Deconstructing inner model theory

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1 Introduction

In this paper we shall repair some errors and fill some gaps in the inner model theory of [2]. The problems we shall address affect some quite basic definitions and proofs.

We shall be concerned with *condensation properties* of canonical inner models constructed from coherent sequences $\vec{E}$ of extenders as in [2]. Condensation results have the general form: if x is definable in a certain way over a level $\mathcal{J}_\alpha^{\vec{E}}$, then either $x \in J_\alpha^{\vec{E}}$, or else from x we can reconstruct $\mathcal{J}_\alpha^{\vec{E}}$ in a simple way.

The first condensation property considered in [2] is the *initial segment condition*, or ISC. In section 1 we show that the version of this condition described in [2] is too strong, in that no coherent $\vec{E}$ in which the extenders are indexed in the manner of [2], and which is such that $L[\vec{E}]$ satisfies the mild large cardinal hypothesis that there is a cardinal which is strong past a measurable, can satisfy the full ISC of [2]. It follows that the coherent sequences constructed in [2] do not satisfy the ISC of [2]. We shall describe the weaker ISC which these sequences do satisfy, and indicate the small changes in the arguments of [2] this new condition requires.

In section 2, we fill a gap in the proof that the standard parameters of a sufficiently iterable premouse are solid. This is Theorem 8.1 of [2], one of its central fine structural results. In section 3, we fill a gap in the proof that the Dodd parameter of a sufficiently iterable premouse is Dodd-solid. This is Theorem 3.2 of [4], and is an important ingredient in the proofs of square in $L[\vec{E}]$ and of weak covering for K. The difficulties we overcome in sections

2 and 3 arise from the need to deal with premouse-like structures which do not satisfy even the weaker ISC we introduce in this paper.

In a sense, all of the difficulties we are addressing here stem from the fact that for coherent sequences indexed as in [2], we do not know how to prove that the comparison process terminates without making use of some form of the ISC. Building on an idea of S. Friedman, Jensen has developed the theory of a different sort of coherent sequence. One can think of a Friedman-Jensen sequence as a dilution of a sequence $\vec{E}$ from [2]; it contains the extenders from $\vec{E}$, interspersed with extenders which only appear on ultrapowers of $\vec{E}$. Jensen's fine structure theory has many similarities to that of [2], but one way it is significantly simpler is that, granting that there are no extenders of superstrong type on $\vec{E}$, one can prove a comparison lemma without appealing in any substantive way to an initial segment condition. Consequently, the problems we are addressing here show up in the Jensen framework only at the level of (in fact many) superstrong cardinals.[1]

2 The initial segment condition

Foremost among the objects of interest in [2] are the structures which arise as stages in certain natural constructions, now known as K^c-constructions. For expository reasons, [2] records certain basic properties of these structures rather early, in the definitions of "good extender sequence" and "premouse" (definitions 1.0.4 and 3.5.1), and only constructs them much later (in section 11), after developing many of their further properties axiomatically. Unfortunately, the structures constructed in section 11 of [2] do not in general have all of the properties laid out in Definition 1.0.4 of [2]. The trouble lies in clause (5) of that definition, the initial segment condition. That clause implies that if E_α is the last extender of a structure $\mathcal{J}_\alpha^{\vec{E}}$ occurring in a K^c-construction, and $F = E_\alpha \restriction \gamma$ is an initial segment of E_α containing properly less information (in that $\mathrm{ult}(\mathcal{J}_\alpha^{\vec{E}}, F) \neq \mathrm{ult}(\mathcal{J}_\alpha^{\vec{E}}, E_\alpha)$), then $F \in J_\alpha^{\vec{E}}$. This much is in fact true, but clause (5) goes on to say exactly how F might be obtained from $\vec{E} \restriction \alpha$, and gets this wrong in the case that F is an extender of a certain exceptional type which we call *type Z*.

[1]In the Jensen framework, solidity was proved by Jensen, and Dodd solidity by Zeman.

Definition 2.1 *Let $\kappa < \nu$ and suppose that $\mathcal{M}$ is transitive and rudimentary closed. Let E be a (κ, ν) pre-extender over $\mathcal{M}$. Then we say that E is* of type Z *if E has a largest generator $\xi < \nu$ which is itself a limit of generators of E, and the length of the trivial completion of $E \restriction \xi$ is the same as the length of the trivial completion of $E \restriction \xi + 1$, i.e.,*

$$\xi^{+Ult(\mathcal{M}, E \restriction \xi)} = \xi^{+Ult(\mathcal{M}, E \restriction \xi+1)}.$$

If E is type Z and ξ is its largest generator, then E and the trivial completion of $E \restriction \xi$ would be given the same index in any good extender sequence on which they appeared. Conversely, let E be an extender over $\mathcal{M}$ which is its own trivial completion, and suppose that $\text{crit}(E)^{+\mathcal{M}} \leq \xi < \nu(E)$, where $\nu(E)$ is the strict sup of the generators of E. Let $\eta = \nu(E \restriction \xi)$, and suppose that $\eta^{+Ult(\mathcal{M}, E \restriction \eta)} = lh(E)$ (i.e., the length of the trivial completion of $E \restriction \xi$ is the same as the length of E.) Then in fact $\eta = \xi$ is a limit of generators of E, and ξ is also the largest generator of E. That is, E is of type Z. Thus it is only in the case of type Z extenders E that the indexing of [2] would require E and some initial segment of E to be given the same index.

Definition 2.2 *An extender sequence $\vec{E}$ satisfies the weak initial segment condition, or weak ISC, iff for any $\alpha \leq dom(\vec{E})$ and any η such that $crit(E_\alpha)^{+J^{\vec{E}}_\alpha} \leq \eta < \nu(E_\alpha)$, we have $E_\alpha \restriction \eta \in J^{\vec{E}}_\alpha$.*

It is easy to see that an extender sequence $\vec{E}$ indexed as in [2] which satisfies the weak ISC cannot have type Z extenders on it. For if E_α is type Z with largest generator ξ, then the weak ISC plus coherence give $E_\alpha \restriction \xi \in Ult_0(J^{\vec{E}}_\alpha, E_\alpha)$. The indexing tells us α is a cardinal of this ultrapower, and yet $E_\alpha \restriction \xi$ collapses α in the ultrapower, a contradiction.

It is also easy to see that the extender sequences constructed in section 11 of [2] do not have type Z extenders on them. This is because in deciding which extender to add with index α to some $\mathcal{M}_\beta$, [2] first minimizes $\nu(E)$ among all candidates E. If E is a candidate of type Z with largest generator $\xi = \nu(E) - 1$, and F is the trivial completion of $E \restriction \xi$, then F is also a candidate, and $\nu(F) < \nu(E)$. Thus E will not be added to the extender sequence being built.

We shall now show, however, that any moderately rich $\vec{E}$ must have on it extenders which have type Z initial segments.[2] In particular, the sequences built in [2] have such extenders on them. It follows that these sequences do not satisfy clause (5) of definition 1.0.4 in [2], the ISC of that paper. Since the main idea here involves no fine structure, we shall formulate it in a "coarse" setting.

Theorem 2.3 *Let M be a transitive, and suppose that M satisfies ZFC plus "F is an extender witnessing that μ is $\kappa + 2$-strong, for some measurable cardinal κ". Then there is an α such that the trivial completion of $F \restriction (\alpha+1)$ is of type Z.*

Proof. We may as well work in M, so let $M = V$. Let F and μ be as described, and let κ be least such that $\mu < \kappa$ and κ is measurable in $Ult(V, F)$. Let

$$k: Ult(V, F \restriction \kappa) \to Ult(V, F)$$

be the natural embedding. Since κ is definable from μ in $Ult(V, F)$, $\kappa \in \mathrm{ran}(k)$, so that $\mathrm{crit}(k) > \kappa$. (It is easy to see that in fact, $\mathrm{crit}(k)$ is the κ^+ of $Ult(V, F \restriction \kappa)$.) Let D be a normal ultrafilter on κ in the sense of $Ult(V, F \restriction \kappa)$, and let $G = k(D)$. Since F was $\kappa + 1$-strong, G is a normal ultrafilter on κ in V. We claim that for G-a.e. α, the trivial completion of $F \restriction (\alpha + 1)$ is of type Z.

In order to show this, let $i: V \to Ult(V, G)$ be the canonical embedding, and let

$$H = \text{ trivial completion of } i(F) \restriction (\kappa + 1).$$

We must see that H has type Z. Since $\kappa \notin \mathrm{ran}\,(i)$, we get $\kappa \neq [a, f]_H$ for all $a \in [\kappa]^{<\omega}$ and $f: [\mu]^{|a|} \to \mu$, so κ is a generator of H. Clearly, κ is a limit of generators, and the largest generator, of H. Thus to show that H is type Z, we must show $Ult(V, H)$ and $Ult(V, H \restriction \kappa)$ compute the same cardinal successor of κ.

