

# Global Weak Solutions of a Polymer Crystal Growth Model

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## Abstract

The aim of this paper is to show the global existence of weak solutions for a moving boundary problem arising in the non-isothermal crystallization of polymers. The main features of our works are (i) the moving interface is shown to be of co-dimension one; (ii) finite Hölder continuous propagation speed yields an intrinsic estimate of finite co-dimension one Hausdorff measure of the moving interface for every time  $t$  in two space dimension; (iii) based upon (ii), we prove Hölder continuity of the temperature  $u$  by a decomposition argument.

**Keywords:** Moving Boundary, Level Set Method, Heat Conduction, Growth, Crystallization, Hausdorff measure, co-dimension one measure estimate, decomposition.

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## 1 Introduction

Crystallization of polymeric materials is a phase-change process in strong interaction with heat conduction. Both of the basic mechanisms involved in the solidification from a melt, namely the nucleation and growth of crystals, are strongly influenced by the temperature and its variations. Vice Versa, the latent heat is rather large for polymeric materials, so that it causes a considerable change in the heat transfer process.

The interaction of solidification and heat conduction is a well-known phenomenon in models of phase-change, and involves important problems in the theory of free and moving boundary problems. In typical examples of moving boundary problems such as the Stefan problem, the unknown boundary is assumed to be isothermal, which leads in a mathematical model to a homogeneous Dirichlet condition for the (appropriately scaled) temperature. For polymers the situation is different, since the phase change does not take place at a fixed temperature (or with kinetic undercooling close to this temperature), but in a rather large temperature range between the thermal melting point  $T_m$  and the glass transition temperature  $T_g$ .

The mathematical modeling of all the mechanisms including nucleation, growth, and deposition is very complicated. It may be appropriate to mathematically study single mechanisms alone at this stage. The modeling of nucleation of new crystals, which can be considered as a heterogeneous Poisson process, has been done in [8], [10]. In this paper, we only consider the effect of crystal growth.

The growth of the crystalline phase  $\Omega = \cup_{i=1}^N \Omega_i$  ( $\Omega_i(t)$  being the single crystal), is determined via nonlinear dynamical relations for the normal velocity  $v$  and the temperature  $u$ ,

$$v = G(u) = \frac{1}{L} [\frac{\partial u}{\partial n}] \quad \text{on} \quad \Gamma \equiv \partial\Omega, \quad (1.1)$$

where  $G$  is a given material function,  $L$  is the latent heat, and  $[\cdot]$  denotes the jump across the boundary. The function  $G$  is non-decreasing until it reaches a maximum at some temperature depending on material, then is non-increasing for larger values. For example,  $G$  attains its maximum at  $80^\circ$  C in case of isotactic polypropylene, see [35, 31, 33]. Motivated by the physics of the problem we shall assume that  $G$  is a non-negative smooth function with positive lower bound and upper bound, i.e., for all  $u, u_1, u_2 \in \mathbb{R}$ .

$$0 < C_1 \leq G(u) \leq C_2 < \infty, \quad (1.2)$$

$$|G(u_1) - G(u_2)| \leq L|u_1 - u_2|. \quad (1.3)$$

Since surface tension in polymers is very small, it is neglected in most treatments, and we follow this approach here.

## 2 Formulation of The Problem

The heat transfer in the material is described by the heat equation in  $\mathbb{R}^n \setminus \partial\Omega \times [0, T]$ , in general with parameters such as conductivity, density, and capacity dependent on the phase. The heat equation is

$$\begin{aligned} c_1 u_t &= -\operatorname{div} q, & q &= -k_1 \nabla u, & \text{in the liquid phase } \Omega \\ c_2 u_t &= -\operatorname{div} q, & q &= -k_2 \nabla u, & \text{in the solid phase } \mathbb{R}^n \setminus \Omega; \end{aligned} \quad (2.1)$$

where  $u$  is the temperature, and  $q$  is the heat flux. For simplicity, we restrict our attention to the case of all parameters being constants in this paper. That is,  $c_1 = c_2 = 1$  and  $K_1 = K_2 = 1$ . On the interface between two phases: solid phase and liquid phase, the Rankine-Hugoniot condition and nonlinear Gibbs-Thompson relation between temperature  $u$  and normal velocity of interface are given by

