(\mathcal{N},\vec{A}),\infty}(\bar{x})$, where $\vec{A} \in \mathcal{A}^{<\omega}$. We have then that

$$\begin{aligned} j(x) &= j(\pi^{M_0}_{(\mathcal{N},\vec{A}),\infty}(\bar{x})) \\ &= \pi^{j(M_0)}_{(\mathcal{N},j(\vec{A})),\infty}(\bar{x}) \\ &= \pi^{j(M_0)}_{(\mathcal{P}^*_\infty,j(\vec{A})),\infty}((\pi^{j(M_0)}_{(\mathcal{N},\vec{A}),(\mathcal{P}^*_\infty,j(\vec{A}))}(\bar{x})) \\ &= \pi^{j(M_0)}_{(\mathcal{P}^*_\infty,j(\vec{A})),\infty}(\sigma(x)). \end{aligned}$$

This proves our claim. □

Now let
$$\mathcal{S} = \text{ collapse of } H^{\mathcal{W}}_{j``\mathcal{A})},$$
and $l\colon \mathcal{S} \to \mathcal{W}$ be the collapse map, and $k\colon \mathcal{P}_\infty \to \mathcal{S}$ the collapse of $\pi^{j``\mathcal{A})}_{\mathcal{P}^*_\infty,\mathcal{W}}\colon H^{\mathcal{P}^*_\infty}_{j``\mathcal{A})} \to H^{\mathcal{W}}_{j``\mathcal{A})}$. By our claim
$$j \restriction \mathcal{P}_\infty = \pi^{j``\mathcal{A})}_{\mathcal{W},\infty} \circ l \circ k.$$
So k factors into j, and thus by 7.8.6, $\mathcal{S}$ is suitable. From the minimality of $\mathcal{W}$, we then have that $l \restriction (\delta^{\mathcal{S}}+1) =$ identity, which is what we want. □

We can now show that the quasi-iterates of $\mathcal{P}^*_\infty$ are "sound", in a certain sense. This allows us to improve 7.8.7.

Lemma 7.8.8 *Let $\mathcal{A}$ be Wadge cofinal in the OD^{M_0} sets of reals, and working in $j(M_0)$, let $\mathcal{W}$ be a quasi-iterate of $\mathcal{P}^*_\infty$; then*

(i) $\mathcal{W} = H^{\mathcal{W}}_{j``\mathcal{A}}$,

(ii) *σ is the identity, so $\mathcal{P}_\infty = \mathcal{P}^*_\infty$,*

(iii) *for any quasi-iterate $\mathcal{Q}$ of $\mathcal{W}$, $\pi^{j``\mathcal{A}}_{\mathcal{W},\mathcal{Q}}\colon \mathcal{W} \to \mathcal{Q}$, and $\pi^{j``\mathcal{A}}_{\mathcal{W},\mathcal{Q}}(\tau^{\mathcal{W}}_{j(B),\nu}) = \tau^{\mathcal{Q}}_{j(B),\mu}$, where μ is the image of ν, for all OD^{M_0} sets of reals B,*

(iv) $j \restriction \mathcal{P}_\infty = \pi^{j``\mathcal{A}}_{(\mathcal{P}^*_\infty,\infty)}$.

Proof. (i): To come.

(ii): This is immediate from the special case of 7.8.8(i) in which $\mathcal{W} = \mathcal{P}^*_\infty$.

(iii): This is immediate from (i) and the definitions.

(iv): This is just the key claim in the proof of 7.8.7, using that σ is the identity. □

We can now show that the embeddings $\pi^{j``\mathcal{A}}_{\mathcal{W},\mathcal{Q}}$ of part (iii) of 7.8.8 comes from an iteration strategy for $\mathcal{P}_\infty$.

Theorem 7.8.9 *Let $\mathcal{A}$ be countable, and Wadge cofinal in the OD^{M_0} sets of reals; then in M (where $j\colon V \to M$) there is a unique fullness-preserving, $j(\mathcal{A})$-guided ω_1-iteration strategy Σ for $\mathcal{P}_\infty$. Moreover, Σ moves the term relations associated to $j(B)$ correctly, for any OD^{M_0} set of reals B.*

Proof. We shall follow the proof of Theorem 5.4.14. There our counterpart to $j(\mathcal{A})$ was a self-justifying system, but 7.8.7 and 7.8.8 give us the consequences of being a self-justifying system which we actually used.

We shall just define Σ on stacks of maximal trees. (Maximal in $j(M_0)$, that is.) On short trees, Σ is Lp-guided.