Notice $H \restriction \kappa = F \restriction \kappa$. We then have

$$\kappa^{+Ult(V,H\restriction\kappa)} = \kappa^{+Ult(Ult(V,F\restriction\kappa),D)},$$

[2]This result is due to Zeman, who also discovered the type Z extenders, and realized that they show the ISC of [2] is too strong.

since going to the ultrapower by D doesn't change the cardinal successor of crit(D). It will suffice then to show that H is an initial segment of the extender derived from $i_D \circ i_{F\restriction\kappa}$, since then the canonical embedding from $Ult(V,H)$ to $Ult(Ult(V,F\restriction\kappa),D)$ has critical point $>\kappa$, so that

$$\kappa^{+Ult(V,H)} \leq \kappa^{+Ult(Ult(V,F\restriction\kappa),D)} = \kappa^{+Ult(V,H\restriction\kappa)},$$

as desired.

But let $a \subseteq \kappa$ be finite, and $X \subseteq [\mu]^{|a|+1}$. We have

$$\begin{aligned} a\cup\{\kappa\} \in i_D \circ i_{F\restriction\kappa}(X) &\leftrightarrow \text{ for } D \text{ a.e. } \alpha(a\cup\{\alpha\} \in i_{F\restriction\kappa}(X)) \\ &\leftrightarrow \text{ for } G \text{ a.e. } \alpha(a\cup\{\alpha\} \in i_{F\restriction\kappa}(X)) \\ &\leftrightarrow \text{ for } G \text{ a.e. } \alpha(X \in F_{a\cup\{\alpha\}}) \\ &\leftrightarrow X \in H_{a\cup\{\kappa\}}, \end{aligned}$$

as desired.

□

There are fine-structural refinements of 2.3, but 2.3 by itself is enough to show that clause (5) of 1.0.4 of [2] is not satisfied by the sequences constructed there. What is the appropriate ISC for these sequences? It turns out that the only initial segments of extenders on such a sequence which provide counterexamples to clause (5) are the type Z ones, so that we need only weaken clause (5) by making an exception for these.[3]

Definition 2.4 *(Correction to [2]* DEF. *1.0.4. clause (5)) (Closure under initial segment) Let ν be the natural length of E_α. If η is an ordinal such that $(\kappa^+)^{J^{\vec{E}}_\alpha} \leq \eta < \nu$, η is the natural length of $E_\alpha \restriction \eta$,* and $E_\alpha \restriction \eta$ is not of type Z*, then one of (a) or (b) below holds:*

(a) There is $\gamma < \alpha$ such that E_γ is the trivial completion of $E_\alpha \restriction \eta$.

(b) $\eta \in S$ and there is a $\gamma < \alpha$ such that $\pi(\vec{E}\restriction\eta)_\gamma$ is the trivial completion of $E_\alpha \restriction \eta$, where $\pi\colon \mathcal{J}^{\vec{E}\restriction\eta}_\eta \leftrightarrow Ult(\mathcal{J}^{\vec{E}\restriction\eta}_\eta, E_\eta)$ is the canonical embedding.

We intend 2.4 to replace [2] DEF. 1.0.4. (5) from now on, which of course changes the meaning of every phrase and statement depending on it. Most importantly, "premouse" gets thereby re-defined.

[3]This observation is due independently to Schindler and Woodin.

Notice that it now follows that no sequence $\vec{E} = (E_\beta : \beta \in S)$ which is good at some α can have a member which is of type Z (and has index $\leq \alpha$). Suppose otherwise, and let $E_\beta \in \vec{E}$ be of type Z. Let ξ be the largest generator of E_β. The trivial completion F of $E_\beta \upharpoonright \xi$ is not of type Z. Moreover, ξ is a cardinal in $J_\beta^{\vec{E}\upharpoonright\beta}$ (for the same reason that $\nu(E_\gamma)$ is a cardinal of $J_\gamma^{\vec{E}}$ whenever $\nu(E_\gamma)$ is a limit ordinal), so $\xi \notin S$. Therefore alternative (a) of 2.4 must apply, which means that F is on $\vec{E}$, and in fact $E_\beta = F$, contradiction. It follows that no premouse has an extender of type Z on its sequence.

We must show that the levels $\mathcal{N}_\alpha$ of the K^c-construction in section 11 of [2] are premice in our new sense. This is in fact what [2] proves in sections 10 and 11. In its "proof" that the full clause (5) of 1.0.4 holds of $\mathcal{N}_\alpha$, [2] simply ignores the possibility that the initial segment in question might have type Z; otherwise the proof is correct.[4] Thus [2] does prove our weaker ISC 2.4 holds of $\mathcal{N}_\alpha$.

We must also show that our weaker ISC 2.4 still suffices for the uses of the ISC in [2]. The only (but still crucial) use of the ISC there lies in the proof that the comparison process terminates. The use of the ISC occurs in the proof of Lemma 7.2, which asserts that if $(\mathcal{T},\mathcal{U})$ is a coiteration, then for all α and β, $E_\alpha^{\mathcal{T}}$ is not an initial segment of $E_\beta^{\mathcal{U}}$. Our weaker ISC suffices here: note that $E_\alpha^{\mathcal{T}}$ is on the sequence of $\mathcal{M}_\alpha^{\mathcal{T}}$, and hence is not of type Z, so that 2.4 still gives $E_\alpha^{\mathcal{T}} \in J_\gamma^{\mathcal{M}_\beta^{\mathcal{U}}}$, for $\gamma = \mathrm{lh}(E_\beta^{\mathcal{U}})$. This leads to the same contradiction reached in [2].

Finally, we must verify that being a premouse is preserved under the appropriate ultrapowers and Skolem hulls. The key here is Lemma 2.5 of [2], according to which "I am a premouse" can be expressed by an rQ sentence. The following two little lemmas are quite helpful here.

Lemma 2.5 *Let $\vec{E} = (E_\beta : \beta \in S)$ be good at α with the possible exception of the initial segment condition 2.4. Let $\alpha \in S$, and let Γ be the set of generators of E_α. Suppose that max(Γ) exists, call it γ, and suppose further that $J_\gamma^{\vec{E}}$ has a largest cardinal, call it κ. Assume that E_γ is the trivial completion of*

[4] This mistake, which was discovered by Schindler, occurs on lines 22 and 23 of page 98. Let us adopt the notation there. The problem is that G might be of type Z with largest generator $\rho - 1$, and $E_\beta^{\mathcal{U}} = E_1^{\mathcal{U}}$ might be the trivial completion of $G \upharpoonright \rho - 1$. Thus $\eta = \gamma$ does not imply $E_\beta^{\mathcal{U}} = G$ at this point, contrary to the authors' claim.

$E_\alpha \restriction \kappa$. *Then either* $\Gamma \cap [\kappa, \gamma) = \emptyset$*, or else* $\Gamma \cap [\kappa, \gamma) = \{\kappa\}$ *and* $E_\alpha \restriction \gamma$ *is of type Z with largest generator* κ*.*

PROOF. Set $\mathcal{M} = J_\alpha^{\vec{E}}$. Consider the maps

$$\sigma: Ult^1 = Ult_0(\mathcal{M}, E_\alpha \restriction \kappa) \to Ult^2 = Ult_0(\mathcal{M}, E_\alpha \restriction \kappa + 1) \text{ and}$$

$$\tau: Ult^2 = Ult_0(\mathcal{M}, E_\alpha \restriction \kappa + 1) \to Ult^3 = Ult_0(\mathcal{M}, E_\alpha).$$

Because E_γ is the trivial completion of $E_\alpha \restriction \kappa$ we have $\gamma = \kappa^{+Ult^1}$, and $\vec{E}^{Ult^1} \restriction \gamma = \vec{E} \restriction \gamma$ (using coherency, [2] DEF. 1.0.4 (4)). As $\gamma \in \Gamma \setminus \kappa + 1$, we have that $\tau \neq id$. But as $\gamma = max(\Gamma)$ and $\tau \restriction \kappa^{+Ult^2} = id$ with $\kappa^{+Ult^2} \geq \kappa^{+Ult^1} = \gamma$, we obtain that $crit(\tau) = \gamma = \kappa^{+Ult^2}$.

CASE 1. $\sigma = id$. Then $\kappa \notin \Gamma$, and in fact $\Gamma \cap [\kappa, \gamma) = \emptyset$.

CASE 2. $\sigma \neq id$. Then $crit(\sigma) = \kappa$, so $\kappa \in \Gamma$, and in fact $\Gamma \cap [\kappa, \gamma) = \{\kappa\}$. Also, $E_\alpha \restriction \kappa + 1$ is then of type Z, because $\kappa^{+Ult^1} = \gamma = \kappa^{+Ult^2}$; and hence $E_\alpha \restriction \gamma$ is of type Z with largest generator κ.

$\square$ (2.5)

Lemma 2.6 *Let* $\vec{E} = (E_\beta : \beta \in S)$ *be good at* α*. Let* $\alpha \in S$*, and let* Γ *be the set of generators of* E_α*. Suppose that* $max(\Gamma)$ *exists, call it* γ*, and suppose that* $E_\alpha \restriction \gamma$ *is of type Z. Let* κ *be the largest generator of* $E_\alpha \restriction \gamma$*. Then* κ *is the largest cardinal of* $J_\gamma^{\vec{E}}$*, and* E_γ *is the trivial completion of* $E_\alpha \restriction \kappa$*.*

We omit the easy proof of 2.6.

