In V, there is a play of $\pi_{\emptyset,x}(G^*(A))$ in which II follows $\pi_{\emptyset,x}(\tau^*)$ and in which II looses namely

I	$n_0, \pi_{\emptyset,x}(\alpha_\emptyset)$	$n_2, \pi_{x\upharpoonright 1,x}(\alpha_{x\upharpoonright 1})$	$\ldots$
II	n_1		$\ldots$

By absoluteness, such a play hence exists in M_x. But this is a contradiction!

It is not hard to show that if there is a measurable cardinal, then every set of reals has an embedding normal form. It is much harder to get embedding normal forms which are sufficiently closed. We now first want to show that all coanalytic sets are determined provided that there is a measurable cardinal.

Definition 13.3 *Let κ be a measurable cardinal, and let U be a $<\kappa$–complete non-trivial ultrafilter on κ. Such U is also called a* measure on κ. *Let γ be an ordinal, or $\gamma = \infty$. Then the system*

$$\mathscr{I} = (M_\alpha, \pi_{\alpha\beta} : \alpha \leq \beta < \gamma)$$

is called the (linear) putative iteration of V of length γ given by U *iff the following hold true.*

(1) $M_0 = V$, and if $\alpha + 1 < \gamma$, then M_α is a (transitive) inner model.
(2) If $\alpha \leq \beta \leq \delta < \gamma$, then $\pi_{\alpha\beta} : M_\alpha \to M_\beta$ is an elementary embedding, and $\pi_{\alpha\delta} = \pi_{\beta\delta} \circ \pi_{\alpha\beta}$.
(3) If $\alpha + 1 < \gamma$, then $M_{\alpha+1} = Ult(M_\alpha; \pi_{\circ\alpha}(U))$ and $\pi_{\alpha\alpha+1}$ is the canonical ultrapower embedding.
(4) If $\lambda < \gamma$ is a limit ordinal, then $(M_\lambda, \pi_{\alpha\lambda} : \alpha < \lambda)$ is the direct limit of $(M_\alpha, \pi_{al\beta} : \alpha \leq \beta < \lambda)$.

The system $\mathscr{I}$ is called the (linear) iteration of V of length γ given by U if either γ is a limit ordinal or else the last model $M_{\gamma-1}$ is well–founded (i.e., transitive).

Notice that by (2), $\pi_{\alpha\alpha} = id$ for all $\alpha < \gamma$. Also, if we write $\kappa_\alpha = \pi_{0\alpha}(\kappa)$ and $U_\alpha = \pi_{0\alpha}(U)$, then

$$M_\alpha \models \text{“}U_\alpha \text{ is a measure on } \kappa_\alpha\text{.”}$$

Therefore, (3) makes sense.

It is easy to verify that $\kappa_\alpha = crit(\pi_{\alpha\beta})$ for $\alpha < \beta < \gamma$ such that M_β is well–founded (i.e., transitive). Every model $M_\alpha, \alpha < \gamma$, is transitive, except possibly the last one. (Of course, if γ is a limit ordinal, then there is no last model.)

Definition 13.4 *Let κ be a measurable cardinal, and let U be a measure on κ. Then V is called* iterable by U and its images *iff for every γ, if*

$$(M_\alpha, \pi_{\alpha\beta} : \alpha \leq \beta < \gamma + 1)$$

is the (linear) putative iteration of V of length $\gamma + 1$ given by U, then M_γ is well–founded (i.e., transitive).

Theorem 13.5 *Let κ be a measurable cardinal, and let U be a measure on κ. Then V is iterable by U and its images.*

Proof: Let γ be an ordinal, and let

$$(M_\alpha, \pi_{\alpha\beta} : \alpha \leq \beta < \gamma + 1)$$

be the (linear) putative iteration of V of length $\gamma + 1$ given by U. Let

$$\sigma : \overline{V} \cong X \prec_{\Sigma_{1000}} V,$$

where $\{\kappa, U, \gamma\} \subset X$, X is countable, and $\overline{V}$ is transitive. Let $\overline{\kappa} = \sigma^{-1}(\kappa)$, $\overline{U} = \sigma^{-1}(U)$, and $\overline{\gamma} = \sigma^{-1}(\gamma)$. We may also set, for $\alpha \in \operatorname{ran}(\sigma) \cap \gamma$,

$$\overline{M}_{\sigma^{-1}(\alpha)} = \sigma^{-1}(M_\alpha),$$

and for $\alpha \leq \beta, \alpha, \beta \in \operatorname{ran}(\sigma) \cap \gamma$,

$$\overline{\pi}_{\sigma^{-1}(\alpha), \sigma^{-1}(\beta)} = \sigma^{-1}(\pi_{\alpha\beta}).^1$$

Then, from the point of view of $\overline{V}$,

$$(\overline{M}_{\alpha, \overline{\pi}_{\alpha\beta}} : \alpha \leq \beta < \overline{\gamma} + 1)$$

is the (linear) putative iteration of $\overline{V}$ of length $\overline{\gamma} + 1$ given by $\overline{U}$.