So let $\mathcal{R}$ be a non-dropping Σ-iterate of $\mathcal{P}_\infty$, via the linear stack of maximal trees $\mathcal{T}$. Since Σ is fullness-preserving, we have that $\mathcal{R}$ is a quasi-iterate of $\mathcal{P}_\infty$. Now let $\mathcal{U}$ be a maximal tree on $\mathcal{R}$, and $\mathcal{W} = \mathcal{M}(\mathcal{U})^+$. We have then that for all $\vec{A} \in \mathcal{A}^{<\omega}$, $\mathcal{W}$ is an $j(\vec{A})$-quasi-iterate of $\mathcal{R}$ in $j(M_0)$. Also,

$$\delta^{\mathcal{U}} = \sup(\{\gamma^{\mathcal{W}}_{j(\vec{A})} \mid \vec{A} \in \mathcal{A}^{<\omega}\}),$$

by 7.8.7(b). Now for each $\vec{A} \in \mathcal{A}^{<\omega}$ we can pick a $j(\vec{A})$-correct branch $b_{\vec{A}}$ of $\mathcal{U}$. Since the $\gamma^{\mathcal{W}}_{j(\vec{A})}$ are cofinal in $\delta(\mathcal{U})$, these branches converge to a cofinal branch b of $\mathcal{U}$, and as in Theorem 5.4.14, we have

(i) $i_b(\delta^{\mathcal{R}}) = \delta(\mathcal{U})$,

(ii) letting $T^{\mathcal{S}}_{\vec{B}}$ be the theory of parameters in $\delta^{\mathcal{S}} \cup \{\tau^{\mathcal{S}}_{B_i,\nu} \mid i < \mathrm{lh}(\vec{B})\}$ in $\mathcal{S}|(\nu^{++})^{\mathcal{S}}$, where ν is the $\mathrm{lh}(\vec{B})$-th cardinal of $\mathcal{S}$ above $\delta^{\mathcal{S}}$, we have

$$i_b(T^{\mathcal{R}}_{j(\vec{A})}) = T^{\mathcal{W}}_{j(\vec{A})},$$

for all $\vec{A} \in \mathcal{A}^{<\omega}$.

By 7.8.8, $\mathcal{R}$ is coded by the $T^{\mathcal{R}}_{j(\vec{A})}$, so $\mathcal{M}^{\mathcal{U}}_b$ is coded by the $i_b(T^{\mathcal{R}}_{j(\vec{A})})$. Thus by (ii) above, there is an embedding $\sigma\colon \mathcal{M}^{\mathcal{U}}_b \to \mathcal{W}$, with $\mathrm{ran}(\sigma) = \bigcup_{\vec{A}\in\mathcal{A}^{<\omega}} H^{\mathcal{W}}_{j(\vec{A})}$. By 7.8.8 then, $\mathrm{ran}(\sigma) = \mathcal{W}$, so σ is the identity. This implies that $i_b(\tau^{\mathcal{R}}_{j(\vec{A}),\nu}) = \tau^{\mathcal{W}}_{j(\vec{A}),i_b(\nu)}$ for all appropriate $\vec{A}$ and ν. Thus b is $j(\vec{A})$-correct for all $\vec{A} \in \mathcal{A}^{<\omega}$, and we can set

$$\Sigma(\mathcal{T}^\frown\mathcal{U}) = b.$$

We leave the easy proof that b is the *unique* fullness-preserving, $j``\mathcal{A}$-correct branch to the reader.

Finally, we must deal with linear stacks $\langle \mathcal{T}_\alpha \mid \alpha < \lambda\rangle$ of maximal trees of limit length. For this, it is enough to show that the direct limit $\mathcal{S}$ along the branches chosen by Σ, under those branch embeddings, is the same as the quasi-limit $\mathcal{W}$ of the $\mathcal{M}(\mathcal{T}_\alpha)^+$. This is done by induction on λ. The key is that, as in the successor case, there is a natural embedding $\sigma\colon \mathcal{S} \to \mathcal{W}$ such that $\sigma \restriction \delta^{\mathcal{S}}$ is the identity, and such that all $\tau^{\mathcal{W}}_{j(\vec{A}),\nu}$ are in the range of σ. We leave the details to the reader. □

We want a fullness-preserving strategy which condenses well. As a first step in that direction, we have

Theorem 7.8.10 (Weak condensation) *Let $\mathcal{A}$ be Wadge cofinal in the OD^{M_0} sets of reals, and Σ be the $j(\mathcal{A})$-guided ω_1 iteration strategy for $\mathcal{P}_\infty$ given by 7.8.9. Let $k\colon \mathcal{P}_\infty \to \mathcal{W}$ be an iteration map by Σ, and suppose there are $l\colon \mathcal{P}_\infty \to \mathcal{S}$ and $t\colon \mathcal{S} \to \mathcal{W}$ be such that $k = t \circ l$. Then $\mathcal{S}$ is suitable.*

Proof. We have that $k = \pi^{j``\mathcal{A}}_{\mathcal{P}_\infty,\mathcal{W}}$, so

$$\begin{aligned} j \restriction \mathcal{P}_\infty &= \pi^{j``\mathcal{A}}_{\mathcal{W},\infty} \circ \pi^{j``\mathcal{A}}_{\mathcal{P}_\infty,\mathcal{W}} \\ &= \pi^{j``\mathcal{A}}_{\mathcal{W},\infty} \circ t \circ l. \end{aligned}$$

Thus k factors into j, and we can apply 7.8.6. □

Finally, we can pull back the existence of a fullness-preserving strategy with weak condensation to V. We get

Theorem 7.8.11 *Let $\mathcal{A}$ be countable, and Wadge cofinal in the OD^{M_0} sets of reals; then there is in V a suitable $\mathcal{N}$ and a fullness-preserving (with respect to the pointclass $(\Sigma^2_1)^{M_0}$), $\mathcal{A}$-guided iteration strategy Σ for $\mathcal{N}$ with the Dodd-Jensen property. Moreover*

