Chapter 1
Naive set theory

GEORG CANTOR (1845 – 1918) discovered set theory. Prior to Cantor, people often thought it to be paradoxical that there are sets which can be put into a bijective correspondence with a proper subset of themselves. For instance, there is a bijection from $\mathbb{N}$ onto the set of all prime numbers. Hence, it seemed, on one hand the set of all primes is "smaller than" $\mathbb{N}$, but on the other hand it is "as big as" $\mathbb{N}$.

Cantor's idea was as follows. Let X and Y be arbitrary sets. Define "X is smaller than Y" (or rather , "Y is not bigger than X") as: there is an injection $f : X \to Y$. Write this as $X \leq Y$. Define "X is of the same size as Y" as: there is a bijection $f : X \to Y$. Write this as $X \sim Y$. Obviously, $X \sim Y$ implies $X \leq Y$. The theorem of CANTOR-SCHRÖDER-BERNSTEIN (cf. Theorem 1.4) will say that $X \sim Y$ follows from $X \leq Y$ and $Y \leq X$. We write $X < Y$ if $X \leq Y$ but not $Y \leq X$.

Notice that if $A \leq B$, i.e., if there is an injection $f : A \to B$, then there is a surjection $g : B \to A$. This is clear if f is already bijective. If not then pick $a_0 \in A$. Define $g : B \to A$ by $g(b) = f^{-1}(b)$, if b is in the range of f, and $g(b) = a_0$ otherwise.

Conversely, if $f : A \to B$ is surjective then there is an injection $g : B \to A$, i.e., $B \leq A$. This is shown by *choosing* for each $b \in B$ some $a \in A$ with $f(a) = b$ and setting $g(b) = a$. This argument is in need of the Axiom of Choice, AC, which we shall present in the next chapter and discuss in detail later on.

To a certain extent, set theory is the study of the cardinality of arbitrary sets, i.e., of the relations $\leq$ and $\sim$. The proof of the following theorem may be regarded as the childbirth of set theory.

Theorem 1.1 (Cantor)

$$\mathbb{N} < \mathbb{R}.$$

Proof. $\mathbb{N} \leq \mathbb{R}$ is trivial. We show that $\mathbb{R} \leq \mathbb{N}$ does not hold.

Assume that there is an injection from $\mathbb{R}$ to $\mathbb{N}$, so that there is then also a surjection $f : \mathbb{N} \to \mathbb{R}$. Write x_n for $f(n)$. In particular, $\mathbb{R} = \{x_n : n \in \mathbb{N}\}$.

Let us now recursively define closed intervals $[a_n, b_n]$ as follows. Put $[a_0, b_0] = [0, 1]$. Suppose $[a_n, b_n]$ to be defined. Pick $[a_{n+1}, b_{n+1}]$ so that $a_n \leq a_{n+1} < b_{n+1} \leq b_n$, $b_{n+1} - a_{n+1} \leq \frac{1}{n+1}$ and $x_n \notin [a_{n+1}, b_{n+1}]$.

Now $\bigcap_{n \in \mathbb{N}}[a_n,b_n] = \{x\}$ for some $x \in \mathbb{R}$ by the Nested Interval Principle. Obviously, $x \neq x_n$ for every n, as $x_n \notin [a_{n+1},b_{n+1}] \supset \{x\}$. Hence $x \notin \{x_n : n \in \mathbb{N}\} = \mathbb{R}$. Contradiction! □

It is not hard to verify that the sets of all integers, of all rationals, and of all algebraic numbers are each of the same size as $\mathbb{N}$ (cf. Problem 1.1). In particular, Theorem 1.1 immediately gives the following.

Corollary 1.2 *There are transcendental numbers.*

For an arbitrary set X, let $\mathscr{P}(X)$ the power set of X, i.e., the set of all subsets of X. The following is a generalization of Theorem 1.1.

Theorem 1.3 *For every X, $X < \mathscr{P}(X)$.*

Proof. We have $X \leq \mathscr{P}(X)$, because $f : X \to \mathscr{P}(X)$ is injective where $f(x) = \{x\}$ for $x \in X$.

