On an Aggregation Model with Long and Short Range Interactions

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April 23, 2003

Abstract

In this paper we investigate well-posedness of a nonlinear degenerate and nonlocal parabolic equation arising from a model for aggregation in presence of short and long range interaction. We show existence of nonnegative weak solutions and uniqueness of entropy solution, a notion we extend to the nonlocal case. Moreover, we consider an approximating viscous case, which is of interest in applications, too.

Keywords: Aggregation, Integro-differential equations, Degenerate parabolic equations.

MSC (AMS 2000): 35K55, 35K65, 45K05.

1 Introduction

This paper is devoted to the mathematical analysis of the Cauchy problem for the following nonlocal degenerate parabolic equation

$$\frac{\partial \rho}{\partial t} - \text{div } (\rho \nabla(\rho - G * \rho)) = 0 \qquad \text{in } \mathbb{R}^d \times (0,T) \tag{1.1}$$

$$\rho = \rho_0 \qquad \text{in } \mathbb{R}^d \times \{0\}. \tag{1.2}$$

This equation may be seen as a mean-field approximation to a system of stochastic particles subject to aggregation at the "macroscale" and repulsion at a "mesoscale". It has been obtained from the analysis carried out in [2] and [11], where a system consisting of a large (but finite) number of particles subject to random dispersal and mutual interaction, was investigated. In particular, this kind of system was used to model the collective

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behavior of individuals which exhibit aggregative behavior [2]. The motivation of such a study is to provide a rigorous limit from the "microscopic" Lagrangian description, based on a system of SDEs, to the "macroscopic" Eulerian description when the total number of individuals in the population becomes sufficiently large; under suitable conditions [12] the evolution of the the limiting distribution of the population may be described in terms of the integro-differential equation (1.1).

The mathematical analysis of (1.1), (1.2) is a challenging problem since the model combines a nonlinear degenerate parabolic differential operator with a nonlocal nonlinearity. Below, we shall not investigate the original model directly, but the alternative formulation

$$\frac{\partial u}{\partial t} + \operatorname{div}\ (u\nabla(G * u)) - \Delta a(u) = 0 \qquad \text{in } \mathbb{R}^d \times (0,T] \tag{1.3}$$

$$u = u_0 \qquad \text{in } \mathbb{R}^d \times \{0\}, \tag{1.4}$$

where a is the function

$$a(u) := \frac{1}{2}u|u|. \tag{1.5}$$

The system (1.3), (1.4) is equivalent to (1.1), (1.2) if the solution u is nonnegative; by the way nonnegative functions are the only solutions of interest due to their interpretation as a population density.

In Section 2, we show the existence of a weak solution, both in the case of a bounded domain Ω and in the case of $\Omega = \mathbb{R}^d$. Moreover, we show that there exists a nonnegative weak solution if the initial value is nonnegative, which is then also a weak solution of (1.1), (1.2).

Section 3 discusses the uniqueness of a solution. In general, we can not expect uniqueness of a weak or even of a strong solution, since this neither holds for hyperbolic-parabolic problems of the form

$$\frac{\partial v}{\partial t} + \operatorname{div}\ F(v) - \Delta a(v) = 0, \tag{1.6}$$

with a nonlinear function $F : \mathbb{R} \to \mathbb{R}^d$ (cf. e.g. [3]) nor for other transport equations with nonlocal nonlinearity (cf. [6]).

However, we may show the uniqueness of the solutions within the class of the entropy solutions. The notion of entropy solution has been carried over to equations of the form (1.6) recently (cf. [3]) and we can adapt it for the nonlocal equation (1.3). The uniqueness of the entropy solution will finally be shown using the so-called "doubling of variables technique". We prove that an entropy solution is a weak solution for our problem.

In Section 4 the existence and uniqueness of a weak solution for the viscous case is shown, as well as the nonnegativity of the solution (for nonnegative initial value) similary to Section 3. As pointed out in [7] in the case of viscosity, entropy conditions are automatically fulfilled. Thus, no extra conditions need to be imposed on the solutions in this case.

1.1 From Stochastic Discrete to Mean-Field Aggregation Models

The starting point of the mathematical modelling is the Lagrangian description of a system of $N \in \mathbb{N} - \{0\}$ particles, subject to interaction forces and random dispersal. Suppose the k-th particle ($k \in \{1, \ldots, N\}$) is located at $X_N^k(t) \in \mathbb{R}^d$, at time $t \geq 0$; each $\{X_N^k(t), t \in \mathbb{R}_+\}$ is a stochastic process in the state space $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^\lceil})$, on the common probability space $(\Omega, \mathcal{F}, P)$. The spatial distribution of the system of N particles at time t, i.e. an Eulerian discrete description, is given by the random measure X_N on $\mathbb{R}^d$

$$X_N(t) = \frac{1}{N} \sum_{k=1}^{N} \epsilon_{X_N^k(t)}.$$

This measure may be considered as the empirical distribution of the system in $\mathbb{R}^d$ at time $t \in \mathbb{R}_+$. The dynamics of the system of interacting particles is given via a system of stochastic differential equations as follows:

$$dX_N^k(t) = F[X_N(t)](X_N^k(t))dt + \sigma_N dW^k(t), \quad k = 1, \ldots, N, \tag{1.7}$$

where the randomness is modelled by additive independentstandard Wiener processes $\{W^k, k = 1, \ldots, N\}$. Furthermore the common variance σ_N^2 might depend on the total number of particles and

$$\lim_{N \to \infty} \sigma_N^2 = \sigma_N^\infty \geq 0. \tag{1.8}$$

The drift term F depends on both the location of the specific particle and the empirical measure of the whole system of particles. It describes the mutual interaction of particles. It is assumed that:

(i) particles tend to aggregate subject to their interaction within a range of size $R > 0$ (finite or not). This corresponds to the assumption that each particle has a limited knowledge of the spatial distribution of its neighbors and interact within a bounded region; this kind of interaction is modeled via an aggregation kernel $G : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ with compact support with radius R. This interaction is called of *McKean-Vlasov* kind.

(ii) particles are subject to repulsion when they come "too close" to each other. Any accumulation in a single point in space is avoided [12]. A repulsion kernel $V_N : \mathbb{R}^d \longrightarrow \mathbb{R}_+$, depending on the total number N of interacting particles as follows. Let V_1 be a continuous probability density on $\mathbb{R}^d$ and consider the scaled kernel

$$V_N(x) = \chi_N^d V_1(\chi_N x), x \in \mathbb{R}^d, \tag{1.9}$$

with a scaling parameter of the form

$$\chi_N = N^{\beta/d}, \tag{1.10}$$

where $\beta \in (0,1)$. It is clear that

$$\lim_{N\to+\infty} V_N = \delta_0, \tag{1.11}$$

where δ_0 is Dirac's delta function.