(a) for any $\mathcal{W}$, $\mathcal{W}$ is a non-dropping Σ-iterate of $\mathcal{N}$ iff $\mathcal{W}$ is a quasi-iterate (in the sense of M_0) of $\mathcal{N}$ iff $\mathcal{W}$ is an $\mathcal{A}$-quasi-iterate of $\mathcal{N}$,

(b) for $\mathcal{W}$ a non-dropping Σ-iterate of $\mathcal{N}$, $H^{\mathcal{W}}_{\mathcal{A}} = \mathcal{W}$,

(c) the Σ-iteration maps are $\mathcal{A}$-quasi-iteration maps, and thus $\mathcal{P}_\infty$ is the direct limit of all Σ-iterates of $\mathcal{N}$, and

(d) if $t\colon \mathcal{N} \to \mathcal{W}$ is an iteration map by Σ, and $t = l \circ k$ where $k\colon \mathcal{N} \to \mathcal{S}$, then $\mathcal{S}$ is suitable.

Proof. $j(\mathcal{A}) = j``\mathcal{A}$ is in M, and in M there is such a $j(\mathcal{A})$-guided strategy for $\mathcal{P}_\infty$. □

7.9 Branch condensation

In this section, we prove

Theorem 7.9.1 (Branch condensation) *Assume* HI*, and let $\mathcal{A}$ be countable and Wadge cofinal in the OD^{M_0} sets of reals; then there is a suitable $\mathcal{N}$ and $\mathcal{A}$-guided strategy Σ for $\mathcal{N}$ having all the properties of 7.8.11, and such that in addition*

(e) Let $\mathcal{R}$ be a Σ-iterate of $\mathcal{N}$, and let $i\colon \mathcal{R} \to \mathcal{W}$ be an iteration map by Σ. Let $\mathcal{T}$ be a maximal tree on $\mathcal{R}$, and b a cofinal, non-dropping branch of $\mathcal{T}$, and suppose there is $t\colon \mathcal{M}_b^{\mathcal{T}} \to \mathcal{W}$ such that $i = t \circ i_b^{\mathcal{T}}$; then $b = \Sigma(\mathcal{T})$.

Corollary 7.9.2 *Let $\mathcal{N}, \Sigma$ be as in 7.9.1; then Σ condenses well.*

Proof. We show that the restriction of Σ to normal trees on $\mathcal{N}$ condenses well, and leave the case of stacks of normal trees to the reader. Let $\mathcal{S}$ be a normal tree on $\mathcal{N}$, and $\mathcal{U}$ be a hull of $\mathcal{S}$, as witnessed by $\sigma\colon \mathrm{lh}(\mathcal{U}) \to \mathrm{lh}(\mathcal{S})$ and $\pi_\gamma\colon \mathcal{M}_\gamma^{\mathcal{U}} \to \mathcal{M}_{\sigma(\gamma)}^{\mathcal{S}}$ for $\gamma < \mathrm{lh}(\mathcal{U})$. If $\mathcal{U}$ is not by Σ, then let $\lambda < \mathrm{lh}(\mathcal{U})$ be least such that $b = [0, \lambda]_U$ is not by Σ. Since $\mathcal{M}_\lambda^{\mathcal{U}}$ is iterable by the pullback of Σ, and since $\mathcal{N}$ is suitable, we must have that $\mathcal{T} = \mathcal{U}|\lambda$ is maximal. Letting $t = \pi_\lambda$ and $\mathcal{W} = \mathcal{M}_{\sigma(\lambda)}^{\mathcal{S}}$, and $i = i_{0,\sigma(\lambda)}^{\mathcal{S}}$, we have $i = t \circ i_b^{\mathcal{T}}$. By refbranchcon, this implies $b = \Sigma(\mathcal{T})$, a contradiction. □

Proof of Theorem 7.9.1. Fix $\mathcal{N}^*$ and Σ which satisfy the conclusion of 7.8.11 with respect to $\mathcal{A}$. The desired $\mathcal{N}$ will be a Σ-iterate of $\mathcal{N}^*$. Note that trivially, every Σ-iterate of $\mathcal{N}_0$ satisfies (a)-(d).

Suppose toward contradiction that no Σ-iterate of $\mathcal{N}^*$ satisfies (e). Then for any such iterate $\mathcal{N}_\alpha$, we can find a Σ-iterates $\mathcal{N}_{\alpha+1}$ of $\mathcal{N}_\alpha$ and $\mathcal{W}$ of $\mathcal{N}_{\alpha+1}$, a maximal tree $\mathcal{T}$ on $\mathcal{N}_{\alpha+1}$, a cofinal, non-dropping branch b of $\mathcal{T}$ such that $b \neq \Sigma(\mathcal{T})$, and a $t\colon \mathcal{M}_b^{\mathcal{T}} \to \mathcal{W}$ such that $i = t \circ i_b^{\mathcal{T}}$. Notice that since $\mathcal{M}_b^{\mathcal{T}}$ embeds into $\mathcal{W}$, it is iterable, and thus since $\mathcal{T}$ is maximal, $\mathcal{M}_b^{\mathcal{T}} = \mathcal{M}(\mathcal{T})^+ = \mathcal{M}_c^{\mathcal{T}}$, where $c = \Sigma(\mathcal{T})$. We can compare $\mathcal{W}$ with $\mathcal{M}_c^{\mathcal{T}}$, using Σ to iterate each of them, and arriving at a common Σ-iterate $\mathcal{N}_{\alpha+2}$. This gives maps k, l such that $k \circ i_b$ and $l \circ i_c$ map $\mathcal{N}_{\alpha+1}$ to $\mathcal{N}_{\alpha+2}$. (Here k is t followed by the coiteration map on $\mathcal{W}$.) We get $k \circ i_b = l \circ i_c$ from the Dodd-Jensen property of Σ.