We have to see that $\mathscr{P}(X) \leq X$ does not hold true. Given an arbitrary $f : X \to \mathscr{P}(X)$, consider $Y = \{x \in X : x \notin f(x)\} \subset X$. If Y were in the range of f, say $Y = f(x_0)$, then we would have that $x_0 \in Y \Longleftrightarrow x_0 \notin g(x_0) = Y$. Contradiction! In particular, f cannot be surjective, which shows that $\mathscr{P}(X) \leq X$ is false. □

Theorem 1.4 (Cantor–Schröder–Bernstein) *Let X and Y be arbitrary. If $X \leq Y$ and $Y \leq X$, then $X {\sim} Y$.*

Proof. Let both $f : X \to Y$ and $g : Y \to X$ be injective. We are looking for a bijection $h : X \to Y$. Let $x \in X$. An X–orbit of x is a finite or infinite sequence of the form

$$g^{-1}(x), f^{-1}(g^{-1}(x)), g^{-1}(f^{-1}(g^{-1}(x))), \ldots$$

For each $n \in \mathbb{N} \cup \{\infty\}$ there is obviously at most one X–orbit of x of length n. Let $n(x)$ the maximal $n \in \mathbb{N} \cup \{\infty\}$ so that there is an X–orbit of x of length n. We put $x \in X_0$ iff $n(x) = \infty$, $x \in X_1$ iff $n(x) \in \mathbb{N}$ is even, and $x \in X_2$ iff $n(x) \in \mathbb{N}$ is odd.

For $y \in Y$ we define the concept of a Y–orbit in an analoguous way, i.e., as a finite or infinite sequence of the form

$$f^{-1}(y), g^{-1}(f^{-1}(y)), f^{-1}(g^{-1}(f^{-1}(y))), \ldots$$

We write $n(y)$ for the maximal $n \in \mathbb{N} \cup \{\infty\}$ so that there is a Y–orbit of y of length n. We set $y \in Y_0$ iff $n(y) = \infty$, $y \in Y_1$ iff $n(y) \in \mathbb{N}$ is odd, and $y \in Y_2$ iff $n(y) \in \mathbb{N}$ is even.

Let us now define $h : X \to Y$ by $h(x) = f(x)$, if $x \in X_0 \cup X_1$, and $h(x) = g^{-1}(x)$, if $x \in X_2$.

The function h is well-defined, as X is the disjoint union of X_0, X_1, and X_2, and because for every $x \in X_2$ there is an X–orbit of x of length 1, i.e., $g^{-1}(x)$ is defined.

The function h is injective: Let $x_1 \neq x_2$ with $h(x_1) = h(x_2)$. Say $x_1 \in X_0 \cup X_1$ and $x_2 \in X_2$. Then obviously $h(x_1) = f(x_1) \in Y_0 \cup Y_2$ and $h(x_2) = g^{-1}(x_2) \in Y_1$. But Y is the disjoint union of Y_0, Y_1, and Y_2. Contradiction!

The function h is surjective: Let $y \in Y_0 \cup Y_2$. Then $y = f(x)$ for some $x \in X_0 \cup X_1$; but then $y = h(x)$. Let $y \in Y_1$. Then $g(y) \in X_2$, so $y = g^{-1}(g(y)) = h(g(y))$. □

Cantor's *Continuum Problem* is the question if there is a set A of real numbers such that

$$\mathbb{N} < A < \mathbb{R}.$$

This problem has certainly always been one of the key driving forces of set theory. A set A is called *at most countable* if $A \leq \mathbb{N}$. A is called *countable* if $A \sim \mathbb{N}$, and A is called *finite* iff $A < \mathbb{N}$. A is called *uncountable* iff $\mathbb{N} < A$.

Cantor's *Continuum Hypothesis* says that the Continuum Problem has a negative answer, i.e., that for every uncountable set A of real numbers, $A \sim \mathbb{R}$.

Cantor initiated the project of proving the Continuum Hypothesis by an induction on the "complexity" of the sets A in question. There is indeed a hierarchy of sets of reals which we shall study in chapter 7. The open and closed sets sit at the very bottom of this hierarchy.

Let $A \subset \mathbb{R}$. A is called *open* iff for every $a \in A$ there are $c < a$ and $b > a$ with $(c,b) = \{x \colon c < x < b\} \subset A$. A is called *closed* iff $\mathbb{R} \setminus A$ is open.