The interaction induced by (1.9) is called *moderate* interaction. This because the range of the interaction is much smaller than the size of the whole space but much larger than the typical distance between two particles. A "mesoscale" is considered in order to have enough particles to perform a law of large numbers. So the drift term F is

$$F[X_N(t)](X_N^k(t)) = [\nabla G * X_N(t)](X_N^k(t)) - [\nabla V_N * X_N(t)](X_N^k(t)). \tag{1.12}$$

The limit (1.11) is the essential reason for the long-range operator $\nabla V_N * X_N$ in (1.12) becoming a classical local operator $\nabla\rho$ in (1.3). The function ρ represents the density of the population as the number of individual increases to infinity. In particular we have the following limit for the meausre $X_N(t)$

$$\lim_{N\to\infty} X_N(t) = \rho(\cdot,t)dx. \tag{1.13}$$

As a consequence, equation (1.3) can be interpreted as describing the time variation of the density of a large population subject to long-range aggregation and "infinitesimal" repulsion interaction. By considering $\sigma_\infty^2 = 0$, the individual randomness is completely lost.

1.2 Notations and Assumptions

In the following we recall some basic notations and definitions of function spaces to be used in the subsequent analysis. Moreover, we give some basic assumptions on the aggregation kernel G and the initial value u_0 in(1.4), which will be used in the subsequent analysis without further notice.

In general, we will denote the time variable by $t \in [0,T]$ (if necessary also by s, τ) and the spatial variable by $x \in \Omega \subset \mathbb{R}^d$ (if necessary also by y, z). Unless further noticed, the domain Ω is assumed to be either a bounded domain with Lipschitz boundary or $\Omega = \mathbb{R}^d$. Furthermore, we shall use the notation $\Omega_T := \Omega \times [0,T] \subset \mathbb{R}^{d+1}$ for the space-time domain. Functions on Ω_T will be denoted by u, v, in some cases also by the original variable ρ for the population density.

Function Spaces

For an open set $D \subset \mathbb{R}^N$, we denote by $C(D)$ the space of continuous functions on D and by $C^k(D)$ the space of k-times continuously differentiable functions equipped with the usual supremum-norms. Moreover, we will use the Lebesgue spaces $L^p(D)$, $1 \leq q \leq \infty$, with

$$\|u\|_{L^p(D)} = \begin{cases} \left(\int_D |u(x)|^p \, dx\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ \text{ess } \sup_{x \in D} |u(x)| & \text{if } p = \infty \end{cases} \tag{1.14}$$

and the Sobolev spaces $W^{k,p}(D)$, $1 \leq p \leq \infty$, $0 \leq k$ of functions with distributional derivatives up to order k in $L^p(D)$. The Sobolev space norms are defined by

$$\|u\|_{W^{k,p}(D)} = \left(\|u\|^p_{L^p(D)} + \sum_{1 \leq |\alpha| \leq k} \|\partial^\alpha u\|^p_{L^p(D)} \right)^{\frac{1}{p}} \tag{1.15}$$

for $1 \leq p < \infty$, and by

$$\|u\|_{W^{k,\infty}(D)} = \max \left\{ \|u\|_{L^\infty(D)}, \sup_{1 \leq |\alpha| \leq k} \|\partial^\alpha u\|_{L^\infty(D)} \right\}. \tag{1.16}$$

Moreover, we will use the standard notations $H^k(D) = W^{k,2}(D)$ and $H^1_0(D)$ for the subspace of functions in $H^1(D)$ with vanishing trace on ∂D. For further details on the spaces $W^{k,p}(D)$ we refer to the monographs by Adams [1] and Evans [8].

Finally, we need the vector valued function spaces on a real interval $I \subset \mathbb{R}$. For this sake, let $u : I \to X$ a function defined almost everywhere in I with values in some Banach space X. If u is continuous, then we say that $u \in C(I; X)$, and equip this space with the supremum norm

$$\|u\|_{C(I;X)} := \sup_{t \in I} \|u(t)\|_X.$$

In an analogous way we define the spaces $C^k(I; X)$, $L^p(I; X)$ and $W^{k,p}(I; X)$ and their norms, whose definition from the vector-valued case is obtained by changing the absolute values of $u(t)$ and its derivatives to the norm of $u(t)$ in the Banach space X. For a detailed discussion of vector valued function spaces we refer to Showalter [15].

The Convolution Kernel G

In the above model for the aggregation kernel, it is assumed that G is a bounded function with finite support, which represents the fact that particles interact only over some finite range. For our analysis, we can relax this assumption to

$$G \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d), \tag{1.17}$$

which implies that $G \in L^p(\mathbb{R}^d)$ for any $p \in [1, \infty]$. Moreover, we will always assume that $G \in H^1(\mathbb{R}^d)$, i.e.,

$$g := \|\nabla G\|_{L^2(\mathbb{R}^d)} < \infty; \tag{1.18}$$

further specific assumptions on the regularity of G will be stated explicitely when needed.

By convolution of a function u, defined on $\mathbb{R}^d$, with the kernel G we mean the function

$$v(x) := (G * u)(x) = \int_{\mathbb{R}^d} G(x-y)\ u(y)\ dy, \qquad x \in \mathbb{R}^d. \tag{1.19}$$

If a function u is defined on a proper subset Ω of $\mathbb{R}^d$ only, the convolution is given by

$$v(x) := (G *_\Omega u)(x) = \int_{\Omega} G(x-y)\ u(y)\ dy, \qquad x \in \mathbb{R}^d, \tag{1.20}$$

i.e., we continue u by zero outside Ω. Subsequently, we will omit the subscript Ω and denote the convolution by $G * u$ for simplicity. Note that due to $G \in L^1(\mathbb{R}^d)$, the convolution $G * u$ is well-defined as a function in $L^2(\mathbb{R}^d)$ due to Plancherel's Theorem. For a comprehensive treatment of convolution operators in L^p-spaces we refer to Champerey [4].

The Initial Value u_0

For the initial value we assume in general that $u_0 \in L^1(\Omega) \cap L^\infty(\Omega)$. Note that this implies in particular that $u_0 \in L^3(\Omega)$, an assumption used subsequently to prove the existence of weak solutions. Moreover we assume partial regularity of u_0, namely

$$u_0^2 \in H^1(\Omega), \tag{1.21}$$

which is needed for the weak form of the nonlinear degenerate differential operator contained in the model equation.

The interpretation of the initial value as a probability density yields two natural assumption for u_0, namely that u_0 is nonnegative (almost everywhere in Ω) and that

$$\int_\Omega u_0(x)\ dx = 1. \tag{1.22}$$

We shall see below that these further assumptions are not needed for the existence of weak solutions, so that we shall not use them unless for specific results such as the existence of nonnegative weak solutions, where they shall be stated explicitly.

2 Existence of Weak Solutions

In the following we investigate the existence of weak solutions of the nonlocal aggregation model (1.3),(1.4). By a weak solution u we mean a function $u \in C([0,T]; L^3(\Omega))$ with $u^2 \in L^2([0,T]; H_0^1(\Omega))$, $u_t \in L^2([0,T]; H^{-1}(\Omega))$ such that the initial condition (1.4) is satisfied and the identity

$$\int_0^T \int_\Omega \left(u \frac{\partial \phi}{\partial t} - \frac{1}{2} \nabla u^2 \nabla \phi + u(\nabla G * u) \nabla \phi \right) \, dx \, dt = 0 \tag{2.1}$$

holds for all $\phi \in C_0^\infty(\Omega)$.