By our hypothesis, $\mathcal{N}_{\alpha+1}$ and Σ satisfy (a)-(d) but not (e), so we can generate $\mathcal{N}_{\alpha+3}$ and $\mathcal{N}_{\alpha+4}$, etc. Taking direct limits at limit ordinals, we generate this way what we shall call a *bad sequence* of length ω_1. It is important that in the formal definition of this concept we record only properties which can be seen in $M_0[g]$, where g is a generic object adding the sequence. So we can refer to the models, embeddings, and suitability, but we cannot mention Σ or $\mathcal{A}$. This leads to

Definition 7.9.3 *A* bad sequence *is a sequence* $\langle(\mathcal{N}_\alpha, \mathcal{T}_\alpha, b_\alpha, c_\alpha, j_\alpha, k_\alpha, l_\alpha) \mid \alpha < \eta\rangle$*, where* $\eta \leq \omega_1$*, such that for all* α*,*

(i) $\mathcal{N}_\alpha$ *is suitable, and* $j_\alpha \colon \mathcal{N}_\alpha \to \mathcal{N}_{\alpha+1}$ *if* $\alpha + 1 < \eta$*,*

(ii) if α *is odd, then* $\mathcal{T}_\alpha$ *is a countable, normal tree on* $\mathcal{N}_\alpha$*, with distinct cofinal non-dropping branches* b_α, c_α *such that* $M^{\mathcal{T}_\alpha}_{b_\alpha} = M^{\mathcal{T}_\alpha}_{c_\alpha}$ *and and* $i_{b_\alpha}(\delta^{\mathcal{N}_\alpha}) = i_{c_\alpha}(\delta^{\mathcal{N}_\alpha}) = \delta\mathcal{T}_\alpha$*,*

(iii) if α *is odd, then* $k_\alpha \colon M^{\mathcal{T}_\alpha}_{b_\alpha} \to \mathcal{N}_{\alpha+1}$ *and* $l_\alpha \colon M^{\mathcal{T}_\alpha}_{b_\alpha} \to \mathcal{N}_{\alpha+1}$*, and* $j_\alpha = k_\alpha \circ i_{b_\alpha} = l_\alpha \circ i_{c_\alpha}$*, and*

(iv) if α *is a limit ordinal, then* $\mathcal{N}_\alpha$ *is the direct limit of the* $\mathcal{N}_\beta$ *for* $\beta < \alpha$*, under the maps* $j_{\gamma,\xi}$ *generated by the* j_β*, for* $\beta < \alpha$

(v) if α *is even (including* α *a limit), then* $(\mathcal{T}_\alpha, b_\alpha, c_\alpha, k_\alpha, l_\alpha) = (0, 0, 0, 0, 0)$.

Our preamble to the definition essentially shows how to construct a bad sequence of length ω_1 such that each $\mathcal{N}_\alpha$ is a Σ-iterate of $\mathcal{N}^*$. We shall show that there is such a sequence in a certain generic extension of M_0. We then reflect this fact to an initial segment of M_0 where we have a sjs on the local Σ^2_1, then use the condensation properties of the associated term relations to reach a contradiction.

The aspect of badness which is not simply a projective property of the sequence, namely the suitability of the $\mathcal{N}_\alpha$, can be certified by embeddings into $\mathcal{P}_\infty$.

Definition 7.9.4 *Let* $\pi \colon \mathcal{N} \to \mathcal{P}$*, where* $\mathcal{N}$ *and* $\mathcal{P}$ *are premice with the first-order properties in suitability, and with* $\mathcal{N}$ *countable. A* $(\pi, \mathcal{N}, \mathcal{P})$*-* certified bad sequence *is a sequence* $\langle(\mathcal{N}_\alpha, \mathcal{T}_\alpha, b_\alpha, c_\alpha, j_\alpha, k_\alpha, l_\alpha, \pi_\alpha) \mid \alpha < \eta\rangle$*, such that for all* α*,*

(i) $\mathcal{N}_0 = \mathcal{N}$*, and* $\pi_0 = \pi$*, and* $j_\alpha \colon \mathcal{N}_\alpha \to \mathcal{N}_{\alpha+1}$ *if* $\alpha + 1 < \eta$*,*

(ii) *if α is odd, then $\mathcal{T}_\alpha$ is a countable, normal tree on $\mathcal{N}_\alpha$, with distinct cofinal non-dropping branches b_α, c_α such that $M^{\mathcal{T}_\alpha}_{b_\alpha} = M^{\mathcal{T}_\alpha}_{c_\alpha}$ and and $i_{b_\alpha}(\delta^{\mathcal{N}_\alpha}) = i_{c_\alpha}(\delta^{\mathcal{N}_\alpha}) = \delta(\mathcal{T}_\alpha)$,*