It is easy to see that if $A \subset \mathbb{R}$ is any non–empty open set, then $\mathbb{R} \leq A$. As $A \leq \mathbb{R}$ is trivial for every $A \subset \mathbb{R}$, we immediately get that $A \sim \mathbb{R}$ for every non–empty open $A \subset \mathbb{R}$ with the help of the Theorem 1.4 of CANTOR–SCHRÖDER–BERNSTEIN. Theorem 1.8 of CANTOR–BENDIXSON will say that $A \sim \mathbb{R}$ for every uncountable closed set $A \subset \mathbb{R}$.

Lemma 1.5 *Let $A \subset \mathbb{R}$. The following are equivalent:*

1. *A is closed.*
2. *For all $x \in \mathbb{R}$, if $a < x < b$ always implies $(a,b) \cap A \neq \emptyset$ then $x \in A$.*

Proof. 1.$\Longrightarrow$2.: Let $x \notin A$. Let $a < x < b$ be such that $(a,b) \subset \mathbb{R} \setminus A$. Then $(a,b) \cap A = \emptyset$.

2.$\Longrightarrow$1.: We prove that $\mathbb{R} \setminus A$ is open. Let $x \in \mathbb{R} \setminus A$. Then there are $a < x < b$ so that $(a,b) \cap A = \emptyset$, i.e., $(a,b) \subset \mathbb{R} \setminus A$. □

Let $A \subset \mathbb{R}$. x is called an *accumulation point* of A iff for all $a < x < b$, $(a,b) \cap (A \setminus \{x\}) \neq \emptyset$. The set of all accumulation points of A is called the *(first) derivative* of A and is abbreviated by A'. Lemma 1.5 readily gives:

Lemma 1.6 *Let $A \subset \mathbb{R}$. The following are equivalent:*

1. *A is closed.*
2. *$A' \subset A$.*

A set $A \subset \mathbb{R}$ is called *perfect* iff $A \neq \emptyset$ and $A' = A$.

Theorem 1.7 *Let $A \subset \mathbb{R}$ be perfect. Then $A \sim \mathbb{R}$.*

Proof. $A \leq \mathbb{R}$ is trivial. It thus remains to be shown that $\mathbb{R} \leq A$. We shall make use of the fact that $\mathbb{R} \sim {}^{\mathbb{N}}\{0,1\}$, where ${}^{\mathbb{N}}\{0,1\}$ is the set of all infinite sequences of 0's and 1's. (Cf. Problem 1.2.) We aim to see that ${}^{\mathbb{N}}\{0,1\} \leq A$.

Let ${}^{*}\{0,1\}$ be the set of all non-empty finite sequences of 0's and 1's, i.e., of all $s : \{0,\ldots,n\} \to \{0,1\}$ for some $n \in \mathbb{N}$. Let us define a function Φ from ${}^{*}\{0,1\}$ to closed intervals as follows.

Let $s_0 : \{0\} \to \{0\}$ and $s_1 : \{0\} \to \{1\}$. As $A \neq \emptyset$ and $A \subset A'$ we easily find

$$a_{s_0} < b_{s_0} < a_{s_1} < b_{s_1}$$

so that

$$[a_{s_0}, b_{s_0}] \cap A \neq \emptyset \text{ und } [a_{s_1}, b_{s_1}] \cap A \neq \emptyset.$$

Set $\Phi(s_0) = [a_{s_0}, b_{s_0}]$ and $\Phi(s_1) = [a_{s_1}, b_{s_1}]$.

Now let $s \in {}^{*}\{0,1\}$ and suppose that $\Phi(s)$ is already defined, where $\Phi(s) = [a_s, b_s]$ with $a_s < b_s$ and $[a_s, b_s] \cap A \neq \emptyset$.