In our analysis we will first consider the initial value problem (1.3), (1.4) in a bounded domain with the boundary condition $u = 0$ on $\partial\Omega$ (which is included in the condition $u^2 \in L^2([0,T]; H_0^1(\Omega))$). The existence of a solutions will follow from a fixed-point argument using Schauder's Theorem. By expansion of the domain and standard weak convergence techniques we will then prove the existence of a weak solution for $\Omega = \mathbb{R}^d$.

2.1 A Nonlinear Degenerate Equation

We first consider the nonlinear degenerate parabolic initial-boundary value problem

$$\frac{\partial u}{\partial t} - \Delta(a(u)) = f \qquad \text{in } \Omega \times (0,T] \tag{2.2}$$

$$u = 0 \qquad \text{on } \partial\Omega \times [0,T] \tag{2.3}$$

$$u = u_0 \qquad \text{in } \Omega \times \{0\}, \tag{2.4}$$

in a bounded domain $\Omega \subset \mathbb{R}^d$, which is usually called *porous medium equation with source term.*

Problems of this type are discussed e.g. by Showalter [15, p.142]. In the following we collect some regularity results. If the right-hand side satisfies $f \in L^{\frac{3}{2}}([0,T]; H^{-1}(\Omega))$ and if $u_0 \in L^3(\Omega)$, there exists a unique weak solution $u \in L^3(\Omega)$ with $u^2 \in H_0^1(\Omega)$. If furthermore $f \in L^2([0,T]; H^{-1}(\Omega))$, one can show that $u \in C([0,T]; L^3(\Omega))$ and that the stability estimates

$$\|u\|_{L^3(\Omega)}^3 + \frac{3}{4} \int_0^t \|\nabla u^2\|_{L^2(\Omega)}^2 \, d\tau \leq \|u_0\|_{L^3(\Omega)}^3 + 3 \int_0^t \|f\|_{H^{-1}(\Omega)}^2 \, d\tau, \tag{2.5}$$

for almost all $t \in (0,T)$, and

$$\left\| \frac{\partial u}{\partial t} \right\|_{L^2([0,T];H^{-1}(\Omega))} \leq 2\|f\|_{L^2([0,T];H^{-1}(\Omega))} \tag{2.6}$$

holds. If we choose in particular $t = T$ in (2.5), we obtain that

$$\frac{1}{T}\|u\|_{L^3(\Omega_T)}^3 + \frac{3}{4}\|\nabla u^2\|_{L^2(\Omega_T)}^2 \leq \|u_0\|_{L^3(\Omega)}^3 + 3\|f\|_{L^2([0,T];H^{-1}(\Omega))}^2. \tag{2.7}$$

The existence and stability results above allow us to define the continuous solution operator

$$\begin{array}{rccc} \mathcal{S}: & L^2([0,T];H^{-1}(\Omega)) & \to & L^2(\Omega_T) \\ & f & \mapsto & u, \end{array} \tag{2.8}$$

which is a compact operator:

Lemma 2.1. *Let Ω be a bounded Lipschitz domain. Then the operator $\mathcal{S}$ defined by* (2.8) *is compact.*

Proof. Suppose f_n is a bounded sequence in $L^2(0,T;H^{-1}(\Omega))$. Then the according sequence $\{u_n\}$ of solutions of (2.2)-(2.4) is uniformly bounded in $H^1([0,T];H^{-1}(\Omega)))$ and $\{u_n^2\}$ is uniformly bounded in $L^2([0,T];H_0^1(\Omega))$. Due to a result about fractional powers of functions in Sobolev spaces (cf. [14, p.365]), we obtain the uniform boundness of u_n in $L^2([0,T];H^{\frac{1}{2}}(\Omega))$. Hence, an embedding result by Lions and Aubin (cf. [15, p.106]) implies that there exists a convergent subsequence $\{u_{n_k}\}$, i.e., the operator $\mathcal{S}$ is compact. ☐

2.2 Existence of a Weak Solution, Ω bounded

Let $\mathcal{F}$ be the operator, which maps v to the solution u of

$$\frac{\partial u}{\partial t} - \text{div }(u\nabla u) = -\text{div }(v(\nabla G) * v) \qquad \text{in } \Omega \times (0,T] \tag{2.9}$$

$$u = 0 \qquad \text{on } \partial\Omega \times [0,T] \tag{2.10}$$

$$u = u_0 \qquad \text{in } \Omega \times \{0\}. \tag{2.11}$$

The above equation has a unique solution $u \in L^3(\Omega)$, if

$$f := -\text{div }(v(\nabla G) * v) \in L^{\frac{3}{2}}([0,T];H^{-1}(\Omega)) \tag{2.12}$$

and $u_0 \in H^{-1}(\Omega)$.

Since $G \in H^1(\mathbb{R}^d)$, we can define the operator

$$\begin{array}{rccc} \mathcal{G}: & L^2(\Omega_T) & \to & L^2([0,T];H^{-1}(\Omega)) \\ & u & \mapsto & \text{div }(u(\nabla G) * u) \end{array} \tag{2.13}$$

Some basic properties of $\mathcal{G}$ are shown in the following lemma:

Lemma 2.2. *The operator $\mathcal{G}$ from* (2.13) *is well-defined, continuous and weakly continuous.*

Proof. Since $\nabla : L^2(\Omega) \to H^{-1}(\Omega)$ is a bounded operator with norm 1, we obtain

$$\begin{aligned} \|\mathcal{G}(u)\| &\leq \|u(\nabla G) * u\|_{L^2([0,T];L^2(\Omega))} \\ &\leq \|u\|_{L^2([0,T];L^2(\Omega))} \|(\nabla G) * u\|_{L^2([0,T];L^\infty(\Omega))}, \end{aligned}$$

where the second estimate is due to the Hölder inequality. A standard result about the Fourier convolution (cf. [4, Theorem 8.24]) shows that

$$\|(\nabla G) * f\|_{L^\infty(\Omega)} \leq c\|\nabla G\|_{L^2(\mathbb{R}^d)}\|f\|_{L^2},$$

for all $f \in L^2(\Omega)$ and some constant $c \in \mathbb{R}^+$. Hence,

$$\|\mathcal{G}(u)\| \leq cg\|u\|^2_{L^2(\Omega_T)} \tag{2.14}$$

i.e., $\mathcal{G}$ is well-defined. The continuity follows from the analogous estimate

$$\begin{aligned}\|\mathcal{G}(u) - \mathcal{G}(\nu)\| &\leq \|(u-\nu)(\nabla G) * u\|_{L^2(\Omega_T)} + \|\nu(\nabla G) * (u-\nu)\|_{L^2(\Omega_T)} \\ &\leq cg\left(\|u\|_{L^2(\Omega_T)} + \|\nu\|_{L^2(\Omega_T)}\right)\|u-\nu\|_{L^2(\Omega_T)}.\end{aligned}$$