Let $\mathcal{M} = \mathcal{J}_\beta^{\vec{E}}$ be a premouse of type II. We then define $\gamma^{\mathcal{M}}$ exactly as on p. 10 of [2] unless the trivial completion G of $E_\beta \restriction \nu^{\mathcal{M}} - 1$ is of type Z; in this case we put $\gamma^{\mathcal{M}} = \nu^{\mathcal{M}} - 1$. Notice that we shall then have that 2.4 (b) cannot occur for $E_\beta \restriction \kappa$ where κ is the largest cardinal of $\mathcal{J}_{\gamma^{\mathcal{M}}}^{\vec{E}}$.

Given 2.5 and 2.6 we can now add to θ_5 on p. 18 of [2] one further disjunction ψ_3 dealing with the possibility that the trivial completion of $E_\beta \restriction \nu^{\mathcal{M}} - 1$ is of type Z. Let ψ_3 be "$\dot{\gamma} = \dot{\nu} - 1$ and $\dot{\gamma} \in dom(\dot{E})$ and $J_{\dot{\gamma}}^{\dot{E}}$ has a largest cardinal and $\forall a, b, \kappa$ (κ is the largest cardinal of $J_{\dot{\gamma}}^{\dot{E}} \wedge \dot{F}(a, b, \kappa) \Rightarrow a \subseteq \dot{E}_{\dot{\gamma}} \wedge \forall \xi < \dot{\gamma}(\xi$ is a generator of $\dot{E}_{\dot{\gamma}} \Rightarrow \xi < \kappa$))." It is straightforward to see that ψ_3 can be written in an $r\Pi_1$ fashion. By 2.5 and 2.6, $\psi_1 \vee \psi_3$ captures 2.4 (a) for the critical restriction of the top extender. This then fixes the proof of [2] Lemma 2.5 (c).

Fixing the rest of [2] (with the exception of §8) goes through routinely. As we remarked above, it is the new, weaker version of closure under initial segment which is actually proved in §10, and so Theorem 10.1 in its new interpretation is what is actually proved there. In §11, [2] Theorem 11.4 needs the additional hypothesis that G (in the notation of [2] p. 102) is not of type Z. Its proof then goes through as before; notice that on p. 104 the use of Theorem 10.1 now still gives the desired conclusion.

It is natural to ask how what we have called the weak ISC (2.2) is related to the more elaborate ISC (2.4) which we have made part of our new definition of "premouse". The answer is that for iterable structures, they are equivalent.

Theorem 2.7 *Let $\vec{E}$ be an extender sequence, with last extender E_α. Suppose $\mathcal{J}_\beta^{\vec{E}}$ is an ω-sound premouse, for all $\beta < \alpha$, and that $\vec{E}$ satisfies clauses (1)-(4) in definition 1.0.4 of [2]. Suppose also that $\mathcal{J}_\alpha^{\vec{E}}$ is $(0, \omega_1, \omega_1 + 1)$-iterable. Then the following are equivalent:*

1. *$\vec{E}$ satisfies the weak ISC,*
2. *$\mathcal{J}_\alpha^{\vec{E}}$ is a premouse.*

We omit the proof of 2.7. The proof involves comparison arguments like the one in the proof of 10.1 of [2], and this is why we need the iterability hypothesis. In view of this result, we might have let the weak ISC serve as our substitute for clause (5) of 1.0.4, instead of using 2.4; in the iterable case, which is all we care about, we get the same objects. In any case, it does seem useful to know exactly how initial segments of an extender E_α on $\vec{E}$ are related to $\vec{E} \restriction \alpha$, and this is what 2.4 tells us.

3 The solidity of the standard parameter

There is still one more problem with [2], in that the proof of Theorem 8.1 may require us to form iteration trees on certain phalanxes in such a way that we obtain structures which do not satisfy even the weak ISC. Let us give such phalanxes a name.

Definition 3.1 *Let $\mathcal{M}$ and $\mathcal{N}$ be either both ppm's or else both sppm's. We call the phalanx $(\mathcal{M}, \mathcal{N}, \alpha)$* anomalous *provided $\alpha = \kappa^{+\mathcal{N}}$ for some*

$\kappa \in Card^{\mathcal{M}}$ *and if* $\beta \geq \alpha$ *is least such that* $\rho_\omega(\mathcal{J}_\beta^{\mathcal{M}}) = \kappa$ *then we have that* $\mathcal{J}_\beta^{\mathcal{M}}$ *is an active type III ppm with* $\nu^{\mathcal{J}_\beta^{\mathcal{M}}} = \kappa$.

Let us adopt the notation of 3.1 for a moment. Suppose we want to iterate $(\mathcal{M}, \mathcal{N}, \alpha)$, and at some stage we are to use an extender $E_\xi^{\mathcal{T}}$ with critical point κ to extend our tree $\mathcal{T}$. The rules for phalanx iteration require us to apply $E_\xi^{\mathcal{T}}$ to the longest possible initial segment of $\mathcal{M}$, which in this case will be $\mathcal{J}_\beta^{\mathcal{M}}$. However, $\mathcal{J}_\beta^{\mathcal{M}}$ is of type III, and if we follow the standard procedure of squashing before taking the ultrapower, we get a structure $(\mathcal{J}_\beta^{\mathcal{M}})^{sq}$ of ordinal height κ. Since $\text{crit}(E_\xi^{\mathcal{T}}) = \kappa$, it is not clear how to make sense of $Ult((\mathcal{J}_\beta^{\mathcal{M}})^{sq}, G)$![5]It can be verified that the only reason for such trouble is in fact an anomaly.[6]

Our solution to this difficulty is to modify the rules for forming iteration trees slightly in this anomalous case. Notice that if $\mathcal{J}_\beta^{\mathcal{N}}$ and κ are as in 3.1 then $\rho_1(\mathcal{J}_\beta^{\mathcal{N}}) = \kappa$.

Definition 3.2 *Let* $\mathcal{B} = (\mathcal{M}, \mathcal{N}, \alpha)$ *be an anomalous phalanx. Let* $\alpha = \kappa^{+\mathcal{N}}$ *where* $\kappa \in Card^{\mathcal{M}}$*, and let* $\beta \geq \alpha$ *be least such that* $\rho_\omega(\mathcal{J}_\beta^{\mathcal{M}}) = \kappa$ *(so* $\rho_1(\mathcal{J}_\beta^{\mathcal{N}}) = \kappa$*). We define what it means to be a k-maximal, normal iteration tree of length* θ *on* $\mathcal{B}$ *in exactly the same way as according to [5] Def. 6.6, except for the following amendment to clause (3):*

Suppose that $\xi + 1 < \theta$ *is such that* $E_\xi^{\mathcal{T}}$ *has critical point* κ *(in which case* $\kappa^{+\mathcal{M}_\xi^{\mathcal{T}}} = \alpha$*). Let* G *be such that*

$$\tilde{G} = \dot{F}^{Ult_0(\mathcal{J}_\beta^{\mathcal{M}}, E_\xi^{\mathcal{T}})},$$

where we want to emphasize that the whole universe of $\mathcal{J}_\beta^{\mathcal{N}}$ *is supposed to be the domain of the ultrapower. Then*

$$\mathcal{M}_{\xi+1}^{\mathcal{T}} = Ult_k(\mathcal{M}, G)$$

with k *being as above.*

[5]The possibility that anomalous phalanxes might arise in the proof of 8.1 of [2], and the problems this causes, were first realized by Jensen. The solution we describe here is due to Schindler and Zeman.

[6]For example, this never happens in an iteration tree on a single premouse; see [2], at the top of page 50, for a proof.

A remark on what is going on here might be in order. As we cannot squash $\mathcal{J}_\beta^{\mathcal{N}}$ and apply $E_\xi^{\mathcal{T}}$ we are forced to apply it to the unsquashed structure. However, the ultrapower thus obtained will not be a premouse, as its top extender, G, does not satisfy the initial segment condition (this was the reason for squashing in the first place). It can easily be seen (cf. the proof of [2] Lemma 9.1) that the natural length of G is the same as the natural length of $E_\xi^{\mathcal{T}}$; i.e., G was added "too late" to the ultrapower's sequence, and its index should thus somehow be counted as being $lh(E_\xi^{\mathcal{T}})$. Therefore we use G right after $E_\xi^{\mathcal{T}}$. (For this reason, although 3.2 calls $\mathcal{T}$ "normal," it is not really normal.) Notice that in any case $\mathcal{M}_{\xi+1}^{\mathcal{T}}$ is a ppm, or a sppm. Moreover, we do not let $Ult_0(\mathcal{J}_\beta^{\mathcal{M}}, E_\xi^{\mathcal{T}})$ appear on $\mathcal{T}$.