(iii) *if α is odd, then $k_\alpha\colon M_{b_\alpha} \to \mathcal{N}_{\alpha+1}$ and $l_\alpha\colon M_{c_\alpha} \to \mathcal{N}_{\alpha+1}$, and $j_\alpha = k_\alpha \circ i_{b_\alpha} = l_\alpha \circ i_{c_\alpha}$, and*

(iv) *if α is a limit ordinal, then $\mathcal{N}_\alpha$ is the direct limit of the $\mathcal{N}_\beta$ for $\beta < \alpha$, under the maps $j_{\gamma,\xi}$ generated by the j_β, for $\beta < \alpha$,*

(v) *$\pi_\alpha\colon \mathcal{N}_\alpha \to \mathcal{P}$, and $\pi_\gamma = \pi_\alpha \circ j_{\gamma,\alpha}$, for all $\gamma < \alpha$,*

(vi) *if α is even (including α a limit), then $(\mathcal{T}_\alpha, b_\alpha, c_\alpha, k_\alpha, l_\alpha) = (0,0,0,0,0)$.*

Let $\pi^*\colon \mathcal{N}^* \to \mathcal{P}_\infty$ be the iteration map by Σ.

Lemma 7.9.5 *Let $\langle(\mathcal{N}_\alpha, \mathcal{T}_\alpha, b_\alpha, c_\alpha, j_\alpha, k_\alpha, l_\alpha, \pi_\alpha) \mid \alpha < \eta\rangle$, be a $(\pi^*, \mathcal{N}^*, \mathcal{P}_\infty)$-certified bad sequence; then $\langle(\mathcal{N}_\alpha, \mathcal{T}_\alpha, b_\alpha, c_\alpha, j_\alpha, k_\alpha, l_\alpha) \mid \alpha < \eta\rangle$, is a bad sequence.*

Proof. This follows easily from weak condensation, 7.8.10. □

One can construct a $(\pi^*, \mathcal{N}^*, \mathcal{P}_\infty)$- certified bad sequence of length η from $(\pi^*, \mathcal{N}^*, \mathcal{P}_\infty)$ and a counting of η, via a simple induction. (Here we use our hypothesis that no quasi-iterate of $\mathcal{N}^*$ satisfies (e).) This gives

Lemma 7.9.6 *Let g be $\mathrm{Col}(\omega, < \omega_1)$-generic over $L[\pi^*, \mathcal{N}^*, \mathcal{P}_\infty]$. Then $L[\pi^*, \mathcal{N}^*, \mathcal{P}_\infty][g] \models$ "there is a $(\pi^*, \mathcal{N}^*, \mathcal{P}_\infty)$-certified bad sequence of length ω_1".*

Proof. We show how to construct a $(\pi^*, \mathcal{N}^*, \mathcal{P}_\infty)$-certified bad sequence of length ω_1, and omit the easy absoluteness argument which shows there is such a sequence in $L[\pi^*, \mathcal{N}^*, \mathcal{P}_\infty][g]$. (Notice that $L[\pi^*, \mathcal{N}^*, \mathcal{P}_\infty][g]$ is correct about which sequences are $(\pi^*, \mathcal{N}^*, \mathcal{P}_\infty)$-certified bad.)

We set $\mathcal{N}_0 = \mathcal{N}^*$, and $\pi_0 = \pi^*$. If $\alpha > 0$ is a limit, then we set $\mathcal{N}_\alpha$ to be the direct limit of the $\mathcal{N}_\xi$ for $\xi < \alpha$ under the $j_{\xi,\gamma}$, and let $\pi_\alpha = \bigcup_{\xi<\alpha} \pi_\xi$.

Now suppose α is even, and we have defined $\mathcal{N}_\alpha$ and $\pi_\alpha\colon \mathcal{N}_\alpha \to \mathcal{P}_\infty$ such that $\pi_0 = \pi_\alpha \circ j_{0,\alpha}$. Since $\mathcal{P}_\infty$ is the direct limit of all Σ-iterates of $\mathcal{N}_0$, we can cover the range of π_α with the range of some map in the direct limit system of all Σ-iterates of $\mathcal{N}_0$. This gives

$$\pi_\alpha = \phi \circ \psi,$$

where $\psi\colon \mathcal{N}_\alpha \to \mathcal{S}$, and $\phi\colon \mathcal{S} \to \mathcal{P}_\infty$, and $\mathcal{S}$ is a Σ-iterate of $\mathcal{N}_0$, and ϕ is the natural map to the direct limit of the system of Σ-iterates.

By hypothesis, (e) does not hold with $\mathcal{N} = \mathcal{S}$, so let $\mathcal{R} = \mathcal{N}_{\alpha+1}$ be part of a counterexample, and let $\rho\colon \mathcal{S} \to \mathcal{N}_{\alpha+1}$ be the Σ-iteration map, and put $j_\alpha = \rho \circ \psi$, and let $\pi_{\alpha+1}\colon \mathcal{N}_{\alpha+1} \to \mathcal{P}_\infty$ be the map of the system of Σ-iterates. For $x \in \mathcal{N}_\alpha$, we have

$$\begin{aligned}\pi_{\alpha+1}(j_\alpha(x)) &= \pi_{\alpha+1}(\rho(\psi(x)))\\ &= \phi(\psi(x))\\ &= \pi_\alpha(x),\end{aligned}$$

because the Σ-iteration maps are unique by Dodd-Jensen.