In order to show weak continuity, assume that $u_n \rightharpoonup u$. The compactness of the linear convolution operator

$$\begin{aligned}\mathcal{C}:\; L^2(\Omega) &\rightarrow L^\infty(\Omega) \\ f &\mapsto (\nabla G) * f\end{aligned}$$

implies that

$$(\mathcal{C}u_n(.,t) - \mathcal{C}u(.,t))^2 \rightarrow 0 \text{ in } L^\infty(\Omega).$$

Furthermore, the estimate

$$\begin{aligned}\|(\mathcal{C}u_n(.,t) - \mathcal{C}u(.,t))^2\|_{L^\infty(\Omega)} &= \|\mathcal{C}(u_n(.,t) - u(.,t))\|^2_{L^\infty(\Omega)} \\ &\leq g^2\|u_n(.,t) - u(.,t)\|^2_{L^2} \\ &\leq g^2(2\|u(.,t)\|_{L^2} + 1)^2,\end{aligned}$$

holds for sufficiently large n. Since $g^2(2\|u(.,t)\|_{L^2}+1)^2$ is in $L^1([0,T])$, Lebesgue's theorem (cf. [15, p. 103]) implies that

$$(\mathcal{C}u_n - \mathcal{C}u)^2 \rightarrow \text{ in } L^1([0,T]; L^\infty(\Omega)),$$

from which we may conclude

$$\mathcal{C}u_n \rightarrow \mathcal{C}u \text{ in } L^2([0,T]; L^\infty(\Omega)).$$

Now a standard argument shows that the product of the weakly convergent sequence u_n and the strongly convergent sequence $\mathcal{C}u_n$ converges weakly in $L^2(\Omega_T)$. □

Proposition 2.3 (Existence for small time). *Let $u_0 \in L^3(\Omega)$ and let T be such that*

$$\sqrt{|\Omega|}T^{\frac{3}{2}}\|u_0\|^{\frac{3}{4}}_{L^3(\Omega)}\left(\frac{3}{2}(cg)^2 + 1\right) \leq 1 \tag{2.15}$$

then there exists a solution $u \in L^2(\Omega_T)$ of (1.3),(1.4), with

$$u^2 \in L^2([0,T]; H^1_0(\Omega)).$$

Proof. Because of the weak continuity of $\mathcal{G}$ (Lemma 2.2) and the compactness and continuity of $\mathcal{S}$ (Lemma 2.1), the concatenation $\mathcal{F}$ is completely continuous, too.

Let $u := \mathcal{F}(\nu)$, then the stability estimate (2.7) implies

$$\frac{1}{3}\|u\|^3_{L^3(\Omega_T)} \leq \frac{T}{2}\|\mathcal{G}(\nu)\|^2_{L^2([0,T];H^{-1}(\Omega))} + \frac{T}{3}\|u_0\|^3_{L^3(\Omega)},$$

and inserting (2.14) yields

$$\frac{1}{3}\|u\|^3_{L^3(\Omega_T)} \leq \frac{T}{2}\|\nu\|^4_{L^2(\Omega_T)}(cg)^2 + \frac{T}{3}\|u_0\|^3_{L^3(\Omega)}$$

Hence, we obtain the L^2-estimate

$$\|u\|^3_{L^2(\Omega_T)} \leq \frac{3\sqrt{|\Omega|}}{2}\|\nu\|^4_{L^2(\Omega_T)}(cg)^2 T^{\frac{3}{2}} + \sqrt{|\Omega|}\|u_0\|^3_{L^3(\Omega)}T^{\frac{3}{2}}.$$

With $M := \|u_0\|^{\frac{3}{4}}_{L^3(\Omega)}$ this implies

$$\begin{aligned}\|u\|^3_{L^2(\Omega_T)} &\leq \sqrt{|\Omega|}T^{\frac{3}{2}}\|u_0\|^{\frac{3}{4}}_{L^3(\Omega)}\left(\frac{3}{2}(cg)^2 + 1\right)M^3 \\ &\leq M^3,\end{aligned}$$

for all ν with $\|\nu\|_{L^2(\Omega_T)} < M$. Applying this estimate we may conclude that $\mathcal{F}$ maps the closed, bounded set

$$\mathcal{M} := \{\ \nu \in L^2(\Omega_T) \mid \|\nu\|_{L^2(\Omega_T)} < M\ \} \tag{2.16}$$

into itself. Thus, we may apply Schauder's fixed point theorem (cf. [5]) and obtain the existence of a fixed point u of

$$u = \mathcal{F}(u),$$

i.e., a weak solution of (1.3)-(1.4) in $L^2(\Omega_T)$. Since the right-hand side satisfies

$$\mathcal{G}(u) \in L^2([0,T]; H^{-1}(\Omega)),$$

we may further deduce $u^2 \in L^2([0,T]; H^1_0(\Omega))$. □

The existence of a solution for arbitrary T can be shown easily by applying Proposition 2.3 subsequently to the time intervals $(0,S], (S,2S], \ldots, ((k-1)S, kS]$, where $k \in N$ and $S \in \mathbb{R}^+$ are chosen such that $T = kS$ and (2.15) is satisfied for S instead of T. The remaining assumption to be verified is that the new initial value for the interval $(jS,(j+1)S]$, $u(.,jS)$ is in $L^3(\Omega)$, but this follows from (2.5). Hence, we have shown:

Theorem 2.4 (Global Existence). *Let $u_0 \in L^3(\Omega)$ and $T \in \mathbb{R}^+$. Then there exists a solution $u \in L^2(\Omega_T)$ of (1.3)-(1.4), such that*

$$u^2 \in L^2([0,T]; H^1_0(\Omega)), u_t \in L^2([0,T]; H^{-1}(\Omega)).$$

2.3 Existence of a Weak Solution, $\Omega = \mathbb{R}^d$

For ∇G in $L^{\frac{4}{3}}(\mathbb{R}^d)$ we obtain (by inserting a smoothed version of u into the weak formulation and taking the limit) the a-priori estimate

$$\frac{1}{2}\|u(.,t)\|_{L^2}^2 + \int_0^t \|\nabla u(.,s)^{\frac{3}{2}}\|_{L^2}^2 \, ds \leq \int_0^t \|\nabla u(.,s)^{\frac{3}{2}}\|_{L^2} \tag{2.17}$$

Since the above estimates for u do not depend on the size of the domain Ω but only on $\|G\|_{H^1(\Omega)}$ and u_0, we may construct a sequence of bounded domains Ω_n that expand to $\mathbb{R}^d$ (e.g. $\Omega_n = B_n(0)$), and the corresponding sequence of solutions $\{u^n\}$ is uniformly bounded. Hence, a standard weak-convergence technique implies the existence of a solution u of

$$\frac{\partial u}{\partial t} - \text{div } (u\nabla u) = -\text{div } (u(\nabla G) * u) \qquad \text{in } \mathbb{R}^d \times (0,T] \tag{2.18}$$

$$u(x,.) \to 0 \qquad \text{as } |x| \to 0 \tag{2.19}$$

$$u = u_0 \qquad \text{in } \mathbb{R}^d \times \{0\}, \tag{2.20}$$

i.e., we obtain the following result:

Corollary 2.5. *Let $\nabla G \in L^{\frac{4}{3}}(\mathbb{R}^d)$, $u_0 \in L^3(\mathbb{R}^d)$ and $T \in \mathbb{R}^+$. Then there exists a weak solution $u \in L^2(\mathbb{R}^d \times [0,T])$ of (2.18)-(2.20), such that*

$$u^2 \in L^2([0,T]; H_0^1(\mathbb{R}^d)), u_t \in L^2([0,T]; H^{-1}(\mathbb{R}^d)).$$

We finally mention that the additional assumption $\nabla G \in L^{\frac{4}{3}}(\mathbb{R}^d)$ is satisfied for each $G \in H^1(\mathbb{R}^d)$, if its support is compact.