We now use this way of iterating anomalous phalanxes to fix the proof of [2] Theorem 8.1. In fact, we may as well fix the proof of the definitive version of this result, which is the following theorem of [3].

Theorem 3.3 *Let $k < \omega$, and let $\mathcal{M}$ be a k-sound, $(k, \omega_1, \omega_1 + 1)$-iterable premouse. Let r be the $k + 1^{st}$ standard parameter of $(\mathcal{M}, u_k(\mathcal{M}))$; then r is $k + 1$-solid and $k + 1$-universal over $(\mathcal{M}, u_k(\mathcal{M}))$.*

Proof. The solidity and universality of r are statements in the first order theory of $\mathcal{M}$, so if they fail, they fail in some countable fully elementary submodel of $\mathcal{M}$. Any countable elementary submodel of $\mathcal{M}$ inherits its $(k, \omega, \omega_1 + 1)$-iterability. Thus we may assume without loss of generality that $\mathcal{M}$ is countable.

The proof in [2] and [3] that r is universal has no gap, because the projectum is a cardinal of $\mathcal{M}$, and therefore the phalanx occurring in that part of the proof is not anomalous. So let us consider solidity.

Let $r = \langle \alpha_0, ..., \alpha_S \rangle$, where the ordinals α_i are listed in decreasing order. Let $\vec{e}$ be an enumeration of the universe of $\mathcal{M}$ such that $e_i = \alpha_i$ for all $i \leq S$. Let Σ be a $(k, \omega_1 + 1)$ iteration strategy for $\mathcal{M}$ having the weak Dodd-Jensen property relative to $\vec{e}$. (See [3].) To see that r is solid at $\alpha := \alpha_i$, we let

$$\mathcal{H} = \mathcal{H}_{k+1}^{\mathcal{M}}(\alpha \cup r \restriction i),$$

and then compare $\mathcal{M}$ with the phalanx $\mathcal{B} = (\mathcal{M}, \mathcal{H}, \alpha)$. We may assume that $\mathcal{B}$ is anomalous, because otherwise the proof in [2] and [3] is correct.

Let $\kappa \in Card^{\mathcal{H}}$ be such that $\alpha = \kappa^{+\mathcal{H}}$. Following [2], we shall denote by $\mathcal{U}$ and $\bar{\mathcal{T}}$ the iteration trees arising from the comparison of $\mathcal{M}$ and $\mathcal{B}$. On the $\mathcal{B}$-side we follow 3.2 in building $\bar{\mathcal{T}}$. That is, suppose that $\bar{\mathcal{T}}$ produces $\mathcal{M}_\gamma^{\bar{\mathcal{T}}}$ such that $E_\nu^{\mathcal{M}_\gamma^{\bar{\mathcal{T}}}} \neq \emptyset$ is part of the least disagreement (i.e., ν is least with $\mathcal{J}_\nu^{\mathcal{M}_\gamma^{\bar{\mathcal{T}}}} \neq \mathcal{J}_\nu^{\mathcal{M}_\gamma^{\mathcal{U}}}$), and we have that $crit(E_\nu^{\mathcal{M}_\gamma^{\bar{\mathcal{T}}}}) = \kappa$. Let $\beta \geq \alpha$ be least with $\rho_1(\mathcal{J}_\beta^{\mathcal{M}}) = \kappa$ (cf. 3.2). Let G be such that

$$\tilde{G} = \dot{F}^{Ult_0(\mathcal{J}_\beta^{\mathcal{M}}, E_\nu^{\mathcal{M}_\gamma^{\bar{\mathcal{T}}}})}.$$

Then we shall put

$$\mathcal{M}_{\gamma+1}^{\bar{\mathcal{T}}} = Ult_k(\mathcal{M}; G).$$

Here, k is as on p. 74 of [2]; we shall have that $\alpha \leq \rho_{k+1}^{\mathcal{M}}$, and hence $crit(G) < \rho_{k+1}^{\mathcal{M}}$.

On the $\mathcal{B}$-side we use the natural $\pi: \mathcal{H} \to \mathcal{M}$ to lift $\bar{\mathcal{T}}$ to a tree on $\mathcal{T}$ on $\mathcal{M}$. We use Σ to choose branches of $\mathcal{T}$ at limit stages, then choose the very same branches to extend $\bar{\mathcal{T}}$. The copying argument from [2] pp. 75 - 79 can be extended slightly so as to prove that $\mathcal{B}$ is iterable by the process we have just described.

There is one wrinkle in the copying argument, even in the case of arbitrary phalanxes $(\mathcal{M}, \mathcal{H}, \alpha)$ such that α is a successor cardinal in $\mathcal{H}$, if α is also the critical point of π. This problem was first noticed by Jensen ([1]), who called such phalanxes anomalous. The issue arises when the coiteration forces us to apply a surviving extender, say $E_\beta^{\bar{\mathcal{T}}}$, whose critical point κ is the cardinal predecessor of α in $\mathcal{H}$. Then $E_\beta^{\bar{\mathcal{T}}}$ is applied to a proper initial segment of $\mathcal{M}$. On the other hand, the $\mathcal{T}$-counterpart $E_\beta^{\mathcal{T}}$ of $E_\beta^{\bar{\mathcal{T}}}$ is applied to $\mathcal{M}$ and measures all subsets of $\kappa = \text{crit}(E_\beta^{\mathcal{T}})$ in $\mathcal{M}$, so $\mathcal{M}$ is *not* truncated at this point, which might ruin the copying of $\bar{\mathcal{T}}$ onto $\mathcal{T}$. Jensen realized that embedding $Ult(M, E_\beta^{\bar{\mathcal{T}}})$ into an appropriate initial segment of $Ult(M, E_\beta^{\mathcal{T}})$ enables us to continue the copying process and that $\mathcal{T}$ is then essentially an iteration tree in the sense of Def. 5.0.6 of [2].

The coiteration $(\bar{\mathcal{T}}, \mathcal{U})$ does terminate, despite the possible failure of our ISC 2.4, since only the very *first* extender used on a branch of $\bar{\mathcal{T}}$ can possibly come from the new clause in 3.2, and hence our ISC can be violated at most once along any branch of $\bar{\mathcal{T}}$.

The proofs of [2] and [3] show that if the final model on the $\mathcal{B}$-side sits above $\mathcal{H}$, then r is indeed solid at α. What needs more argument is the assertion that the final model does sit above $\mathcal{H}$. So suppose it sits above $\mathcal{M}$.

Using the weak Dodd-Jensen property of Σ, we get that the branches $[0,\theta]_U$ and $[-1,\theta]_T$ do not drop, and the final models and branch embeddings along them are identical (that is, $\mathcal{M}^{\mathcal{U}}_\theta = \mathcal{M}^{\bar{\mathcal{T}}}_\theta$ and $i^{\mathcal{U}}_{0,\theta} = i^{\bar{\mathcal{T}}}_{-1,\theta}$).[7]

Let F be the first extender used on $[0,\theta]_{\mathcal{U}}$, and let G be the first extender used on $[-1,\theta]_{\bar{\mathcal{T}}}$. We may assume that G violates the initial segment condition, as otherwise we get a contradiction as before. That is, we may assume that for some $\xi < lh(G)$ with $G \upharpoonright \xi$ not being of type Z we have that the trivial completion of $G \upharpoonright \xi$ is not on the sequence of $\mathcal{M}^{\bar{\mathcal{T}}}_\theta$. This can only happen when G comes from the new clause in 3.2. Hence, if we let $\gamma+1$ be least in $(-1,\theta]_{\bar{\mathcal{T}}}$ then we shall have that

$$\tilde{G} = \dot{F}^{Ult_0(\mathcal{J}^{\mathcal{M}}_\beta, E_\nu^{\mathcal{M}^{\bar{\mathcal{U}}}_\gamma})}, \text{ some } \nu, \text{ and}$$

$$\mathcal{M}^{\bar{\mathcal{T}}}_{\gamma+1} = Ult_k(\mathcal{M}, G).$$

The following claim says that although G is "bad," its action can be split into the action of two "good" extenders. Let $H = E_\nu^{\mathcal{M}^{\bar{\mathcal{T}}}_\gamma}$.

Claim A. We can also write $\mathcal{M}^{\bar{\mathcal{T}}}_{\gamma+1} = Ult_k(Ult_k(\mathcal{M}, E^{\mathcal{M}}_\beta), H)$.

Proof. For $a \in [\kappa]^{<\omega}$, $b \in [\kappa,\nu)^{<\omega}$, and appropriate $f = f_{\tau,q}$ for $q \in \mathcal{M}$ and $\tau \in Sk_k$ (resp. SK_k), we may put

$$[b, [a,f]^{\mathcal{M}}_{E^{\mathcal{M}}_\beta}]_H^{Ult_k(\mathcal{M},E^{\mathcal{M}}_\beta)} \mapsto [a \cup b, \tilde{f}]^{\mathcal{M}}_G,$$

where $\tilde{f}(u \cup v) = f(u)(v)$. All generators of $E^{\mathcal{M}}_\beta$ are smaller than κ, whereas $crit(H) = \kappa$. It is thus straightforward to check that this assignment defines an $\in$-isomorphism from $Ult_k(Ult_k(\mathcal{M}, E^{\mathcal{M}}_\beta), H)$ onto $Ult_k(\mathcal{M}, G) = \mathcal{M}^{\bar{\mathcal{T}}}_{\gamma+1}$.