$$Lv = [q]n \quad (2.2)$$

$$v = G(u) \quad (2.3)$$

where  $v$  is the normal velocity of the interface  $\Gamma$  between  $\Omega_1$  and  $\Omega_2$ ,  $L$  is latent heat coefficient, and  $n$  is the unit normal vector at the interface. For the sake of simplicity, let  $L = 1$ . Introducing a phase function  $\varphi$  as in [32] and [3], let  $\{\varphi < 0\}$  denote solid phase, and  $\{\varphi > 0\}$  denote liquid phase. Therefore  $\{\varphi = 0\}$  is the interface between two phases. On the interface  $\{\varphi = 0\}$ , the unit normal vector and velocity, respectively, are

$$n = \frac{\nabla \varphi}{|\nabla \varphi|}, \quad v = -\frac{\varphi_t}{|\nabla \varphi|}.$$

Hence (2.3) can be re-written as a Hamilton-Jacobi equation as

$$\varphi_t + G(u)|\nabla \varphi| = 0. \quad (2.4)$$

(2.1) and (2.2) are equivalent to the following differential equation in the distributional sense.

$$\partial_t(u + \chi(\varphi)) - \Delta u = 0, \quad (2.5)$$

where

$$\chi(\varphi) = \operatorname{sign} \varphi = \begin{cases} -1, & \text{if } \varphi < 0, \\ [-1, 1], & \text{if } \varphi = 0, \\ 1, & \text{if } \varphi > 0. \end{cases} \quad (2.6)$$

(2.5)-(2.4) is the weak formulation of the front propagation problem (2.1)-(2.3) for the non-isothermal crystallization in polymeric materials.

Let us make a comparison with the classical Stefan problem. In the classical case– Stefan problem, the equation (2.3) is replaced by an isothermal condition, i.e.,

$$u = 0. \quad (2.7)$$

Therefore the weak formulation of the classical Stefan problem (2.1), (2.2), and (2.7) is as follows:

$$\partial_t(u + \chi(u)) - \Delta u = 0, \quad (2.8)$$

There are vast literatures in the study of Stefan problems (see e.g. [25, 22, 28, 40]). However, the best known regularity of the temperature  $u$  for (2.8) is merely continuous (see [11, 17, 41, 34]) and the codimension one property of moving front is still open.

Because of the non-isothermal dynamical relation (2.3) on the interface,  $\text{sign}(u)$  in the Stefan problem (2.8) is replaced by  $\text{sign}(\varphi)$  in (2.5) where  $\varphi$  is governed by the Hamilton-Jacobi equation (2.4). Earlier works by Avrami [1], Kolmogorov [24], and Evans [20] in the 1930's and 1940's addressed spatially homogeneous growth of crystals, where the phase could be computed analytically. Recently the modeling of non-isothermal crystallization has been extensively examined in [31, 35, 38, 33, 18, 19, 6, 8, 9, 10]. The local existence of a single smooth moving front in (2.1)-(2.3) was shown under the assumption that initial interface is a small perturbation of a sphere in Friedman-Velazquez [23]. In practice there are many crystals created by nucleation so that a situation with multiple crystals should be studied instead. As crystalline grow, they definitely infringe each other in finite time as long as there is positive lower bound on the propagation speed. Consequently the moving front may not be smooth. To capture the physical features of front propagation we study global weak solution of (2.4) and (2.5). In this paper, we prove the global existence of a weak solution of (2.4) and (2.5) even for large initial data. In particular we show that temperature is Hölder continuous in space and the moving front is a Lipschitz graph in time-space. The main novel features of our work are:

- (i) the moving interface is shown to be of co-dimension one;
- (ii) finite Hölder continuous propagation speed yields an intrinsic estimate of finite co-dimension one Hausdorff measure of the moving interface for every time  $t$  in two space dimension;
- (iii) based upon (ii), we prove Hölder continuity of the temperature  $u$  by a decomposition argument.