2.4 Existence of a Nonnegative Weak Solution

In order to show the existence of a nonnegative solution, we consider the problem

$$\frac{\partial u}{\partial t} - \text{div } \left(\sqrt{\epsilon + u^2}\nabla u - u\nabla G * u\right) = 0 \qquad \text{in } \Omega \times (0,T], \tag{2.21}$$

$$u = u_0 \qquad \text{in}\Omega \times \{0\} \tag{2.22}$$

with homogenous Dirichlet boundary conditions. The involved parabolic differential operator is nondegenerate for $\epsilon > 0$ and therefore easier to investigate than the original one with $\epsilon = 0$.

Lemma 2.6. *Let $\epsilon > 0$ and $u_0 \geq 0$ a.e. in Ω, then (2.21), (2.22) with homogenous Dirichlet boundary values on $\partial\Omega \times [0,T]$ has a unique weak solution $u^\epsilon \in L^\infty([0,T]; L^2(\Omega)) \cap L^3(\Omega_T)$, which is nonnegative.*

Proof. The existence and uniqueness of a weak solution for $\epsilon > 0$ follows from a standard argument for nondegenerate parabolic equations. Hence, we may define $f^\epsilon := -\nabla G * u^\epsilon$ and consider the problem

$$\frac{\partial u}{\partial t} - \text{div} \left(\sqrt{\epsilon + u^2}\nabla u + uf^n\right) = 0 \tag{2.23}$$

with initial value $u = u_0^n$ in $\Omega \times \{0\}$, where $f^n \in C_0^\infty(\Omega_T)$ and $u_0^n \in C_0^\infty(\Omega)$ are sequences such that

$$\|f^\epsilon - f^n\|_{L^\infty(\Omega_T)} \to 0, \qquad \|u_0 - u_0^n\|_{L^\infty(\Omega_T)} \to 0$$

as $n \to \infty$. Again, the weak solution $u^{\epsilon,n}$ of the homogenous Dirichlet boundary value problem for (2.23) exists and is unique for $\epsilon > 0$. Moreover, the theory of quasilinear parabolic equations (cf. [10]) implies that u^ϵ is a classical solution if $\partial\Omega$ is sufficiently smooth.

Now let $v = e^{-\lambda t}u^{\epsilon,n}$, with

$$\lambda > \sup_{\Omega_T} \text{div } f^\epsilon$$

and assume that v achieves its minimum in $(x_0, t_0) \in \Omega_T$. Then, since v solves

$$\frac{\partial v}{\partial t} - \text{div} \left(\sqrt{\epsilon + e^{2\lambda t}|v|^2}\nabla v\right) - \nabla v.f^\epsilon + (\lambda - \text{div } f^\epsilon)v = 0$$

and $\nabla v = 0$, $\frac{\partial v}{\partial t} = 0$ at (x_0, t_0), we obtain

$$(\lambda - \text{div } f^\epsilon)v(x_0, t_0) = \sqrt{\epsilon + e^{2\lambda t}|v|^2}\Delta v(x_0, t_0).$$

The second order necessary condition for a minimum at (x_0, t_0) implies that the right-hand side is nonnegative and consequently $v(x_0, t_0) \geq 0$. Hence, either $v \geq 0$ in Ω_T or v has no minimum in Ω_T. The latter implies that

$$\inf_{\Omega_T} v \geq \inf_{\partial\Omega_T} v = \min\left\{0, \inf_{\Omega\times\{T\}} v\right\},$$

where we have used the homogenous boundary and nonnegative initial values of v. If v would achieve a minimum in $x_1 \in \Omega$ at time T with $v(x_1, T) < 0$, we deduce by similar reasoning as above ($\nabla v(x_1, T) = 0$, $\Delta v(x_1, T) \geq 0$) that

$$\frac{\partial v}{\partial t}(x_1, T) = \sqrt{\epsilon + e^{2\lambda t}|v|^2}\Delta v(x_1, T) - (\lambda - \text{div } f^\epsilon)v(x_1, T) > 0,$$

but this would imply the existence of a time $t_1 < T$ such that $v(x_1, t_1) < v(x, T)$ and therefore contradict the assumption that $v(x_1, T)$ is a minimum. Thus, v is nonnegative a.e. in Ω_T in both cases and consequently

$$u^{\epsilon,n} = e^{-\lambda t}v \geq 0 \qquad \text{a.e. in } \Omega_T.$$

For $n \to \infty$, the norm of $u^{\epsilon,n}$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^3(\Omega_T)$ and hence, there exists a weakly convergent subsequence in $L^3(\Omega_T)$, whose limit is nonnegative and a weak solution of

$$\frac{\partial u}{\partial t} - \text{div }\left(\sqrt{\epsilon + u^2}\nabla u + u f^\epsilon\right) = 0 \tag{2.24}$$

with $u = u_0$ in $\Omega \times \{0\}$ and $u = 0$ on $\partial\Omega \times (0,T)$. Since the solution of (2.24) is unique it must be equal to u^ϵ and thus, u^ϵ is nonnegative. □

Theorem 2.7. *Let $u_0 \geq 0$ a.e. in Ω, then there exists a nonnegative weak solution of* (1.3)-(1.4).

Proof. With the notation

$$a_\epsilon(u) = \frac{1}{2}\left(u\sqrt{u^2+\epsilon} + \epsilon \ln(u + \sqrt{u^2+\epsilon})\right),$$

we may rewrite (2.21) as

$$\frac{\partial u}{\partial t} - \Delta a_\epsilon(u) - \text{div }(u \nabla G * u) = 0.$$

A standard variational argument shows that u^ϵ is is uniformly bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^3(\Omega_T)$ as $\epsilon \to 0$ and hence there exists a weakly convergent subsequence u^{ϵ_k} in $L^2(\Omega_T) \cap L^3(\Omega_T)$, whose limit $\overline{u}$ is a nonnegative weak solution of (1.3)-(1.4). □

2.5 Further Properties of Nonnegative Weak Solutions

Now we investigate some properties of solutions of the model equations (1.3)-(1.4), in connection with their interpretation of a population density. One easily verifies that the model (1.3)-(1.4) conserves the density, i.e.,

$$\int_\Omega u(x,t)\ dx = \int_\Omega u_0(x)\ dx. \tag{2.25}$$

Thus, for nonnegative solutions, we may conclude that $u(.,t)$ is in $L^1(\Omega)$ for almost all t:

Theorem 2.8. *Let u be a nonnegative solution of* (1.3), (1.4), *then* $u \in L^\infty(0,T;L^1(\Omega))$.