We shall now recursively, for $\alpha < \overline{\gamma} + 1$, construct embeddings

$$\sigma_\alpha : \overline{M}_\alpha \to_{\Sigma_{1000}} V$$

such that whenever $\alpha \leq \beta < \overline{\gamma} + 1$, then

$$\sigma_\beta \circ \overline{\pi}_{\alpha\beta} = \sigma_\alpha.$$

We set $\sigma_0 = \sigma$. Now let $\delta < \overline{\gamma}$, and suppose all $\sigma_\alpha, \alpha < \delta$, are already construed such that $\sigma_\beta \circ \overline{\pi}_{\alpha\beta} = \sigma_\alpha$ for all $\alpha \leq \beta < \delta$.

Let us first suppose δ to be a limit ordinal. We then define $\sigma_\delta : \overline{M}_\delta \to V$ by setting

$$\sigma_\delta(x) = \sigma_\alpha \circ \overline{\pi}^{-1}_{\alpha\delta}(x),$$

whenever $x \in \operatorname{ran}(\overline{\pi}_{\alpha\delta})$. It is easy to verify that σ_δ is well-defined and Σ_{1000}-elementary, and that $\sigma_\beta \circ \overline{\pi}_{\alpha\beta} = \sigma_\alpha$ for all $\alpha \leq \beta \leq \delta$.

Now suppose δ to be a successor ordinal, say $\delta = \xi + 1$. Set $\overline{\kappa}_\xi = \overline{\pi}_{0\xi}(\overline{\kappa})$ and $\overline{U}_\xi = \overline{\pi}_{0\xi}(\overline{U})$.

If φ is a formula, and $f_1, \ldots, f_k \in {}^{\overline{\kappa}_\xi}\overline{M}_\xi \cap \overline{M}_\xi$, then we write $X_{\varphi, f_1, \ldots, f_k}$ for

$$\{\eta < \overline{\kappa}_\xi : \overline{M}_\xi \models \varphi(f_1(\eta), \ldots, f_k(\eta))\}.$$

[1] Recall that for a proper class X, $\sigma^{-1}(X) = \bigcup\{\sigma^{-1}(X \cap V_\alpha) : \alpha \in \operatorname{ran}(\sigma)\}$.

By Łoś, $X_{\varphi,f_1,\ldots,f_k} \in \overline{U}_\xi$ iff

$$\overline{M}_{\xi+1} \models \varphi([f_1],\ldots,[f_k]).$$

Because U is $< \aleph_1$–complete, $\bigcap \sigma_\xi'' \overline{U}_\xi \neq \emptyset$, say $\rho \in \bigcap \sigma_\xi'' \overline{U}_\xi$.

Let us define $\sigma_\delta : \overline{M}_\delta \to V$ by setting

$$\sigma_\delta([f]) = \sigma_\xi(f)(\rho).$$

This is well–defined and Σ_{1000}–elementary, because if φ is Σ_{1000} and $f_1,\ldots,f_k \in {}^{\kappa_\xi}\overline{M}_\xi \cap \overline{M}_\xi$, then

$$\begin{array}{c} \overline{M}_{\xi+1} \models \varphi([f_1],\ldots[f_k]) \text{ iff} \\ X_{\varphi,f_1,\ldots,f_k} \in \overline{U}_\xi \text{ iff} \\ \rho \in \sigma_\xi(X_{\varphi,f_1,\ldots,f_k}) = \{\eta < \kappa : V \models \varphi(\sigma_\xi(f_1)(\eta),\ldots,\sigma_\xi(f_k)(\eta))\} \text{ iff} \\ V \models \varphi(\sigma_\xi(f_1)(\eta),\ldots,\sigma_\xi(f_k)(\eta)). \end{array}$$

It is also easy to verify that $\sigma_\delta = \overline{\pi}_{\xi\delta} \circ \sigma_\xi$ and hence $\sigma_\beta \circ \overline{\pi}_{\alpha\beta} = \sigma_\alpha$ for all $\alpha \leq \beta \leq \delta$.