Now we need to interpret the weak solution  $(\varphi, u)$  of (2.4) and (2.5). The temperature  $u$  satisfies (2.5) in the standard distributional sense. The best available Hölder regularity of the temperature  $u$  is weaker than Lipschitz continuity due to the non-smoothness of the moving interface, so is the normal velocity of moving interface. In reality, there is also thermal noise in the heat transfer. Then the temperature  $u$  may at best be Hölder continuous, and consequently the phase function  $\varphi$  has to be understood in  $L^\infty$  sense see [13, 14, 15] since the level set equation (2.4) allows a discontinuity to develop in finite time even if the initial phase function is smooth (because of the non-uniqueness of ODE solutions given a Hölder continuous right hand side).

Denote by  $\Gamma(t) = \partial\Omega(t)$ , where  $\Omega(t) = \{\varphi(\cdot, t)\}$ . Our Main Theorem is stated as follows:

**Theorem 2.1.** *Assume that the almost everywhere continuous initial phase function  $\varphi_0$  satisfies:  $\varphi_0 < -1$  if  $\varphi_0 \leq 0$ , and  $\varphi_0 > 1$  if  $\varphi_0 > 0$ . Assume that the initial front  $\Gamma(0) = \cup_{i=1}^m \Gamma_i^0$ , where each  $\Gamma_i^0$  is a curve satisfying a local flattening condition: for any points  $A$  and  $B$  on  $\Gamma_i^0$  with  $L(\overline{AB}) < 1$ ,  $\sup_{x \in \overline{AB}} \text{dist}(x, \overline{AB}) \leq \xi L(\overline{AB})^{1+\beta}$ , where  $\overline{AB}$  is the connecting part of  $\Gamma_i^0$  between the point  $A$  and  $B$ . Moreover, let  $\|u_0\|_{C^{0,1}(\mathbb{R}^2)} \leq C_0$ . Then for any  $T > 0$ , the temperature  $u$  of (3.1)-(3.3) satisfies*

$$\|u\|_{L^\infty(\mathbb{R}^2 \times [0, T])} \leq C, \quad (2.9)$$

$$|u(x, t_1) - u(x, t_2)| \leq C(\gamma, C_0) |t_1 - t_2|^\gamma \quad \text{for any } \gamma \in (0, \frac{1}{2}), \quad (2.10)$$

$$|u(x_1, t) - u(x_2, t)| \leq C(\alpha, C_0) |x_1 - x_2|^\alpha \quad \text{for any } \alpha \in (0, 1), \quad (2.11)$$

for any  $x, x_1, x_2 \in \mathbb{R}^2$  and  $t_1, t_2 > 0$ . The phase function  $\varphi \in L^\infty(\mathbb{R}^2 \times (0, T]) \cap BV_{loc}(\mathbb{R}^2 \times (0, T])$  satisfies  $\partial\{\varphi < 0\} = \partial\{\varphi > 0\}$ . Furthermore  $\partial\{\varphi < 0\}$  is a Lipschitz graph  $t = \gamma(y, \varphi)$  in time-space, i.e.  $|\gamma(y_1, \varphi) - \gamma(y_2, \varphi)| \leq \frac{|y_1 - y_2|}{C_1}$ . For each time  $t$ ,  $\Gamma(t) = \partial\{\varphi(\cdot, t) < 0\}$  satisfies

$$\mathcal{H}^1(\Gamma(t) \cap B_r(y_0)) \leq C(m, \xi, \beta, C_0, C_1, C_2, r), \quad (2.12)$$

for any  $y_0 \in \mathbb{R}^2$ . As  $r \rightarrow 0$ ,  $\mathcal{H}^1(\Gamma(t) \cap B_r(y_0)) = O(r) \rightarrow 0$  for each  $t \in (0, T]$ .