Another important result is the regularity of nonnegative solutions (and since we want to compute a population density, these are the only ones of interest):

Theorem 2.9. *Every nonnegative solution u of* (1.3)-(1.4) *satisfies*

$$u \in L^2(0,T;H_0^1(\Omega)).$$

Proof. We define the nonnegative function $u := \frac{u}{R}$, where

$$R := \int_\Omega u_0(x)\ dx.$$

(S-t) The identity (2.25) shows that

$$\int_\Omega u(x,t)\ dx = 1, \qquad \forall t \in [0,T)$$

and hence, the *Kullback-Leibler divergence*

$$E(t) := \int_\Omega \left(u(x,t)\ln u(x,t) + 1 - u(x,t)\right)\ dx \tag{2.26}$$

exists and is nonnegative for all $t \in [0,T)$. Since u solves

$$\frac{\partial u}{\partial t} - \text{div}\ (u\nabla(u - G * u)) = 0,$$

standard smoothing techniques allow to obtain the estimate

$$\begin{aligned} E(t) + R\int_0^t \|\nabla u\|_{L^2}^2\ ds &\le \int_\Omega u(x,t)\ dx + R\int_0^t |\langle(\nabla G) * u, \nabla u\rangle_{L^2}|\ ds \\ &\le 1 + \frac{R}{2}\int_0^t \left(\|\nabla u\|_{L^2}^2 + \|(\nabla G) * u\|_{L^2}^2\right)\ ds. \end{aligned}$$

Since u is nonnegative, we obtain by using Fourier's theorem (cf. [4])

$$\begin{aligned} \|(\nabla G) * u\|_{L^2} &\le \|\nabla G\|_{L^2}\|u\|_{L^1(\Omega)} \\ &= \|\nabla G\|_{L^2}\int_\Omega u(x,t)\ dx \\ &= \|\nabla G\|_{L^2} \end{aligned}$$

for all $t \in (0,T)$, and hence,

$$\begin{aligned} E(t) + \frac{R}{2}\int_0^t \|\nabla u\|_{L^2}^2\ ds &\le 1 + \frac{R}{2}\int_0^t \|\nabla G\|_{L^2}^2\ ds \\ &= 1 + \frac{Rt}{2}\|\nabla G\|_{L^2}^2. \end{aligned}$$

Using the fact that $\|\nabla G\|_{L^2}^2$ is equivalent to the H^1-norm on $H_0^1(\Omega)$ (cf. [8]), we finally obtain

$$\|u\|_{L^2(0,T;H^1(\Omega))}^2 \le C(1+T)$$

fo some nonnegative constant C, i.e., $u \in L^2(0,T;H_0^1(\Omega))$. ☐

Finally, we show that there exists a bounded weak solution:

Theorem 2.10. *Let $u_0 \in L^\infty(\Omega)$ be a nonnegative initial value satisfying (1.22). Then there exists a nonnegative weak solution $u \in C(0,T;L^1(\Omega)) \cap L^\infty(\Omega_T)$.*

Proof. Above we have deduced the existence of a nonnegative solution. By an analogous approximation technique we can show that this weak solution is bounded by the essential supremum of the initial value almost everywhere in Ω_T. Finally, due to conservation of the mean value of $u(.,t)$ we have that $u \in L^\infty(0,T;L^1(\Omega))$. □

3 Uniqueness of Entropy Solutions

So far we have shown that the initial-boundary value problem (1.3)-(1.4) has a weak solution u, which is nonnegative and satisfies $u \in L^\infty(0,T;L^1(\Omega)) \cap L^\infty(\Omega_T)$, but we did not pose the question of uniqueness. In general we cannot expect uniqueness of weak solutions for a nonlocal transport equation, a problem that has been discussed recently by Diekmann et. al. [6].

For nonnegative weak solutions of (1.3),(1.4) the only uniqueness result we can prove is for the special case of homogenous initial values:

Lemma 3.1. *Let u be a solution of* (1.3), (1.4) *with*

$$u = 0 \qquad \text{in } \Omega \times \{0\},$$

then $u = 0$ a.e. in Ω_T.

Proof. The proof of Proposition 2.3 shows that u satisfies an estimate of the form

$$\|u\|^3_{L^2(\Omega_t)} \;\leq\; \sqrt{|\Omega|}t^{\frac{3}{2}}\|u_0\|^{\frac{3}{4}}_{L^3(\Omega)}\left(\frac{3}{2}(cg)^2+1\right)M^3 = 0$$

for sufficiently small t, which implies directly $u = 0$ in Ω_t if $u^0 = 0$. By applying the same result succesively to the time intervals $[t,2t]$,$[2t,3t]$, ... we may conclude $u = 0$ in Ω_T. □

In order to prove a uniqueness result for general initial values, we restrict the class of feasible solutions to the class of entropy solutions. This means that in general there is uniqueness for weak solutions, but in the class of solutions satisfying conditions of entropy dissipation, the solution is unique.

In the next definition we introduce the concept of *entropy solutions* to our system. Entropy solutions have been introduced recently for equations of the form (1.6) (cf. e.g. [3, 9]). We adapt this notion to our system, closelying follow this approach with a simple modification enforced by the nonlocal convolution operator, which leads us to the following definition (for simplicity, we always assume $\Omega = \mathbb{R}^d$ in the remainder of this section). Then we prove then a solution which is entropy solution is also a weak solution. Finally the main result is given.

Definition 3.2. An entropy solution of (1.3) is a measurable function u on Ω_T satisfying the following conditions:

(i) $u \in L^\infty(\Omega_T) \cap C(0,T; L^1(\mathbb{R}^d))$

(ii) $u^2 \in L^2(0,T; H^1(\Omega))$

(iii) For all $c \in \mathbb{R}$ and all nonnegative test functions $\phi \in C_0^\infty(\Omega_T)$, the following entropy inequality holds:

$$\int_0^T \int_{\mathbb{R}^d} \Big(|u-c| \frac{\partial \phi}{\partial t} + \text{ sign}(u-c) \ \ ((u-c)(\nabla G * u)) \nabla \phi + \frac{1}{2}|u^2 - c^2| \Delta \phi - \text{ sign}(u-c) \ \text{div}(c(\nabla G * u))\phi \Big) \ dx \ dt \le 0 \quad (3.1)$$