Let $s : \{0,\ldots,n\} \to \{0,1\}$. Write $s^\frown 0$ for the unique $t : \{0,\ldots,n+1\} \to \{0,1\}$ with $t(i) = s(i)$ for $i \leq n$ and $t(n+1) = 0$; write $s^\frown 1$ for the unique $t : \{0,\ldots,n+1\} \to \{0,1\}$ with $t(i) = s(i)$ for $i \leq n$ and $t(n+1) = 1$. Because $A \subset A'$, we easily find

$$a_s < a_{s^\frown 0} < b_{s^\frown 0} < a_{s^\frown 1} < b_{s^\frown 1} < b_s,$$

so that

$$[a_{s^\frown 0}, b_{s^\frown 0}] \cap A \neq \emptyset, [a_{s^\frown 1}, b_{s^\frown 1}] \cap A \neq \emptyset,$$
$$b_{s^\frown 0} - a_{s^\frown 0} \leq \frac{1}{n+1}, \text{ and } b_{s^\frown 1} - a_{s^\frown 1} \leq \frac{1}{n+1}.$$

Set $\Phi(s^\frown h) = [a_{s^\frown h}, b_{s^\frown h}]$ for $h = 0,1$.

We may now define an injection $F : {}^{\mathbb{N}}\{0,1\} \to A$. Let $f \in {}^{\mathbb{N}}\{0,1\}$. Then

$$\bigcap_{n \in \mathbb{N}} [a_{f \restriction \{0,\ldots,n\}}, b_{f \restriction \{0,\ldots,n\}}] = \{x\}$$

for some $x \in \mathbb{R}$ by the Nested Interval Principle. Set $F(f) = x$. Obviously, $F(f) \in A$, as $F(f)$ is an accumulation point of A and $A' \subset A$. Also, F is certainly injective. □

Theorem 1.8 (Cantor–Bendixson) *Let $A \subset \mathbb{R}$ be closed. Then there are sets $A_0 \subset \mathbb{R}$ and $P \subset \mathbb{R}$ so that:*

1. *A is the disjoint union of A_0 and P,*
2. *A_0 is at most countable, and*
3. *P is perfect, or else $P \neq \emptyset$.*

Proof. An $x \in \mathbb{R}$ is called a *condensation point* of A iff $(a,b) \cap A$ is uncountable for all $a < x < b$.

Let P be the set of all condensation points of A, and let $A_0 = A \setminus P$. As A is closed, $P \subset A' \subset A$. It remains to be shown that 2. and 3. both hold true. We shall make use of the fact that $\mathbb{Q}\sim\mathbb{N}$ (cf. Problem 1.1) and that $\mathbb{Q}$ is dense in $\mathbb{R}$, i.e., that for all x, $y \in \mathbb{R}$ with $x < y$ there is some $z \in \mathbb{R}$ with $x < z < y$.

2.: Let $x \in A_0$. Then there are $a_x < x < b_x$ with $a_x, b_x \in \mathbb{Q}$ and such that $(a_x, b_x) \cap A$ is at most countable. Therefore,

$$A_0 \subset \bigcup_{x \in A_0} (a_x, b_x) \cap A.$$

As $\mathbb{Q}\sim\mathbb{N}$, there are at most countably many sets of the form $(a_x, b_x) \cap A$, and each of them is at most countable. Hence A_0 is at most countable (cf. Problem 1.3).

3.: Suppose that $P \neq \emptyset$. We first show that $P \subset P'$. Let $x \in P$. Let $a < x < b$. We have that $(a,b) \cap A$ is uncountable. Suppose that $(a,b) \cap (P \setminus \{x\}) = \emptyset$. For each $y \in ((a,b) \setminus \{x\}) \cap A$ there are then $a_y < y < b_y$ with $a_y, b_y \in \mathbb{Q}$ so that $(a_y, b_y) \cap A$ is at most countable. But then we have that

$$(a,b) \cap A \subset \{x\} \cup \bigcup_{y \in (a,b) \setminus \{x\}} (a_y, b_y) \cap A$$

is at most countable. Contradiction!