$\square$ (Claim A)

Neither F nor $E^{\mathcal{M}}_\beta$ can be of type Z, as they both come from good extender sequences. Thus if $F \neq E^{\mathcal{M}}_\beta$ then we could use 2.4 to get that F or $E^{\mathcal{M}}_\beta$ would be on the sequence of $\mathcal{M}^{\mathcal{U}}_\theta = \mathcal{M}^{\bar{\mathcal{U}}}_\theta$, which gives the standard contradiction.

[7]The trees here are padded, so that they have the same length.

We may therefore assume that $F = E_\beta^{\mathcal{M}}$. Let $\delta + 1$ be least in $(0, \theta]_{\mathcal{U}}$ (so $F = E_\delta^{\mathcal{U}}$). By $F = E_\beta^{\mathcal{M}}$ we have that that

$$\mathcal{M}_{\delta+1}^{\mathcal{U}} = Ult_k(\mathcal{M}, F) = Ult(\mathcal{M}; E_\beta^{\mathcal{M}}).$$

We need just one more observation.

Claim B. $i_{\delta+1,\theta}^{\mathcal{U}} = i_{\gamma+1,\theta}^{\bar{\mathcal{T}}} \circ i_H$, where i_H denotes the map from taking the k-ultrapower of $Ult_k(\mathcal{M}, E_\beta^{\mathcal{M}})$ by H.

Proof. Writing $i_{E_\beta^{\mathcal{M}}}$ for the map from taking the k-ultrapower of $\mathcal{M}$ by $E_\beta^{\mathcal{M}}$, Claim A buys us that

$$i_{\delta+1,\theta}^{\mathcal{U}} \circ i_{0,\delta+1}^{\mathcal{U}} = i_{0,\theta}^{\mathcal{U}} = i_{0,\theta}^{\bar{\mathcal{T}}} = i_{\gamma+1,\theta}^{\bar{\mathcal{T}}} \circ i_H \circ i_{E_\beta^{\mathcal{M}}},$$

so that for $x = i_{0,\delta+1}^{\mathcal{U}}(f)(a) \in \mathcal{M}_{\delta+1}^{\mathcal{U}} = Ult_k(\mathcal{M}, E_\beta^{\mathcal{M}})$ with $a \in [\beta]^{<\omega}$ and appropriate f we get that $i_{\delta+1,\theta}^{\mathcal{U}}(x) = i_{0,\theta}^{\mathcal{U}}(f)(a) = i_{\gamma+1,\theta}^{\bar{\mathcal{T}}} \circ i_H \circ i_{E_\beta^{\mathcal{M}}}(f)(a) = i_{\gamma+1,\theta}^{\bar{\mathcal{T}}} \circ i_H(x)$.

□ (Claim B)

Now let E^* be the second extender used on $[0, \theta]_{\mathcal{U}}$. We then have that E^* is compatible with H by Claim B. However, neither E^* nor H can be of type Z, so that we can use 2.4 once more to get the standard contradiction.

We have shown that the final model of the $\mathcal{B}$-side does not sit above $\mathcal{M}$. We can now argue that $\mathcal{H} \in \mathcal{M}$ just as is done in [2]. This completes our repair.

4 The solidity of the Dodd parameter

In this section we shall fill a rather substantial gap in the proof of Theorem 3.2 of [4].[8] Fix, for the rest of this section, an active ppm $\mathcal{M}$, with F the last extender of $\mathcal{M}$, and $\mu = \text{crit}(F)$. We shall show that the Dodd parameter of $\mathcal{M}$ is solid. For the reader's convenience, we include some of the relevant definitions.(See [4], §3.)

[8]That "proof", and the arguments we give here completing it, are due to Steel.

If $\xi < \text{lh}(F)$ and $t \in [\text{lh}(F) - \xi]^{<\omega}$, then we take $F \restriction (\xi \cup t)$ to mean $\{(a, x) \in F \mid a \in [\xi \cup t]^{<\omega} \wedge x \in J_\xi^{\mathcal{M}}\}$. This is, perhaps, slightly different from the usual definition. Of course, when $\xi \geq (\mu^+)^{\mathcal{M}}$, then $F \restriction (\xi \cup t) = \{(a, x) \in F \mid a \in [\xi \cup t]^{<\omega}\}$.

We say that an ordinal $\xi < \text{lh}(F)$ is an F-generator relative to a given parameter t iff $\xi \neq [b, f]_F^{\mathcal{M}}$ for any $f\colon [\mu]^{|b|} \to \mu$, $f \in |\mathcal{M}|$, and $b \in [\xi \cup t]^{<\omega}$. As with the notion it generalizes, this depends solely on F, and not on $\mathcal{M}$.

Let $s = \{s_0 > \cdot \cdot \cdot > s_k\} \in [OR^{\mathcal{M}}]^{<\omega}$ be as long as possible with $s_i \geq (\mu^+)^{\mathcal{M}}$ and s_i the largest F-generator relative to $s|i$. So, if $\mathcal{M}$ is not type II, then $s = \emptyset$. If $\mathcal{M}$ is type II, then $s_0 = \dot{\nu}^{\mathcal{M}} - 1$, and s_1, should it exist, is the largest $\xi > (\mu^+)^{\mathcal{M}}$ such that ξ is an F-generator relative to $\{s_0\}$; it is easy to see that if s_1 exists, then $s_1 < s_0$, and so on. We call s the *Dodd parameter* of $\mathcal{M}$.

Let τ be the supremum of $(\mu^+)^{\mathcal{M}}$ and the F-generators relative to s. So τ is a limit ordinal, and it is not difficult to show that τ is in fact a cardinal of $\mathcal{M}$. We call τ the *Dodd projectum* of $\mathcal{M}$.

The following is a generalization of Theorem 3.2 of [4]. It says about the Dodd parameter and projectum what Theorem 8.1 of [2] says about the standard parameter and projectum.

Theorem 4.1 *Suppose $\mathcal{M}$ is a 1-sound, $(0, \omega_1, \omega_1 + 1)$-iterable active premouse, with last extender F, Dodd projectum τ, and Dodd parameter $s = \{s_0 > \cdot \cdot \cdot > s_k\} \in [OR^{\mathcal{M}}]^{<\omega}$. Then*

(A) For $i \leq k$, $F \restriction (s_i \cup s|i) \in |\mathcal{M}|$.

(B) For $\xi < \tau$, $F \restriction (\xi \cup s) \in |\mathcal{M}|$.

This result is actually stronger than 3.2 of [4], because we have dropped the assumption that $\mathcal{M}$ is 1-small. The ideas of [3] enable us to deal with arbitrary sufficiently iterable mice.

The proof of part (B) of 4.1 which is given in [4] is correct. We shall outline enough of the "proof" of part (A) given there that we can locate its gap, and then we shall fill that gap.

Let $\mathcal{M}, F, \tau, s$, and i constitute a counterexample to (A), with i as small as possible. By the weak ISC (2.2), we have $i > 0$. By taking a Skolem hull, we may assume that $\mathcal{M}$ is countable. Fix an enumeration $\vec{e} = \langle e_n | n < \omega\rangle$

of the universe of $\mathcal{M}$ such that $e_l = s_l$ for all $l \leq k$. Let Σ be a $(0, \omega_1 + 1)$-iteration strategy for $\mathcal{M}$ which has the weak Dodd-Jensen property relative to $\vec{e}$. (See [3].)

Let $\mathcal{P} = Ult(\mathcal{M}, F \restriction (s_i \cup s|i))$, and $\mathcal{Q} = Ult(\mathcal{M}, F)$, and let $j: \mathcal{M} \longrightarrow \mathcal{P}$ and $k: \mathcal{P} \longrightarrow \mathcal{Q}$ be the natural maps.

Let $\mathcal{N}$ be the unique candidate for a premouse with the following properties:

- $OR^{\mathcal{N}} = [\{s_0\}, id^+]^{\mathcal{M}}_{F_i} = k^{-1}((s_0)^+)^{\mathcal{P}}$ (recall that $\dot{\nu}^{\mathcal{M}} = s_0 + 1$)
- $\langle |\mathcal{N}|, \in, \dot{E}^{\mathcal{N}} \rangle = \mathcal{P} \| OR^{\mathcal{N}}$
- $\dot{F}^{\mathcal{N}} = E_j \restriction OR^{\mathcal{N}}$ (where E_j is the extender derived from j)

$\dot{F}^{\mathcal{N}}$ is essentially F_i, with its information re-arranged so that it can go on the sequence of the premouse-like structure $\mathcal{N}$. We must set $\dot{\mu}^{\mathcal{N}} = \mu$ and $\dot{\nu}^{\mathcal{N}} = k^{-1}(\dot{\nu}^{\mathcal{M}})$, but we have no value for $\dot{\gamma}^{\mathcal{N}}$, because we do not know that $\mathcal{N}$ satisfies even the weak ISC.