(iv) Essentially, as $t \downarrow 0$,

$$\int_{\mathbb{R}^d} |u(x,t) - u_0(x)| \ dx \to 0.$$

The only difference to the definition of an entropy solution for an equation of the form (1.6) as introduced by Carrillo [3] is in the entropy inequality, which is given originally by

$$\int_0^T \int_{\mathbb{R}^d} \Big(|u-c| \frac{\partial \phi}{\partial t} + \text{ sign}(u-c) \ \ (F(u) - F(c)) \nabla \phi + \frac{1}{2}|u^2 - c^2| \Delta \phi - \text{ sign}(u-c) \ \text{div}(F(c))\phi \Big) \ dx \ dt \le 0$$

for all nonnegative test functions $\phi \in C_0^\infty(\Omega_T)$. If we would adapt this definition directly by replacing the nonlinear function F by the nonlinear operator $F : u \mapsto u(\nabla G * u)$, the entropy inequality read

$$\int_0^T \int_{\mathbb{R}^d} \Big(|u-c| \frac{\partial \phi}{\partial t} + \text{ sign}(u-c) \ \ (u(\nabla G * u)) \nabla \phi + \frac{1}{2}|u^2 - c^2| \Delta \phi \phi \Big) \ dx \ dt \le 0$$

due to the fact that $\nabla G * c = 0$ for each constant function c. Thus, it seems more natural to adapt the notion of an entropy solution for systems of the form

$$\frac{\partial v}{\partial t} + \text{div } f(x,t,v) - \Delta a(v) = 0, \quad (3.2)$$

introduced by Karlsen and Risebro [9], where $f(x,t,u) = g(x,t)u$ in our case. The vectorial function g is defined by $g = \nabla G * u$, where u is implicitely assumed to be given in this part. The rationale behind this approach is that by construction a fixed point map as the concatenation of $u \mapsto g$ and $g \mapsto v$, where v is the unique entropy solution of (3.2), one can hope to obtain at least a subsequence converging to the entropy solution of (1.3), (1.4).

In order to be coherent with the definition of weak solutions given above, we verify that each entropy solution is also a weak solution:

Proposition 3.3. *Let u be an entropy solution of* (1.3), (1.4). *Then u is also a weak solution.*

Proof. First, we choose $c > \|u\|_{L^\infty(\Omega_T)}$, then the entropy inequality becomes

$$\int_0^T \int_{\mathbb{R}^d} \Big((c-u)\frac{\partial \phi}{\partial t} - (u(\nabla G * u))\,\nabla \phi - \frac{1}{2}(u^2 - c^2)\Delta\phi \Big)\; dx\; dt \le 0,$$

where we have used Gauss' Theorem on Ω to eliminate the terms containing $c(\nabla G * u)$. Because of

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^d} c\frac{\partial \phi}{\partial t}\; dx\; dt = \int_{\mathbb{R}^d} c(\phi(x,T) - \phi(x,0))\; dx &= 0 \\ \int_0^T \int_{\mathbb{R}^d} (u^2 - c^2)\Delta\phi\; dx\; dt = \int_0^T \int_{\mathbb{R}^d} \nabla(u^2 - c^2)\nabla\phi\; dx\; dt, & \end{aligned}$$

where we have used $u^2 \in L^2(0,T;H^1(\Omega))$ and the compact support of ϕ, we obtain that for all nonnegative $\phi \in C_0^\infty(\Omega_T)$ the inequality

$$\int_0^T \int_{\mathbb{R}^d} \Big(-u\frac{\partial \phi}{\partial t} - (u(\nabla G * u))\,\nabla \phi + \frac{1}{2}\nabla(u^2)\nabla\phi \Big)\; dx\; dt \le 0.$$

Similary, by choosing $c < -\|u\|_{L^\infty(\Omega_T)}$, we may deduce the reverse inequality and thus,

$$\int_0^T \int_{\mathbb{R}^d} \Big(-u\frac{\partial \phi}{\partial t} - (u(\nabla G * u))\,\nabla \phi - \frac{1}{2}\nabla(u^2)\nabla\phi \Big)\; dx\; dt = 0$$

for all nonnegative test functions ϕ. For general ϕ we can construct two sequences of nonnegative test functions $\phi_k^+, \phi_k^- \in C_0^\infty(\Omega)$ such that $\phi_k^+ \to \max\{\phi, 0\}$, $\phi_k^- \to -\min\{\phi,0\}$ in $C(\Omega_T)$ and

$$\begin{aligned} 0 &= \int_0^T \int_{\mathbb{R}^d} \Big(-u\frac{\partial \phi_k^+}{\partial t} - (u(\nabla G * u))\,\nabla \phi_k^+ - \frac{1}{2}\nabla(u^2)\nabla\phi_k^+ \Big)\; dx\; dt \\ &\quad - \int_0^T \int_{\mathbb{R}^d} \Big(-u\frac{\partial \phi_k^-}{\partial t} - (u(\nabla G * u))\,\nabla \phi_k^- - \frac{1}{2}\nabla(u^2)\nabla\phi_k^- \Big)\; dx\; dt \\ &\to \int_0^T \int_{\mathbb{R}^d} \Big(-u\frac{\partial \phi}{\partial t} - (u(\nabla G * u))\,\nabla \phi - \frac{1}{2}\nabla(u^2)\nabla\phi \Big)\; dx\; dt, \end{aligned}$$

which completes the proof. □

A natural way to circumvent problems caused by the discontinuity of the sign function is to replace it by a continuous approximation sign_ϵ such as

$$\text{sign}_\epsilon(\tau) := \begin{cases} -1 & \text{if } \tau < -\epsilon \\ \frac{\tau}{\epsilon} & \text{if } -\epsilon \le \tau \le \epsilon \\ 1 & \text{if } \tau > \epsilon, \end{cases} \tag{3.3}$$

and to consider the limit $\epsilon \to 0$. This leads us to the so-called *entropy dissipation property* (cf. [3]):

Lemma 3.4. *Let u be an entropy solution of (1.3), (1.4) and let $\Delta G \in L^1(\mathbb{R}^d)$. Then, for any nonnegative function $\phi \in C_0^\infty(\Omega_T)$ and $c \in \mathbb{R}$, we have that*

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^d} \Big(|u-c|\frac{\partial \phi}{\partial t} + \ & sign(u-c)\ \left((u-c)(\nabla G * u)\right) \nabla\phi\ + \\ & \frac{1}{2}|u^2-c^2|\Delta\phi - \ sign(u-c)\ div(c(\nabla G * u))\phi \Big)\ dx\ dt \\ = \frac{1}{4}\lim_{\epsilon\downarrow 0}\int_0^T \int_{\mathbb{R}^d} & |\nabla(u^2)|^2\ sign_\epsilon'(u^2-c^2)\phi\ dx\ dt \end{aligned} \tag{3.4}$$

Proof. In an analogous way to the proof of Lemma 2.2 in [9] we can show that

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^d} \Big(|u-c|\frac{\partial \phi}{\partial t} + \ & \text{sign}(u-c)\ \left((u-c)(\nabla G * u)\right) \nabla\phi\ + \\ & \frac{1}{2}|u^2-c^2|\Delta\phi - \ \text{sign}(u-c)\ \text{div}(c(\nabla G * u))\phi \Big)\ dx\ dt \\ = \frac{1}{4}\lim_{\epsilon\downarrow 0}\int_0^T \int_{\mathbb{R}^d} & |\nabla(u^2)|^2\ \text{sign}_\epsilon'(u^2-c^2)\phi\ dx\ dt - I_1 - I_2 \end{aligned}$$

with I_1 and I_2 given by

$$\begin{aligned} I_1 &= \frac{1}{4}\lim_{\epsilon\downarrow 0}\int_0^T \int_{\mathbb{R}^d} (u-c)(\nabla G * u)\nabla(u^2)\text{sign}_\epsilon'(u^2-c^2)\phi\ dx\ dt \\ I_2 &= \frac{1}{4}\lim_{\epsilon\downarrow 0}\int_0^T \int_{\mathbb{R}^d} \big(2(u-c)(\nabla G * u) - \nabla(u^2)\big)\ \text{sign}_\epsilon(u^2-c^2)\nabla\phi\ dx\ dt \end{aligned}$$