The remainder of our counterexample to (e) is a Σ-iteration map $i\colon \mathcal{N}_{\alpha+1} \to \mathcal{W}$, a maximal tree $\mathcal{T} = \mathcal{T}_{\alpha+1}$ on $\mathcal{N}_{\alpha+1}$, a cofinal, non-dropping branch $b = b_{\alpha+1}$ of $\mathcal{T}$ such that $b \neq c$, where $c = c_{\alpha+1} = \Sigma(\mathcal{T})$, and a $t\colon \mathcal{M}_b \to \mathcal{W}$ such that $i = t \circ i_b$. Since $\mathcal{M}_b^{\mathcal{T}}$ embeds into $\mathcal{W}$, it is iterable, and thus since $\mathcal{T}$ is maximal, $\mathcal{M}_b^{\mathcal{T}} = \mathcal{M}(\mathcal{T})^+ = \mathcal{M}_c(\mathcal{T})$. We can compare $\mathcal{W}$ with $\mathcal{M}_c^{\mathcal{T}}$, using Σ to iterate each of them, and arriving at a common Σ-iterate $\mathcal{N}_{\alpha+2}$ with iteration maps $l = l_{\alpha+1}\colon \mathcal{M}_c \to \mathcal{N}_{\alpha+2}$ and $u\colon \mathcal{W} \to \mathcal{N}_{\alpha+2}$. Set $k = k_{\alpha+1} = u \circ t$. We have that $k \circ i_b$ and $l \circ i_c$ map $\mathcal{N}_{\alpha+1}$ to $\mathcal{N}_{\alpha+2}$. Clearly $l \circ i_c$ is an iteration map by Σ, but since $k \circ i_b = u \circ i$, it is also an iteration map by Σ. Thus $k \circ i_b = l \circ i_c$.

Finally, let $j_{\alpha+1} = l \circ i_c$, and let $\pi_{\alpha+2}\colon \mathcal{N}_{\alpha+2} \to \mathcal{P}_\infty$ be the map of the Σ-system. It is easy to see that $\pi_{\alpha+1} = \pi_{\alpha+2} \circ j_{\alpha+1}$.

Although we have used Σ to show the desired extension of a given $(\pi^*, \mathcal{N}^*, \mathcal{P}_\infty)$- certified bad sequence exists, the properties of the extension we wish to have can be verified by a tree obtained from a counting of the given sequence and $(\pi^*, \mathcal{N}^*, \mathcal{P}_\infty)$. Thus there is a $(\pi^*, \mathcal{N}^*, \mathcal{P}_\infty)$- certified bad sequence in $L[\pi^*, \mathcal{N}^*, \mathcal{P}_\infty][g]$. □

Our next goal is to show that if g is $\mathrm{Col}(\omega, <\omega_1)$-generic over V, then $M_0[g] \models$ "there is a bad sequence of length ω_1." This then gives a first order property of M_0 which we can reflect to some Wadge level where we have a self-justifying system.

Unfortunately, $\pi^* \notin M_0$, so it is not at all clear one can obtain a bad sequence in $M_0[g]$. This is where we need HI(c).

Lemma 7.9.7 *The following is true in M_0: There is a countable structure $(R_0, \in, \mu, \rho, \mathcal{N}, \mathcal{Q}, t)$ such that*

(i) R_0 *is a transitive model of* ZFC$^-$ + *"μ is a normal measure on a measurable cardinal",*

(ii) *if g is any generic over R_0 for* $\mathrm{Col}(\omega, < \mathrm{crit}(\mu))$*, then t^g is a $(\rho, \mathcal{N}, \mathcal{Q})$-certified bad sequence, and*

(iii) *R_0 is linearly iterable by μ, and if $i\colon R_0 \to S$ is an iteration map from a countable length iteration, and g is S-generic over* $\mathrm{Col}(\omega, < i(\mathrm{crit}(\mu)))$*, then the projection of $i(t)^g$ (dropping out the last coordinate) is truly a bad sequence.*

Proof. Let $\mathcal{N} = \mathcal{N}^*$. Let R witness HI(c), with respect to A coding $\mathcal{P}_\infty$ and X coding $\pi^*, \mathcal{N}^*$. Let μ^* be a normal measure of R on ω_1^V, and let t^* be the term in R for a $(\pi^*, \mathcal{N}^*, \mathcal{P}_\infty)$-certified bad sequence in $R[g]$, for any g is on $\mathrm{Col}(\omega, < \omega_1^V)$. In R, let $Y \prec H_\eta^R$ with Y countable, where $H_\eta^R \models$ ZFC$^-$, and $\psi\colon R_0 \cong Y$ be the collapse map, suppose $\psi((\mu, \rho, \mathcal{N}^*, \mathcal{Q}, t)) = (\mu^*, \pi^*, \mathcal{N}^*, \mathcal{P}_\infty, t^*)$. We claim that $(R_0, \in, \mu, \rho, \mathcal{N}^*, \mathcal{Q}, t)$ has properties (i)-(iii) in M_0.