By the definition of s_i, $OR^{\mathcal{N}} > \text{crit}(k) = s_i > (\mu^+)^{\mathcal{N}} = (\mu^+)^{\mathcal{M}}$.

Let $\psi = k \restriction |\mathcal{N}|$. By the coherence of F, $\psi: |\mathcal{N}| \longrightarrow |\mathcal{M}|$. In fact, since $k(OR^{\mathcal{N}}) = OR^{\mathcal{M}}$, as a map from $\langle |\mathcal{N}|, \in, \dot{E}^{\mathcal{N}} \rangle$ to $\langle |\mathcal{M}|, \in, \dot{E}^{\mathcal{M}} \rangle$, ψ is fully elementary. It is shown in [4] that ψ is a cofinal and $r\Sigma_1$ elementary (in the language with $\in$, $\dot{F}$, $\dot{\mu}$, and $\dot{\nu}$, but without $\dot{\gamma}$) embedding from $\mathcal{N}$ to $\mathcal{M}$.

We now compare $\mathcal{M}$ with the phalanx $(\mathcal{M}, \mathcal{N}, s_i)$. Let $\mathcal{U}$ be the tree on $\mathcal{M}$ we obtain this way, using Σ to choose branches at limit stages and iterating away the least disagreement at successor stages. Let $\bar{\mathcal{T}}$ be the tree on $(\mathcal{M}, \mathcal{N}, s_i)$ we obtain this way; we define this tree simultaneously with a copied tree $(\text{id}, \psi)\bar{\mathcal{T}} = \mathcal{T}$ on $\mathcal{M}$. We use $\mathcal{M}_\alpha$ for the models of $\mathcal{U}$, $\bar{\mathcal{N}}_\alpha$ for the models of $\bar{\mathcal{T}}$, and $\mathcal{N}_\alpha$ for the models of $\mathcal{T}$. We have copy maps $\psi_\alpha: \bar{\mathcal{N}}_\alpha \to \mathcal{N}_\alpha$, and we choose branches at limit stages in $\mathcal{T}$ by using Σ; the same branch works as a choice for $\bar{\mathcal{T}}$ in virtue of the ψ's. (So $\bar{\mathcal{T}}$ and $\mathcal{T}$ have the same tree order.) We index so that $\bar{\mathcal{N}}_{-1} = \mathcal{M} = \mathcal{N}_{-1}, \psi_{-1} = \text{id}$, $\bar{\mathcal{N}}_0 = \mathcal{N}$, $\mathcal{N}_0 = \mathcal{M}$, and $\psi_0 = \psi$.

[4] shows that this comparison terminates (claim 2, p. 178), a fact which is not entirely routine because we have no ISC for $\dot{F}^{\mathcal{N}}$. Let $\text{lh}(\mathcal{U}) = \sigma + 1$ and $\text{lh}(\bar{\mathcal{T}}) = \text{lh}(\mathcal{T}) = \theta + 1$. The proof of [4] is correct in the case that $0 T \theta$, that is, when the last model on the phalanx side is above $\mathcal{N}$. The problem lies in the assertion, on line 20 of page 178, that the usual arguments using

the Dodd-Jensen lemma show that $-1T\theta$ is impossible. The rest of our work here will be devoted to showing $-1T\theta$ is impossible; the usual arguments are not enough.

So suppose $-1T\theta$, that is, the final model on the phalanx side of our comparison is above $\mathcal{M}$ in $\bar{\mathcal{T}}$. The "usual arguments", for which the weak Dodd-Jensen property of Σ suffices, do show that the branches $[-1,\theta]_T$ and $[0,\sigma]_U$ do not drop, that the final models on these branches are the same ($\bar{\mathcal{N}}_\theta = \mathcal{M}_\sigma$), and that these branches give rise to the same embedding ($i_{-1,\theta} = j_{0,\sigma}$, where we are writing $i_{\alpha,\beta}$ for $i^{\bar{\mathcal{T}}}_{\alpha,\beta}$ and $j_{\alpha,\beta}$ for $i^{\mathcal{U}}_{\alpha,\beta}$). As usual, this implies that the first extender G used on $[-1,\theta]_T$ in $\bar{\mathcal{T}}$ is compatible with the first extender E used on $[0,\sigma]_U$ in $\mathcal{U}$. What [4] overlooks, however, is that we do not get the usual contradiction (based on Lemma 7.2 of [2]) at this point, because G may not satisfy even the weak ISC of this paper. We have $G \neq E$ because they were part of a disagreement when they were used, and we have that G is not an initial segment of E, because $E = E^{\mathcal{U}}_\alpha$ does satisfy our ISC 2.4 in the model $\mathcal{M}^{\mathcal{U}}_\alpha$ from which it is taken. (Notice that $\mathcal{U}$ is an ordinary iteration tree on an ordinary premouse.) But we need more argument to deal with the case E is a proper initial segment of G.

So assume that E is a proper initial segment of G, so that the weak ISC fails for G. There are two ways this might happen. First, our phalanx $(\mathcal{M}, \mathcal{N}, s_i)$ might be anomalous, and

$$G = \dot{F}^{Ult_0(\mathcal{J}^{\mathcal{M}}_\beta, E^{\bar{\mathcal{T}}}_\xi)}$$

might be the stretch of an unsquashed type III extender, as in 3.2 (here $\beta = s_i^{+\mathcal{N}}$). In this case, we reach a contradiction just as we did in the last section. We used in that argument the weak ISC for $E^{\mathcal{M}}_\beta$ and for $E^{\bar{\mathcal{T}}}_\xi$. The former holds since $\mathcal{M}$ is a premouse. The latter holds unless $E^{\bar{\mathcal{T}}}_\xi$ is $i_{0,\xi}(\dot{F}^{\mathcal{N}})$, the last extender of $\bar{\mathcal{N}}_\xi$. But then we have $\mathrm{crit}(i_{0,\xi}) \geq s_i > \mu$ by the rules for $\bar{\mathcal{T}}$, so $\mathrm{crit}(E^{\bar{\mathcal{T}}}_\xi) = \mu$. As $(\mu^+)^{\mathcal{M}} \leq s_i$, we can't be in the anomalous case. Therefore the weak ISC also holds for $E^{\bar{\mathcal{T}}}_\xi$, and we have a contradiction as in the last section.

The other possibility is that

$$G = i_{0,\xi}(\dot{F}^{\mathcal{N}}) = \dot{F}^{\bar{N}_\xi}.$$

Since we don't know the weak ISC holds of $\dot{F}^{\mathcal{N}}$, we don't know it holds of G, so we don't get the usual contradiction. The rest of our argument is devoted to this case. Notice again that because $\text{crit}(i_{0,\xi}) \geq s_i$, $\text{crit}(G) = \mu$, and in fact $F \restriction (s_i \cup s \restriction i)$ "is" (i.e. codes the same embedding as) $G \restriction (s_i \cup t)$, for some t. But G is part of the extender H derived from the branch embedding $i_{-1,\theta} = j_{0,\sigma}$. We shall show that $\mathcal{M}$ knows enough about $j_{0,\sigma}$ that it can compute $G \restriction (s_i \cup t)$, so that $F \restriction (s_i \cup s \restriction i) \in \mathcal{M}$, as desired.

Recall that $E = E^{\mathcal{U}}_{\alpha}$, so that $\alpha + 1$ is the U-successor of 0 on $[0, \sigma]_U$. Let

$$\eta := \text{least } \gamma \in (0, \sigma]_U \text{ s.t. } \text{crit}(j_{\gamma,\sigma}) \geq j_{0,\gamma}(\mu),$$

or $\eta = \sigma$ if there is no such γ. If $\alpha + 1 < \eta$, then we set

$$I := \text{extender of length } j_{0,\eta}(\mu) \text{ derived from } j_{\alpha+1,\eta},$$

and if $\alpha + 1 = \eta$ we leave I undefined. Here I is a "long" extender, in that it may have measures not concentrating on its critical point. Each component measure of I concentrates on $[\mu^*]^{<\omega}$, for some $\mu^* < j_{0,\alpha+1}(\mu)$.

Lemma 4.2 *If I is defined, then for any finite $c \subseteq j_{0,\eta}(\mu)$, $I_c \in \mathcal{M}_{\alpha+1}$.*

PROOF. We show by induction on $\beta \in [\alpha + 1, \eta]_U$ that every component H_c of the length $j_{0,\beta}(\mu)$ extender H derived from $j_{\alpha+1,\beta}$ is in $\mathcal{M}_{\alpha+1}$. The base case $\beta = \alpha + 1$ is trivial, and the case β is a limit ordinal is quite easy, so we omit it. So suppose $\beta = \delta + 1$, and let τ be the U-predecessor of β. Let $\nu = \nu(E^{\mathcal{U}}_{\delta})$. It will clearly be enough to show that every component of the extender of length ν derived from $j_{\alpha+1,\beta}$ is a member of $\mathcal{M}_{\alpha+1}$, since U is non-overlapping, so that the extender of length $j_{0,\sigma}(\mu)$ derived from $j_{\alpha+1,\beta}$ has no generators above ν.