This work was announced in [36]. The organization of the remainder of this paper is as follows. In Section 3, the existence of weak solutions of approximate equations is shown. In particular, we prove that the moving front of the approximate solution is a Lipschitz graph in time-space with Lipschitz constant independent of the approximate parameter  $\epsilon$ . Then we prove the intrinsic estimate of co-dimension one Hausdorff measure of the moving front for every time  $t$ , as long as the normal velocity is Hölder continuous, in Section 4. Though the idea behind the proof is simple, the argument is quite involved due to its geometric nature. The readers merely interested in the regularity estimate of temperature may skip the proof of Theorem 4.4 in the first round of reading. Based upon the estimate established in Section 4, we employ a decomposition argument to derive an a-priori  $C^\alpha$  continuity estimate for the temperature  $u$  independent of  $\epsilon$  by exploring the finiteness of the propagation speed of the moving front in Section 5. In Section 6, the existence of a weak solution of (2.4) and (2.5) is shown in the passage to the limit of approximate solutions. In the appendix, the theory of  $L^\infty$  solutions of Hamilton-Jacobi equations is recalled. In order to interpret the moving front driven by Hölder continuous normal velocity studied in Section 4, various of properties of  $L^\infty$  solutions of the level set equation are shown.

### 3 Initial Value Problem and its Approximation

The propagation of moving fronts of the non-isothermal polymer crystal growth problem is formulated as the initial value problem

$$\frac{\partial}{\partial t}(u + \chi(\varphi)) - \Delta u = 0 \quad \text{in } R^n \times [0, T], \quad (3.1)$$

$$\frac{\partial \varphi}{\partial t} + G(u)|\nabla \varphi| = 0 \quad \text{in } R^n \times [0, T], \quad (3.2)$$

$$u(x, 0) = u_0(x), \quad \varphi(x, 0) = \varphi_0(x), \quad (3.3)$$

where  $\chi$  is defined via (2.6).

Besides the assumptions (1.2) and (1.3), let us assume that

(A1)  $m(\partial\{\varphi_0 < 0\} \cap \partial\{\varphi_0 > 0\} \cap \{\varphi_0 = 0\}) = 0$ , and  $\{\varphi_0 < 0\}$  is a bounded subset in  $\mathbb{R}^n$ .

To study (3.1)-(3.3), we consider the following approximate equation

$$\frac{\partial}{\partial t}(u + \chi_\epsilon(x, t, \varphi)) - \Delta u = 0 \quad \text{in } R^n \times [0, T], \quad (3.4)$$

$$\frac{\partial \varphi}{\partial t} + G(u)|\nabla \varphi| = 0 \quad \text{in } R^n \times [0, T], \quad (3.5)$$

$$u(x, 0) = u_0(x), \quad \varphi(x, 0) = \varphi_0(x). \quad (3.6)$$

Here  $\chi_\epsilon$  is defined by

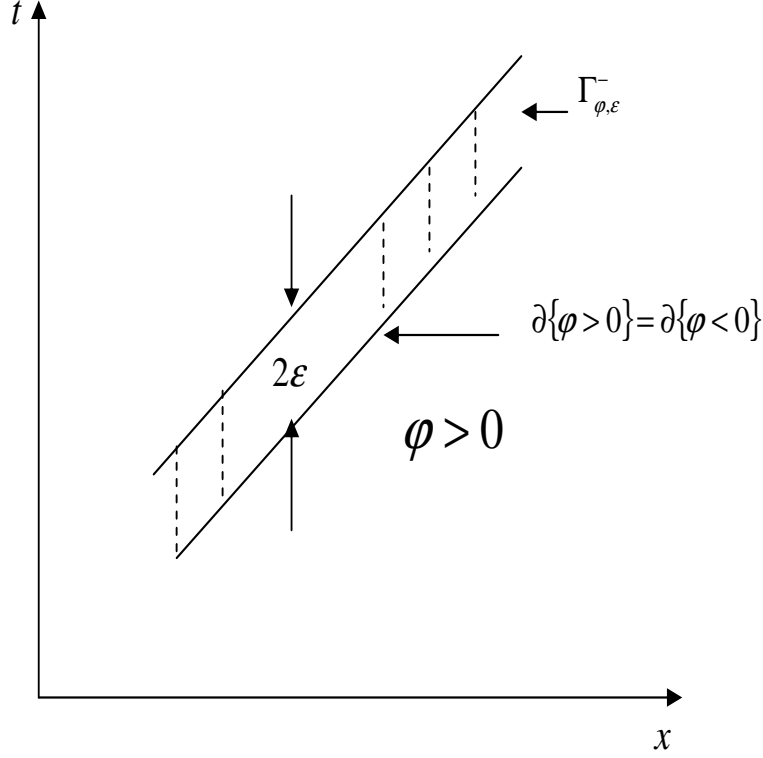
$$\chi_\epsilon(x, t, \varphi) = \begin{cases} -1, & \text{if } t - \gamma(x, \varphi) > 2\epsilon, \\ -1 + \frac{\gamma(x, \varphi) - t}{\epsilon}, & \text{if } 0 \leq t - \gamma(x, \varphi) \leq 2\epsilon, \\ 1, & \text{if } \varphi(x, t) > 0, \end{cases} \quad (3.7)$$