Parts (i) and (ii) are obvious. For (iii), let let $i\colon R_0 \to S$ come from iterating by μ and its images countably many times. Let $\phi\colon S \to R$ be some realization map, so that $\psi = \phi \circ i$, and let g be S-generic over $\mathrm{Col}(\omega, < i(\mathrm{crit}(\mu))$. By the elementarity of ϕ, we have that $i(t)^g$ is a $(i(\rho), \mathcal{N}^*, i(\mathcal{Q}))$-certified bad sequence. But letting $i(t)^g = \langle (\mathcal{N}_\alpha, \mathcal{T}_\alpha, b_\alpha, c_\alpha, j_\alpha, k_\alpha, l_\alpha, \pi_\alpha) \mid \alpha < \eta \rangle$,, we can simply compose our realizations into $\mathcal{Q}$ with ϕ to get $\langle (\mathcal{N}_\alpha, \mathcal{T}_\alpha, b_\alpha, c_\alpha, j_\alpha, k_\alpha, l_\alpha, \phi \circ \pi_\alpha) \mid \alpha < \eta \rangle$. We claim this is a $(\pi^*, \mathcal{N}^*, \mathcal{P}_\infty)$-certified bad sequence. Since i is elementary, this is pretty clear, except perhaps for the requirement that $\phi \circ \pi_0 = \pi^*$. There we have

$$\begin{aligned} \phi(\pi_0(x))) &= \phi(i(\rho)(x)) \\ &= \phi(i(\rho(x))) \\ &= \psi(\rho(x)) \\ &= \pi^*(x), \end{aligned}$$

for $x \in \mathcal{N}^*$. The first line comes from $i(\rho) = \pi_0$, which is one of the properties of certified badness of $i(t)^g$ in $S[g]$. The second comes from $i(x) = x$, the third from $\psi = \phi \circ i$, and the last from $\psi(x) = x$.

But the projection of any $(\pi^*, \mathcal{N}^*, \mathcal{P}_\infty)$- certified bad sequence is bad from the point of view of M_0. This proves (iii) holds in M_0. □

We have just shown that $M_0 \models \varphi_0$, where φ_0 is the sentence:

φ_0: There is a suitable $\mathcal{N}$ and a countable structure $(R_0, \in, \mu, \rho, \mathcal{N}, \mathcal{Q}, t)$ such that (i)-(iii) of 7.9.7 hold.

Now let β be least such that for some α,

$$L_\beta(P_\alpha(\mathbb{R})) \models \mathsf{ZF}^- \wedge \varphi_0.$$

Let $W = L_\beta(P_\alpha(\mathbb{R}))$. Using the minimality of W, we have a sjs $\langle A_i \mid i < \omega\rangle$ such that the universal $(\Sigma^2_1)^W$ set is A_0, and each A_i is OD^W. (This is a result of Woodin; we gave a similar argument in the proof of 7.7.15.) We have also that $W, \vec{A}$ are $\boldsymbol{\Delta^2_1}$ in M_0.

Let $\mathcal{N}$ and $(R_0, \in, \mu, \rho, \mathcal{N}, \mathcal{Q}, t)$ witness that $W \models \varphi_0$. Working in M_0, let T be a tree on some $\omega \times \lambda$ such that $p[T]$ is a universal Σ^2_1 set. Let r_0 be a real coding $(R_0, \in, \mu, \rho, \mathcal{N}, \mathcal{Q}, t)$, and such that W and $\vec{A}$ are $(\Delta^2_1(r_0))^{M_0}$, and there are Skolem functions for the structure $(\mathrm{HC}, \in, W, \vec{A})$ in $(\Delta^2_1(r_0))^{M_0}$. The existence of such Skolem functions implies that for any $G \in V$ which is generic over $L[T, r_0]$ for a poset in $L_{\omega_1^V}[T, r_0]$,

$$(\mathrm{HC}^L[T, r_0, G], \in, W \cap L[T, x, G], \vec{A} \cap L[T, x, G]) \prec (\mathrm{HC}, \in, W, \vec{A}).$$

Now let $g \in V$ be $L[T, r_0]$-generic for $\mathrm{Col}(\omega, <\omega_1^{L[T,r_0]})$. In $L[T, r_0, g]$ we can form $i\colon R_0 \to S$ by iterating $\omega_1^{L[T,r_0]}$ times. Let s be $i(t)^g$, projected onto its first 6 coordinates; say

$$s = \langle (\mathcal{N}_\alpha, \mathcal{T}_\alpha, b_\alpha, c_\alpha, j_\alpha, k_\alpha, l_\alpha) \mid \alpha < \eta\rangle,$$

where $\eta = \omega_1^{L[T,r_0,g]}$. Since $(R_0, \in, \mu, \rho, \mathcal{N}, \mathcal{Q}, t)$ has properties (i)-(iii) in W, we have that $W \models s$ is bad. Let $\mathcal{M}$ be the direct limit of the $\mathcal{N}_\alpha$ for $\alpha < \eta$, under the $j_{\alpha,\beta}$. Let $j_{\alpha,\infty}\colon \mathcal{N}_\alpha \to \mathcal{M}$ be the direct limit map.