Fix $d \subseteq \text{crit}(E^{\mathcal{U}}_{\delta})$ and $e \subseteq (\nu - \text{crit}(E^{\mathcal{U}}_{\delta}))$, both finite. We must show

$$\{X \subseteq [\mu^*]^{<\omega} \mid X \in \mathcal{M}_{\alpha+1} \wedge d \cup e \in j_{\alpha+1,\beta}(X)\} \in \mathcal{M}_{\alpha+1},$$

for the appropriate $\mu^* < j_{0,\alpha+1}(\mu)$. For this, it is enough to show that $(E^{\mathcal{U}}_{\delta})_e \in \mathcal{M}_{\tau}$. For in that case, we have

$$(E^{\mathcal{U}}_{\delta})_e = j_{\alpha+1,\tau}(h)(a)$$

for some finite $a \subseteq \text{crit}(E^{\mathcal{U}}_{\delta})$ and $h \in \mathcal{M}_{\alpha+1}$. But then, for any $X \subseteq [\mu^*]^{<\omega}$ such that $X \in \mathcal{M}_{\alpha+1}$,

$$
\begin{aligned}
d \cup e \in j_{\alpha+1,\beta}(X) &\leftrightarrow e \in j_{\alpha+1,\beta}(X)_d \\
&\leftrightarrow e \in j_{\tau,\beta}(j_{\alpha+1,\tau}(X)_d) \\
&\leftrightarrow (j_{\alpha+1,\tau}(X))_d \in (E_\delta^{\mathcal{U}})_e \\
&\leftrightarrow (j_{\alpha+1,\tau}(X))_d \in j_{\alpha+1,\tau}(h)(a) \\
&\leftrightarrow d \cup a \in j_{\alpha+1,\tau}(\{v \mid (X)_{v_d} \in h(v_a)\}) \\
&\leftrightarrow \{v \mid (X)_{v_d} \in h(v_a)\} \in J_{(d\cup a)},
\end{aligned}
$$

where J is the extender derived from $j_{\alpha+1,\tau}$, and v_d and v_a are $\pi''d$ and $\pi''a$, for π the unique order isomorphism from $d \cup a$ to v. (So v ranges over $[\mu^*]^{|d\cup a|}$ in the formulae above.) Since $J_{(d\cup a)} \in \mathcal{M}_{\alpha+1}$ by our induction hypothesis, we are done.

We now finish the proof of 4.2 by showing $(E_\delta^{\mathcal{U}})_e \in \mathcal{M}_\tau$.

Suppose that $\delta = \tau$. Then since $\mathrm{crit}(E_\delta^{\mathcal{U}}) < j_{0,\delta}(\mu)$ (because $\delta = \tau < \eta$), $E_\delta^{\mathcal{U}}$ is not the last extender $\dot{F}^{\mathcal{M}_\delta}$ of $\mathcal{M}_\delta$. So $E_\delta^{\mathcal{U}} \in \mathcal{M}_\tau$, which is more than we need. Thus we may assume $\tau < \delta$.

If $(E_\delta^{\mathcal{U}})_e \in \mathcal{M}_\delta$, then as it is coded by a subset of $\mathrm{lh}(E_\tau^{\mathcal{U}})$ and $E_\tau^{\mathcal{U}} \in \mathcal{M}_\tau$, the basic lemma on the agreement of models in an iteration tree implies $(E_\delta^{\mathcal{U}})_e \in \mathcal{M}_\tau$.(Note that $E_\tau^{\mathcal{U}}$ cannot be the last extender of $\mathcal{M}_\tau$, since otherwise $\tau = \eta$.) Thus we may assume that $E_\delta^{\mathcal{U}}$ is the last extender of $\mathcal{M}_\delta$.

Let $\rho \in [0, \delta]_U$ be largest such that $\rho \le \tau$.

Case 1. $D^{\mathcal{U}} \cap (\rho, \delta]_U = \emptyset$.

If $\rho = \tau$, then

$$\mathrm{crit}(E_\delta^{\mathcal{U}}) = \mathrm{crit}(\dot{F}^{\mathcal{M}_\delta}) = j_{\rho,\delta}(\mathrm{crit}(\dot{F}^{\mathcal{M}_\rho})) \ge j_{0,\tau}(\mu),$$

so $\tau = \eta$, a contradiction.

If $\rho < \tau$, let $\gamma + 1 \in [0, \delta]_U$ be such that ρ is the U-predecessor of $\gamma + 1$. Then $\tau \le \gamma$ by the definition of ρ, so $\nu(E_\tau^{\mathcal{U}}) \le \nu(E_\gamma^{\mathcal{U}})$. As $\mathrm{crit}(E_\delta^{\mathcal{U}}) \in \mathrm{ran}(j_{\rho,\delta})$, $\mathrm{crit}(E_\delta^{\mathcal{U}}) \notin [\mathrm{crit}(E_\gamma^{\mathcal{U}}), \mathrm{lh}(E_\gamma^{\mathcal{U}})]$. But $\mathrm{crit}(E_\delta^{\mathcal{U}}) < \nu(E_\tau^{\mathcal{U}})$, so $\mathrm{crit}(E_\delta^{\mathcal{U}}) < \mathrm{crit}(E_\gamma^{\mathcal{U}})$. But then $\tau = \mathrm{pred}_U(\delta+1) \le \mathrm{pred}_U(\gamma+1) = \rho$, a contradiction. This finishes the proof in case 1.

Case 2. Otherwise.

Let $\gamma + 1$ be largest in $D^{\mathcal{U}} \cap (\rho, \delta]_U$. Let λ be the U-predecessor of $\gamma + 1$.

We claim first that $(E^{\mathcal{U}}_\delta)_e \in \mathcal{M}_\lambda$. For let J be the extender derived from $j_{\gamma+1,\delta} \circ j^*_{\gamma+1}$, the canonical embedding from the proper initial segment $\mathcal{M}^*_{\gamma+1}$ of $\mathcal{M}_\lambda$ to which we dropped to $\mathcal{M}_\delta$. One can show by an induction like the one above that every component J_d of J is boldface $r\Sigma_1$ over $\mathcal{M}^*_{\gamma+1}$, and hence a member of $\mathcal{M}_\lambda$.(Recall that in an iteration tree, every extender is close to the model to which it is applied, so the components of the individual extenders used in J are boldface $r\Sigma_1$ over the models to which they are applied.) Also, we have $\dot{F}^{\mathcal{M}^*_{\gamma+1}} \in \mathcal{M}_\lambda$. Since $E^{\mathcal{U}}_\delta = j_{\gamma+1,\delta}(j^*_{\gamma+1}(\dot{F}^{\mathcal{M}^*_{\gamma+1}}))$, we get $(E^{\mathcal{U}}_\delta)_e \in \mathcal{M}_\lambda$.

Now if $\rho < \lambda$, then $\tau < \lambda$ by the definition of ρ, so the agreement between $\mathcal{M}_\lambda$ and $\mathcal{M}_\tau$ implies that $(E^{\mathcal{U}}_\delta)_e \in \mathcal{M}_\tau$, as desired. Assume then that $\rho = \lambda$.

If also $\rho = \tau$, then $(E^{\mathcal{U}}_\delta)_e \in \mathcal{M}_\tau$ again. So assume $\rho < \tau$.

Now we argue as in case 1. Since $\text{crit}(E^{\mathcal{U}}_\delta)$ is in the range of $j_{\gamma+1,\delta} \circ j^*_{\gamma+1}$, $\text{crit}(E^{\mathcal{U}}_\delta) \notin [\text{crit}(E^{\mathcal{U}}_\gamma), \text{lh}(E^{\mathcal{U}}_\gamma)]$. But $\tau \leq \gamma$ and $\text{crit}(E^{\mathcal{U}}_\delta) < \nu(E^{\mathcal{U}}_\tau)$. Thus $\text{crit}(E^{\mathcal{U}}_\delta) < \text{crit}(E^{\mathcal{U}}_\gamma)$. But then $\tau = \text{pred}_U(\delta+1) \leq \text{pred}_U(\gamma+1) = \rho$, so $\rho = \tau$, a contradiction.

This finishes our proof in case 2. We have shown that $(E^{\mathcal{U}}_\delta)_e \in \mathcal{M}_\tau$, which completes the proof of 4.2.

□

Recall that $E = E^{\mathcal{U}}_\alpha$ is the first extender used on the main branch of $\mathcal{U}$.