Let us finally show that $P' \subset P$. Let $x \in P'$. Then $(a,b) \cap (P \setminus \{x\}) \neq \emptyset$ for all $a < x < b$. Let $y \in (a,b) \cap (P \setminus \{x\})$. Then $(a,b) \cap A$ is uncountable. Hence $x \in P$.
□

There is a different proof of the Theorem of CANTOR–BENDIXSON which brings the concept of an "ordinal number" into play. Let $A \subset \mathbb{R}$ be given. Define A^1 as A', A^2 as A'', etc., i.e., A^{n+1} as $(A^n)'$ for $n \in \mathbb{N}$. It is easy to see that each A^n is closed, and

$$\ldots \subset A^{n+1} \subset A^n \subset \ldots \subset A^1 \subset A.$$

If there is some n with $A^{n+1} = A^n$ then $P = A^n$ and $A_0 = A \setminus P$ are as desired. Otherwise we have to continue this process into the transfinite. Let

$$A^\infty = \bigcap_{n \in \mathbb{N}} A^n, A^{\infty+1} = (A^\infty)', \ldots, A^{\infty+n+1} = (A^{\infty+n})',$$

$$A^{\infty+\infty} = \bigcap_{n \in \mathbb{N}} A^{\infty+n}, \ldots \text{etc.}$$

It can be shown that there is a "number" α so that $A^{\alpha+1} = A^\alpha$. If $A^\alpha \neq \emptyset$, then A^α is perfect, and $A \setminus A^\alpha$ is always at most countable.

Such "numbers" are called *ordinal numbers* (cf. Definition 3.2). We need an axiomatization of set theory, though, in order to be able to introduce them rigorously. With their help we shall be able to prove much stronger forms of the Theorem of CANTOR–BENDIXSON (cf. Theorem 7.11).

Problems

1.1. Show that the sets of all integers, of all rationals, and of all algebraic numbers are each countable, i.e., of the same size as $\mathbb{N}$. Show also that there are only countably many continous $f\colon \mathbb{R} \to \mathbb{R}$.

1.2. Show that $\mathbb{R} \sim {}^{\mathbb{N}}\{0,1\}$, where ${}^{\mathbb{N}}\{0,1\}$ is the set of all infinite sequences of 0's and 1's.

1.3. Let, for each $n \in \mathbb{N}$, A_n be a countable set. Show that $\bigcup_{n\in\mathbb{N}} A_n$ is countable.

1.4. Let $n \in \mathbb{N}$. Construct a set $A \subset \mathbb{R}$ such that $A^n \neq \emptyset$, but $A^{n+1} = \emptyset$. Also construct a set $A \subset \mathbb{R}$ such that $A^{\infty+n} \neq \emptyset$, but $A^{\infty+n+1} = \emptyset$.

1.5. Let $A \subset \mathbb{R}$ be closed. Show that the pair (A_0, P) as in the statement of the Theorem 7.11 of CANTOR–BENDIXSON are unique.

1.6. A set $A \subset \mathbb{R}$ is called *nowhere dense* iff $\mathbb{R} \setminus A$ has an open subset which is dense in $\mathbb{R}$.
(a) Show that A is nowhere dense iff for all a, $b \in \mathbb{R}$ with $a < b$ there are a', $b' \in \mathbb{R}$ with $a \leq a' < b' \leq b$ and $[a', b'] \cap A = \emptyset$.
(b) Show that if each A_n is open and dense, $n \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}} A_n$ is dense. (This is the BAIRE *Categoricity Theorem.*)
(c) Show that $\mathbb{R}$ is not the countable union of nowhere dense sets.
(d) For a, $b \in \mathbb{R}$ with $a < b$ let

$$[a,b]^{\frac{2}{3}} = [a, \frac{2}{3}a + \frac{1}{3}b] \cup [\frac{1}{3}a + \frac{2}{3}b, b],$$

and for a_0, $b_0, \ldots, a_k$, $b_k \in \mathbb{R}$ with $a_i < b_i$ for all $i \leq k$ let

$$([a_0,b_0] \cup \ldots \cup [a_k,b_k])^{\frac{2}{3}} = [a_0,b_0]^{\frac{2}{3}} \cup \ldots \cup [a_k,b_k]^{\frac{2}{3}}.$$

Finally, let, for a, $b \in \mathbb{R}$ with $a < b$, $[a,b]_0 = [a,b]$, $[a,b]_{n+1} = ([a,b]_n)^{\frac{2}{3}}$, and

$$[a,b]_\infty = \bigcap_{n\in\mathbb{N}} [a,b]_n.$$

$[a,b]_\infty$ is called *Cantor's Discontinuum.* Show that for all a, $b \in \mathbb{R}$ with $a < b$, $[a,b]_\infty$ is nowhere dense, and $[a,b]_\infty \sim \mathbb{R}$.