But now we cannot have that $(\overline{M}_\alpha, \overline{\pi}_{\alpha\beta} : \alpha \leq \beta < \overline{\gamma}+1)$ has a last ill–founded model, so that $(M_\alpha, \pi_{\alpha\beta} : \alpha \leq \beta < \gamma)$ cannot have a last ill–founded model: otherwise a witness to the fact that M_γ is ill–founded would be in the range of σ.

Lemma 13.6 *Let κ be a measurable cardinal, and let U be a normal measure on κ. Let*

$$(M_\alpha, \pi_{\alpha\beta} : \alpha \leq \beta \in OR\}$$

be the (linear) iteration of $V = M_0$ which is given by U. For $\alpha \in OR$, set $U_\alpha = \pi_{0\alpha}(U)$ and $\kappa_\alpha = crit(U_\alpha) = \pi_{0\alpha}(\kappa)$. Let $\alpha \leq \beta$, and let $\varphi : \alpha \to \beta$ be order preserving. There is then a natural elementary embedding

$$\pi^\varphi_{\alpha\beta} : M_\alpha \to M_\beta,$$

called the shift map *given by φ such that $\pi^\varphi_{\alpha\beta}(\kappa_\alpha) = \kappa_\beta$, and for all $\overline{\alpha} < \alpha$, $\pi^\varphi_{\alpha\beta}(\kappa_{\overline{\alpha}}) = \kappa_{\varphi(\overline{\alpha})}$ and in fact $\pi^\varphi_{\alpha\beta} \circ \pi_{\overline{\alpha}\alpha} = \pi_{\overline{\beta}\beta} \circ \pi^{\varphi \restriction \overline{\alpha}}_{\overline{\alpha}\overline{\beta}}$ for all $\overline{\beta}$ with $ran(\varphi \restriction \overline{\alpha}) \subset \overline{\beta} < \beta$.*

Proof by induction on β. The statement is trivial for $\beta = 0$. Now let $\beta > 0$.

Let us first suppose β to be a successor ordinal, say $\beta = \overline{\beta}+1$. If $\overline{\beta} \notin ran(\varphi)$, then we may construe φ as a map from α to $\overline{\beta}$ and simply set $\pi^\varphi_{\alpha\beta} = \pi_{\overline{\beta}\beta} \circ \pi^\varphi_{\alpha\overline{\beta}}$. Let us thus assume $\overline{\beta} \in ran(\varphi)$, which implies that α is a successor ordinal as well, say $\alpha = \overline{\alpha}+1$, and $\varphi(\overline{\alpha}) = \overline{\beta}$. We then define $\pi^\varphi_{\alpha\beta}$ by setting

$$\pi^\varphi_{\alpha\beta}(\pi_{\overline{\alpha}\alpha}(f)(\kappa_{\overline{\alpha}})) = \pi_{\overline{\beta}\beta} \circ \pi^{\varphi\restriction\overline{\alpha}}_{\overline{\alpha}\overline{\beta}}(f)(\kappa_{\overline{\beta}}),$$

where $f \in M_{\overline{\alpha}}, f : \kappa_{\overline{\alpha}} \to M_{\overline{\alpha}}$. This is well–defined because if ψ is a formula and $f \in M_{\overline{\alpha}}, f : \kappa_{\overline{\alpha}} \to M_{\overline{\alpha}}, \ldots$, then

$$\begin{gathered} M_\alpha \models \psi(\pi_{\overline{\alpha}\alpha}(f)(\kappa_{\overline{\alpha}}), \ldots) \Leftrightarrow \\ \{\xi < \kappa_{\overline{\alpha}} : M_{\overline{\alpha}} \models \psi(f(\xi), \ldots)\} \in U_{\overline{\alpha}} \Leftrightarrow \\ \{\xi < \kappa_{\overline{\beta}} : M_{\overline{\beta}} \models \psi(\pi^{\varphi \restriction \overline{\alpha}}_{\overline{\alpha}\overline{\beta}}(f)(\xi), \ldots)\} \in U_{\overline{\beta}}, \text{ by using } \pi^{\varphi \restriction \overline{\alpha}}_{\overline{\alpha}\overline{\beta}}, \Leftrightarrow \\ M_\beta \models \psi(\pi_{\overline{\beta}\beta} \circ \pi^{\varphi \restriction \overline{\alpha}}_{\overline{\alpha}\overline{\beta}}(f)(\kappa_{\overline{\beta}}), \ldots). \end{gathered}$$

It is easy to verify, using the inductive hypotheses, that $\pi^{\varphi}_{\alpha\beta}$ is as desired.