If we define the function Q_ϵ via

$$Q_\epsilon(z) = \int_0^\epsilon \text{sign}_\epsilon'(r-c^2)(\sqrt{r}-c)\ dr$$

then we can rewrite I_1 (using Gauss' Theorem) as

$$\begin{aligned} I_1 &= \frac{1}{4}\lim_{\epsilon\downarrow 0}\int_0^T \int_{\mathbb{R}^d} \nabla Q_\epsilon(u^2)(\nabla G * u)\phi\ dx\ dt \\ &= -\frac{1}{4}\lim_{\epsilon\downarrow 0}\int_0^T \int_{\mathbb{R}^d} Q_\epsilon(u^2)\left((\Delta G * u)\phi + (\nabla G * u).\nabla\phi\right)\ dx\ dt. \end{aligned}$$

With the same arguments as in the proof of Lemma 2.2 in [9] we can now argue that $Q_\epsilon(z)$ tends to zero as $\epsilon \downarrow 0$ for all $z \geq 0$ and employ the Lebesgue dominated convergence theorem to show that $I_1 = 0$. In a similar way one can verify that $I_2 = 0$ and thus, we may conclude that (3.4) holds. □

The entropy dissipation property is the fundamental requirement for the main result of this section, namely the uniqueness of the of an entropy solution:

Theorem 3.5 (Uniqueness). *Let u, v be two entropy solutions of* (1.3), (1.4). *Then $u = v$ a.e. in $\Omega_T = \mathbb{R}^d \times (0,T)$.*

Proof. The proof can be carried out in an analogous way to the proof of Theorem 1.1 in [9], using the entropy dissipation property (3.4). □

4 The Viscous Case

In the following we consider the viscous approximation to the partial differential equation (1.3), i.e.,

$$u_t - \epsilon\Delta u - \frac{1}{2}\Delta u^2 = -\operatorname{div}(u(\nabla G) * u). \tag{4.1}$$

Viscous approximations to nonlinear hyperbolic or degenerate parabolic differential equations are frequently used as a tool for proving the existence of weak solutions (cf. e.g. [8]). With respect to the interpretation of (1.3) as the mean-field limit of a system of interacting particles, the viscous case is of interest for itself. This can be seen from an inspection of the derivation of the original model (1.3) in [12], where one has to assume that the mean free path of each particle (i.e. the coefficient of the Brownian term in the stochastic differential equations describing the dynamics of the system) tends to zero as the number N of total particles increases to infinity. Hence, the limiting behavior becomes completely deterministic and no memory of the individual stochastic behavior appears.In a specific applications to population biology, this assumption is well justified (cf. [2]), but for possible different applications it might be violated.

If one does not assume a priori that the individual variance σ_N^2 tends to zero, but $\sigma_N \to \sigma_\infty$ for some positive real value σ_∞, then the limiting behavior is described by the partial differential equation (4.1) with $\epsilon = \frac{\sigma_\infty^2}{2}$. The diffusion term in (4.1) is thus a trace of the individual stochastic behavior that remains in the deterministic mean-field model.

Similarly to the case $\epsilon = 0$ one can show the existence of a solution by a fixed point method (noticing that the results about the general porous media equation (2.2) are applicable with $a(u) = \epsilon u + u|u|$). In addition, one can easily prove that $u \in L^2(0,T; H_0^1(\Omega)) \cap L^\infty(\Omega_T))$ for positive ϵ.

Theorem 4.1. *Under the conditions of Theorem 2.4, there exists a unique nonnegative weak solution* $u \in L^2(0,T;H_0^1(\Omega))$ *of* (4.1) *for each initial value* $u_0 \in L^2(\Omega_T)$*, where* Ω *is either a bounded set or* $\mathbb{R}^d$*.*

Proof. Suppose u and μ are two different solutions of (4.1) and let s be the minimal $t \in [0,T]$, such that u and μ differ in $L^2(s,S;H_0^1(\Omega))$ for any $S > s$. Multiplying the difference of (4.1) for u and μ by $-\Delta^{-1}(u-\mu)$ and integrating over $\Omega \times (s, s+\tau)$ yields

$$\frac{1}{2}\|u-\mu\|_{H^{-1}}^2|_{s+\tau} + \epsilon \int_s^{s+\tau} \left(\|u-\mu\|_{L^2}^2 + \int_\Omega (|u|+|\mu|)(u-\mu)^2\,dx\right)\,dt$$

$$\leq \int_s^{s+\tau} \|u-\mu\|_{H^{-1}}\|\nabla G\|_{L^2}(\|u\|_{L^2}(t) + \|\mu\|_{L^2})\|u-\mu\|_{L^2}(t)\,dt,$$

where we have used standard estimates for the H^{-1}-norm and similar estimates for the convolutive products as above. Since u and μ are bounded in $L^\infty(0,T;L^2(\Omega)$ and the term $(|u|+|\mu|)(u-\mu)^2$ is nonnegative, there exists a real number η such that

$$\frac{1}{2}\|u-\mu\|_{H^{-1}}^2(s+\tau) + \epsilon \int_s^{s+\tau} \|u-\mu\|_{L^2}^2(t)\,dt$$

$$\leq \eta \int_s^{s+\tau} \|u-\mu\|_{H^{-1}}(t)\|u-\mu\|_{L^2}(t)\,dt$$

$$\leq \frac{\epsilon}{2}\int_s^{s+\tau} \|u-\mu\|_{L^2}^2(t)\,dt + \frac{\eta^2}{2\epsilon}\int_s^{s+\tau} \|u-\mu\|_{H^{-1}}^2(t)\,dt.$$

Now integration with respect to τ from s to S yields

$$\int_s^S \left[\left(\frac{\epsilon - \eta^2(S-t)}{2\epsilon}\right)\|u-\mu\|_{H^{-1}}^2 + \frac{\epsilon(S-t)}{2}\|u-\mu\|_{L^2}^2\right](t)\,dt \leq 0,$$

which contradicts $u \neq \mu$ in $L^2(s,S;H_0^1(\Omega))$ for arbitrary $S > s$. □

A similar argument to the proof of Theorem 2.9 is possible in the viscous case, but we have to consider the function $\mu^\epsilon = u + 2\epsilon$ and the modified divergence

$$E^\epsilon(t) = \int_\Omega (\mu^\epsilon(x,.)\ln\mu^\epsilon(x,.) + 1 - \mu^\epsilon(x,.))\,dx.$$

Analogous reasoning implies the estimate

$$E^\epsilon(t) - E^\epsilon(0) + \frac{1}{2}\int_0^t \|\nabla u(.,s)\|_{L^2(\Omega)}^2\,ds \leq \frac{1}{2}\|\nabla G\|^2\int_0^t \|u(.,s)\|_{L^2(\Omega)}^2\,ds.$$

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