

Dilemmas and truths in set theory

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Set Theory and Cantor's Continuum Hypothesis

- ▶ Set theory started with the following theorem of Georg Cantor.
- ▶ Cantor (Nov 11, 1873, in a letter to R. Dedekind): \mathbb{R} is uncountable. I.e., there are uncountably many real numbers.
- ▶ Cantor's first proof of this used nested intervals.
- ▶ But **how many** real numbers are there?
- ▶ Continuum Hypothesis (CH): For every uncountable $A \subset \mathbb{R}$ there is a bijection $f: \mathbb{R} \rightarrow A$.
- ▶ Cantor's Program: Show CH by "induction on the complexity" of $A \subset \mathbb{R}$.

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- ▶ Cantor–Bendixson (1883): Every uncountable *closed* $A \subset \mathbb{R}$ contains a perfect subset.
- ▶ Young (1906): Every uncountable G_δ - oder F_σ -set $A \subset \mathbb{R}$ contains a perfect subset.
- ▶ Aleksandrov/Hausdorff (1916): Every uncountable *Borel* set $A \subset \mathbb{R}$ contains a perfect subset.
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Cantor's Generalized Continuum Hypothesis

- ▶ In addition to sets of natural numbers, of reals, of sets of reals, etc., Cantor started considering sets *in general*.
- ▶ “By a ‘set’ we understand any gathering-together M of determined well-distinguished objects m of our intuition or of our thought, into a whole.” (Cantor, 1995)
- ▶ This idea leads to the **cumulative hierarchy** of sets.
- ▶ For every set x whatsoever, the *power set* $\mathcal{P}(x)$ exists.

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- ▶ Cantor's Theorem (1892): Let x be any set. There is no surjection $f: x \rightarrow \mathcal{P}(x)$.
- ▶ This time, Cantor's proof uses a diagonal argument.
- ▶ *How big* is $\mathcal{P}(x)$ in comparison to x ?
- ▶ Generalized Continuum Hypothesis (GCH): For every infinite set x and every $A \subset \mathcal{P}(x)$, there is either a surjection $f: x \rightarrow A$ or else a bijection $f: \mathcal{P}(x) \rightarrow A$.
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The axiom system ZFC (Zermelo–Fraenkel with choice)

- ▶ Any two sets with the same elements are equal.
- ▶ For all x and y , $\{x, y\}$, $\bigcup x$, and $\mathcal{P}(x)$ exist.
- ▶ There is an infinite set.
- ▶ **Separation.** For all x and for all formulae $\varphi(y)$, $\{y \in x : \varphi(y)\}$ exists.
- ▶ **Replacement.** For all x and for all formulae $\varphi(y, z)$ such that for all $y \in x$ there is a unique z with $\varphi(y, z)$, $\{z : \exists y \in x \varphi(y, z)\}$ exists.
- ▶ Every x with $\emptyset \notin x$ admits a choice function.
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- ▶ The usual formulation of ZFC allows the formulae φ in Separation and Replacement to contain *parameters*.
- ▶ It may be shown, though, that these parameters are not needed:
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- ▶ ZFC formalizes the idea (albeit somewhat indirectly) that the universe of set theory arises from nothing (\emptyset) through the operations $x \mapsto \mathcal{P}(x)$ and $x \mapsto \bigcup x$ in a cumulative fashion:
- ▶ If we define $V_\alpha = \bigcup \{\mathcal{P}(V_\beta) : \beta < \alpha\}$ for ordinals α , then ZFC proves that every x is an element of some V_α . The V_α 's are called *ranks*.
- ▶ Provably, there is no set of all sets. (By Cantor's Theorem: if v were such a set, then there would be a surjection from v onto $\mathcal{P}(v)$.)
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Classes and Truth

- ▶ The introduction of classes is tantamount to adding a *truth predicate* to the language of set theory.
- ▶ **BGC** (Bernays–Gödel with choice) results from ZFC by adding a new sort of variables, class variables X, Y, \dots , and demanding that the universe of all classes is closed under the logical operations; instead of talking about formulae in Separation and Replacement we now talk about classes.
- ▶ **A philosophical credo.** In contrast to sets, classes do not exist *de re*, they just exist *de dicto*. Otherwise the collection of all classes would just be another rank of the set theoretical universe, and what appeared to be classes are in fact sets.

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$$\varphi \longleftrightarrow \ulcorner \varphi \urcorner \text{ is true}$$

for set theoretical φ may be proven in BGC.

- ▶ Sch (2002): The Tarski *sentence* of negation,

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- ▶ Tarski (1936)/Mostowski (1950): Whereas the truth predicate for set theory cannot be defined in the language of ZFC, it may be defined in the language of BGC in a Δ_1^1 fashion. All instances of the Tarski schema

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- ▶ Sch (1993): We may in fact define a Tarskian hierarchy of meta-languages for the language of set theory.
- ▶ Each language \mathcal{L}_α comes with a new sort of variables for classes of type α . We demand that if $\varphi(x)$ is from \mathcal{L}_β , some $\beta < \alpha$, then

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Large cardinals

- ▶ Replacement may be construed as a “large cardinal axiom.” It says that for every formula φ there is a rank V_α which reflects φ , i.e.,

$$\varphi(x_1, \dots, x_k) \longleftrightarrow V_\alpha \models \varphi(x_1, \dots, x_k)$$

for all $x_1, \dots, x_k \in V_\alpha$.

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- ▶ Here is a list of some of the large cardinal concepts which are on the market nowadays.
- ▶ Inaccessible $<$ Mahlo $<$ weakly compact $<$ measurable $<$ strong $<$ Woodin $<$ subcompact $<$ supercompact $<$ I_0 $<$...
- ▶ Shelah/Woodin (1990): If there are infinitely many Woodin cardinals, then CH holds for all projective sets.
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Bernays' System of Class Theory

- ▶ Bernays has formulated a system of class theory which proves the existence of inaccessible and Mahlo cardinals via reflection principles.
- ▶ Bernays' System B_{refl} is BGC together with the following schema of reflection. For every formula φ in the language of BGC with no class quantifiers,

$$\forall X \varphi(X) \rightarrow \exists \text{ a transitive } u \forall x \subset u \varphi^u(x \cap u).$$

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- ▶ Our only apparently good arguments for the existence of large cardinals are based on reflection principles.
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The Consistency of Large Cardinals

- ▶ On the other hand, there are many statements which imply the *consistency* of the existence of large cardinals with ZFC, in fact the existence of canonical inner models with such large cardinals.
- ▶ One example is given by a violation of GCH:
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