Now suppose β to be a limit ordinal. If φ is not cofinal in β, say $ran(\varphi) \subset \overline{\beta} < \beta$, then we may construe φ as a map from α to $\overline{\beta}$ an simply set $\pi^{\varphi}_{\alpha\beta} = \pi_{\overline{\beta}\beta} \circ \pi^{\varphi}_{\alpha\overline{\beta}}$. Let us thus assume that φ is cofinal in β, which implies that α is a limit ordinal as well. We then define $\pi^{\varphi}_{\alpha\beta}$ by setting

$$\pi^{\varphi}_{\alpha\beta}(\pi_{\overline{\alpha}\alpha}(x)) = \pi_{\varphi(\overline{\alpha})\beta} \circ \pi^{\varphi \restriction \overline{\alpha}}_{\overline{\alpha}\varphi(\overline{\alpha})}(x).$$

Notice that each $y \in M_\alpha$ is of the form $\pi_{\overline{\alpha}\alpha}(x)$, where $\overline{\alpha} < \alpha$ and $x \in M_{\overline{\alpha}}$. Moreover, if $\pi_{\overline{\alpha}\alpha}(x) = \pi_{\overline{\alpha}'\alpha}(x')$, where $\overline{\alpha} < \overline{\alpha}'$, then

$$\begin{gathered} \pi_{\overline{\alpha}\alpha'}(x) = x' \text{ and } \pi_{\varphi(\overline{\alpha})\beta} \circ \pi^{\varphi \restriction \overline{\alpha}}_{\overline{\alpha}\varphi(\overline{\alpha})} = \\ \pi_{\varphi(\overline{\alpha}')\beta} \circ \pi_{\varphi(\overline{\alpha})\varphi(\overline{\alpha}')} \circ \pi^{\varphi \restriction \overline{\alpha}}_{\overline{\alpha}\varphi(\overline{\alpha})}(x) = \\ \pi_{\varphi(\overline{\alpha}')\beta} \circ \pi^{\varphi \restriction \overline{\alpha}'}_{\overline{\alpha}'\varphi(\overline{\alpha}')} \circ \pi_{\overline{\alpha}\overline{\alpha}'}(x) \text{ by the inductive hypothesis,} \\ = \pi_{\varphi(\overline{\alpha}')\beta} \circ \pi^{\varphi \restriction \overline{\alpha}'}_{\overline{\alpha}'\varphi(\overline{\alpha}')}(x'), \end{gathered}$$

so that the definition of $\pi^{\varphi}_{\alpha\beta}(y)$ is independent from the choice of $\overline{\alpha} < \alpha$ and $x \in M_\alpha$ with $y = \pi_{\overline{\alpha}\alpha}(x)$. It is easy to verify the inductive hypothesis.

Theorem 13.7 *Let κ be a measurable cardinal, and let $A \subset {}^\omega\omega$ be coanalytic. Then A has a κ–closed embedding normal form.*

Proof: Recall that if A is coanalytic, then there is a map $s \mapsto <_s$, where $s \in {}^{<\omega}\omega$ and $<_s$ is a linear oder on $lh(s)$, such that $<_t \restriction lh(s) = <_s$ whenever $s \subset t$ and setting $<_x = \bigcup\{<_s : s \subset x\}$ for $x \in {}^\omega\omega$, we have that for all $x \in {}^\omega\omega$,

$$x \in A \Leftrightarrow <_x \text{ is a well–order.}$$

Let $s \subsetneq t$, where $lh(t) = lh(s) + 1$. Write $n = lh(s)$. Suppose that n is the k^{th} element of $\{0, \ldots, n\}$ according to $<_t$, i.e.,

$$m_0 <_t \ldots <_t m_{k-1} <_t n <_t m_{k+1} <_t \ldots <_t m_n.$$

We then define $\varphi(s,t) : n \to n+1$ by $\varphi(s,t)(l) = l$ for $l < k$ and $\varphi(s,t)(l) = l+1$ for $l \geq k$.