Lemma 4.3 *For any finite $a \subseteq lh(E)$, $E \restriction (s_i \cup a) \in \mathcal{M}$.*

PROOF. First, if E is not the last extender of $\mathcal{M}_\alpha$, so that $E \in \mathcal{M}_\alpha$, then the lemma follows easily because $E \restriction (s_i \cup a)$ is coded by a subset of s_i belonging to $\mathcal{M}_\alpha$, and all extenders used in $\mathcal{U}$ have length at least s_i. (Either $E^{\mathcal{U}}_0 \in \mathcal{M}$, and from it $\mathcal{M}$ can compute this subset of s_i, or $E^{\mathcal{U}}_0 = \dot{F}^{\mathcal{M}}$, in which case $\mathcal{M}$ agrees with $\mathcal{M}_\alpha$ to $\text{lh}(\dot{F}^{\mathcal{M}})$, which is a cardinal of $\mathcal{M}_\alpha$.)

So we assume $E = \dot{F}^{\mathcal{M}}$.

CLAIM 1. $[0,\alpha]_U \cap D^U \neq \emptyset$.

PROOF. Recall that $G = i_{0,\xi}(\dot{F}^{\mathcal{N}})$ is the last extender of $\bar{\mathcal{N}}_\xi$, and that E is a proper initial segment of G. So $\mathcal{M}_\alpha$ would be a proper initial segment of $\bar{\mathcal{N}}_\xi$ *if* $\bar{\mathcal{N}}_\xi$ satisfied our ISC. It does not, but we have the copy map $\psi_\xi \colon \bar{\mathcal{N}}_\xi \to \mathcal{N}_\xi$, and $\mathcal{N}_\xi$ is a model of a tree on $\mathcal{M}$, so $\mathcal{N}_\xi$ does satisfy our ISC. Letting

$$\gamma := \sup(\psi''_\xi(\mathcal{M}_\alpha \cap \text{OR}),$$

we have that

$$\psi_\xi \restriction \mathcal{M}_\alpha \colon \mathcal{M}_\alpha \to \mathcal{J}_\gamma^{\mathcal{N}_\xi}$$

is cofinal and $r\Sigma_1$ elementary. Note here that we can apply ψ_ξ to the fragments E^* of some amenable-to-$\mathcal{M}_\alpha$ coding of E, then assemble these $\psi_\xi(E^*)$'s into an initial segment of the last extender of $\mathcal{N}_\xi$, and that this initial segment must be on the sequence of $\mathcal{N}_\xi$ because E was on the sequence of $\mathcal{M}_\alpha$, so that clause 5(b) of 2.4 is ruled out.

But then if $[0,\alpha]_U$ does not drop, then $\psi_\xi \circ j_{0,\alpha}$ is a cofinal, $r\Sigma_1$ elementary embedding from $\mathcal{M}$ to a proper initial segment of a Σ-iterate of $\mathcal{M}$. This contradicts the weak Dodd-Jensen property of Σ.

□

Claim 2. If $k \colon \mathcal{P} \to \mathcal{Q}$ is the canonical embedding along part of a branch of an iteration tree, and all the ultrapowers taken in k are Σ_0, then any boldface $r\Sigma_1^{\mathcal{Q}}$ subset of crit(k) is also boldface $r\Sigma_1^{\mathcal{P}}$.

Claim 2 is proved by an easy induction, making use of the fact that the extenders used are close to the models to which they are applied. We omit further detail.

Now let $\gamma+1$ be largest in $D^{\mathcal{U}} \cap [0,\alpha]_U$, and let δ be the U-predecessor of $\gamma+1$. We may assume $E \restriction (s_i \cup a) \notin \mathcal{M}_\alpha$, otherwise we are done, as above. Thus

$$\rho_1(\mathcal{M}_\alpha) \leq s_i.$$

It follows that

$$\rho_1(\mathcal{M}_\alpha) = \rho_1(\mathcal{M}^*_{\gamma+1}) \text{ and } \rho_1(\mathcal{M}^*_{\gamma+1}) \leq \text{crit}(j_{\gamma+1,\alpha} \circ j^*_{\gamma+1}).$$

Therefore, all ultrapowers used in

$$k := j_{\gamma+1,\alpha} \circ j^*_{\gamma+1}$$

are Σ_0. If $s_i \leq \text{crit}(k)$, we can use claim 2 to conclude that $E \restriction (s_i \cup a)$ is definable over $\mathcal{M}^*_{\gamma+1}$, hence in $\mathcal{M}_\delta$, hence in $\mathcal{M}$.

Thus we may assume $\text{crit}(k) = \text{crit}(E^{\mathcal{U}}_\gamma) < s_i$. This implies $\delta = 0$. Note that $s_i \leq \text{crit}(j_{\gamma+1,\alpha})$, so claim 2 tells us that $E \restriction (s_i \cup a)$ is boldface $r\Sigma_1$ over $\mathcal{M}_{\gamma+1}$. Because $\mathcal{M}_{\gamma+1} = Ult(\mathcal{M}^*_{\gamma+1}, E^{\mathcal{U}}_\gamma)$ and $\mathcal{M}^*_{\gamma+1} \in \mathcal{M}$, it is enough to show

CLAIM 3. For any finite b, $E^{\mathcal{U}}_{\gamma} \restriction (s_i \cup b) \in \mathcal{M}$.
(We then use this claim with $[b, f]$ being a parameter from which $E \restriction (s_i \cup a)$ is definable over $\mathcal{M}_{\gamma+1}$.)

But claim 3 is just like what we are trying to prove about $E = E^{\mathcal{U}}_{\alpha}$, except that we can't use the Dodd-Jensen argument we used for E to see that E is not the non-dropping image of $\dot{F}^{\mathcal{M}}$. However, $E^{\mathcal{U}}_{\gamma}$ cannot be such an image, for $\mu < \mathrm{crit}(E^{\mathcal{U}}_{\gamma}) < s_i$. (To see that $\mu < \mathrm{crit}(E^{\mathcal{U}}_{\gamma})$, note that $\mu = \mathrm{crit}(E) = \mathrm{crit}(\dot{F}^{\mathcal{M}_\alpha})$ is in the range of k.) Thus we can repeat the argument above, with $E^{\mathcal{U}}_{\alpha}$ replaced by $E^{\mathcal{U}}_{\gamma}$. Since $\gamma < \alpha$, this process eventually ends, and we get lemma 4.3.

□

For $k < i$, let $t_k = i_{0,\xi}(\psi^{-1}(s_k))$ and $t = \{t_0 > \cdot\cdot\cdot > t_{i-1}\}$, so that

$$G \restriction (s_i \cup t) = F \restriction (s_i \cup s \restriction i),$$

up to the obvious isomorphism.

Lemma 4.4 $t \subseteq j_{0,\eta}(\mu)$.

PROOF. It is an old observation, due to Martin, that if H is the (κ, λ)-extender derived from $\pi: P \to Q$, where

$$\lambda = \sup\{\pi(f)(\kappa) \mid f \in P \text{ and } f: \kappa \to \kappa\},$$

then H is of superstrong type, i.e.

$$i_H(\kappa) = \lambda.$$

It follows that if H is any (κ, λ)-extender on the sequence of a premouse, then $\lambda < i_H(f)(\kappa)$ for some $f: \kappa \to \kappa$. In particular, this is true with $H = \dot{F}^{\mathcal{N}_\xi}$, and ψ_ξ is elementary enough that it is true with $H = G$.

So fix $f: \mu \to \mu$ in $\mathcal{M}$ such that $t \subseteq i_G(f)(\mu)$. Then we have

$$t \subseteq i_G(f)(\mu) \le i_{-1,\theta}(f)(\mu) = j_{0,\sigma}(f)(\mu) = j_{0,\eta}(f)(\mu),$$

where the last equality holds because

$$j_{0,\eta}(f)(\mu) < j_{0,\eta}(\mu) \le \mathrm{crit}(j_{\eta,\sigma}).$$

The lemma now follows.

□

We are ready to finish the proof. Let us assume $\alpha + 1 < \eta$; the proof when $\alpha + 1 = \eta$ is similar but simpler. Fix a and f such that $I_t = [a, f]_E$; notice here that I_t makes sense by 4.4, and that $I_t \in Ult(\mathcal{M}, E)$ by 4.2. We have $E \restriction (s_i \cup a) \in \mathcal{M}$, and we can use it to compute $G \restriction (s_i \cup t)$ in $\mathcal{M}$ as follows. Let $b \subseteq s_i$ be finite, and $X \subseteq [\mu]^{|b \cup t|}$ be in $\mathcal{M}$; then

$$\begin{aligned} X \in G_{(b\cup t)} &\leftrightarrow b \cup t \in i_{-1,\sigma}(X) \\ &\leftrightarrow b \cup t \in j_{0,\theta}(X) \\ &\leftrightarrow t \in j_{\alpha+1,\eta}(j_{0,\alpha+1}(X)_b) \\ &\leftrightarrow \text{ for } I_t \text{ a.e. } u, u \in j_{0,\alpha+1}(X)_b \\ &\leftrightarrow \text{ for } E_{(a\cup b)} \text{ a.e. } v, \text{ for } f(v_a) \text{ a.e. } u, u \in X_{v_b}. \end{aligned}$$

Here, once again, v_a and v_b are the sets sitting indide v the way a and b sit inside $a \cup b$. This equivalence shows $G \restriction s_i \cup t \in \mathcal{M}$, and thereby completes the proof of part (A) of the theorem.

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