Now let $h \in V$ be $L[T, r_0, g]$-generic for $\Pi_{\alpha\in[\eta,\eta_1]} \mathrm{Col}(\omega, \eta)$, where $\eta_1 = \omega_2^{L[T,r_0]}$. Letting S_1 be the result of iterating R_0 η_1 times, we get $i_1\colon S \to S_1$, and this lifts to $i_1\colon S[g] \to S_1[g][h]$. It follows that $i_1(i(t))^{g\times h}$ projects to a sequence u of length η_1 which extends s, and is also bad in W. But this means that $\mathcal{M} = \mathcal{N}^u_\eta$, from which we conclude

$$W \models \mathcal{M} \text{ is suitable }.$$

We have that $\langle \tau^{\mathcal{M}}_{A_i,\delta^{\mathcal{M}}} \mid i < \omega\rangle \in L[T, r_0, g]$, because $\langle A_i \cap L[T, r_0, g] \mid i < \omega\rangle \in L[T, r_0, g]$. Working in $L[T, r_0, g]$, we then get an odd ordinal $\alpha < \eta$ such that

$$\forall i < \omega(\tau^{\mathcal{M}}_{A_i,\delta^{\mathcal{M}}} \in \mathrm{ran}(j_{\alpha,\infty})),$$

from which it follows by the condensation properties of the capturing terms for a sjs that

$$j_\alpha(\tau^{\mathcal{N}_\alpha}_{A_i,\delta^{\mathcal{N}_\alpha}}) = \tau^{\mathcal{N}_{\alpha+1}}_{A_i,\delta^{\mathcal{N}_{\alpha+1}}}.$$

But $j_\alpha = k_\alpha \circ i_{b_\alpha} = l_\alpha \circ i_{c_\alpha}$, so by condensation again, for all i

$$i_{b_\alpha}(\tau^{\mathcal{N}_\alpha}_{A_i,\delta^{\mathcal{N}_\alpha}}) = i_{b_\alpha}(\tau^{\mathcal{N}_\alpha}_{A_i,\delta^{\mathcal{N}_\alpha}}) = \tau^{\mathcal{R}}_{A_i,\delta^{\mathcal{R}}},$$

where $\mathcal{R} = \mathcal{M}_{b_\alpha} = \mathcal{M}_{c_\alpha}$. But $\sup(\{\gamma^{\mathcal{R}}_{A_i,\delta^{\mathcal{R}}} \mid i < \omega\}) = \delta^{\mathcal{R}} = \delta(\mathcal{T}_\alpha)$ as usual, and this implies $b_\alpha = c_\alpha$. This is a contradiction, completing the proof of 7.9.1. □

7.10 Conclusion

We can now complete the proof of Theorem 7.1.3.

Lemma 7.10.1 *For any set of reals A in M_0, there is a scale on A all of whose associated prewellorders are in M_0.*

Proof. Let x be a real such that A is $\mathrm{OD}(x)^{M_0}$. Relativising 7.9.1 to x, we get a countable, Wadge cofinal collection $\mathcal{A}$ of $\mathrm{OD}(x)^{M_0}$ sets of reals containing A, and an x-mouse $\mathcal{N}$ with a fullness-preserving, $\mathcal{A}$-guided strategy Σ with properties (a)-(e) of 7.9.1. In particular, Σ condenses well and has the Dodd-Jensen property. One can now get a scale on A by a straightforward adaptation of the construction of [44, §2]. □

Corollary 7.10.2 *For any set of reals A in M_0, there is an sjs $\vec{B}$ such that $A = B_0$, and each $B_i \in M_0$.*

Now let $\vec{B}$ be an sjs with each B_i in M_0, and with B_0 being the universal $(\Sigma^2_1)^{M_0}$ set. Thus $\vec{B} \notin M_0$.

Lemma 7.10.3 $L(\vec{B}, \mathbb{R}) \models \mathsf{AD}$.

Proof. This is a core model induction like the one for $L(\mathbb{R})$. Things to note are:

(i) The pattern-of-scales results generalize from $L(\mathbb{R})$ to $L(\vec{B}, \mathbb{R})$. For this, it is important that $\vec{B}$ is a sjs. The Friedman games involved in our closed game representations will involve player I proving facts of the form "$y \in B_i$", and for this, he should use the tree of the scale on B_i coded into $\vec{B}$.

(ii) W^*_α has the same statement as before, with $J_\alpha(\vec{B}, \mathbb{R})$ replacing $J_\alpha(\mathbb{R})$.

(iii) in W_α, the lightface capturing mice are now $\vec{B}$-mice. These are obtained by adding extenders and closing under the operation $\mathcal{M} \mapsto \mathcal{M} \oplus \vec{B}$. Here $\mathcal{M} \oplus \vec{B}$ is $(\mathcal{M}^+, T)$, where the "+" operation is as above, with respect to the pointclass $(\Sigma^2_1)^{M_0}$, and $T(i, \tau)$ holds iff $\tau = \tau^{\mathcal{M}^+}_{B_i,\nu}$, for ν the i-th cardinal of $\mathcal{M}^+$ above $o(\mathcal{M})$.

(iv) Let S code up the trees of the scales on the B_i's which are given by $\vec{B}$. Then $j(S) \in V$. [Exercise, using that each B_i is OD from a real in M_0.] This enables us to show that $j(K) \in V$ at the successor steps in our core model induction.

We shall give no further detail here. □

Since $\vec{B} \notin M_0$, we get $L(\vec{B}, \mathbb{R}) \models \theta_0 < \theta$. This completes our proof of Theorem 7.1.3.

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