where  $\gamma(x, \varphi) = \inf\{s : \varphi(x, s) < 0\}$  for a monotone decreasing function  $\varphi$  in time  $t$ .

**Remark 3.1.** Observe that the solution of the level set equation (3.5) is monotone decreasing in time, thus  $\gamma$  is well-defined.

Let us make the following notations. Suppose that  $A \subset R^n$ , and  $b \subset R^n$ , then the distance between sets  $A$  and  $B$  is given by

$$\text{dist}(A, B) = \sup_{x \in A} \text{dist}(x, B) + \sup_{y \in B} \text{dist}(y, A), \quad (3.8)$$



Let  $T$  be any given positive number. Denote by  $\Gamma_{\phi}^{-}$  the boundary of  $\{\phi < 0\}$  in  $R^n \times (0, T]$ , by  $\Gamma_{\phi}^{+}$  the boundary of  $\{\phi > 0\}$  in  $R^n \times (0, T]$  for a function  $\phi$  monotone decreasing in time  $t$ . Let us denote the fattened boundary by

$$\Gamma_{\varphi, \epsilon}^{-} = \{(x, t) \in R^n \times (0, T] \mid -1 < \chi_{\epsilon}(x, t, \varphi) < 1\}, \quad (3.9)$$

$$\Gamma_{\varphi, \epsilon}^{-}(t) = \{x \in R^n \mid -1 < \chi_{\epsilon}(x, t, \varphi) < 1\}. \quad (3.10)$$

**Lemma 3.2.** *Suppose that the initial temperature satisfies  $u_0 \in C^2(R^n) \cap L^{\infty}(R^n)$ , and an initial almost everywhere continuous phase function  $\varphi_0 \in L^{\infty}(R^n)$  satisfies the assumption (A1) with the following property:  $\varphi_0 < -1$  if  $\varphi_0 \leq 0$ , and  $\varphi_0 > 1$  if  $\varphi_0 > 0$ . The definition of almost everywhere continuous function is in the Appendix A. The material function  $G$  satisfies assumptions (1.2) and (1.3). Then for every  $\epsilon > 0$ , there exists a solution  $u^{\epsilon} \in W_{x,t}^{2,1,p}(R^n \times (0, T]) \cap C^{1+\alpha, \frac{1+\alpha}{2}}(R^n \times [0, T])$  for any  $p \in (1, \infty)$  and  $\varphi^{\epsilon} \in L^{\infty}(R^n \times (0, T]) \cap BV(R^n \times (0, T])$  to (3.4)-(3.6) with  $\partial\{\varphi^{\epsilon} < 0\} = \partial\{\varphi^{\epsilon} > 0\}$ . Furthermore  $\partial\{\varphi^{\epsilon} < 0\}$  is a Lipschitz graph in time-space independent of  $\epsilon$ , i.e.  $|\gamma(y_1, \varphi^{\epsilon}) - \gamma(y_2, \varphi^{\epsilon})| \leq \frac{|y_1 - y_2|}{C_1}$ .*

Proof: Observe that the following equations

$$\frac{\partial}{\partial t}(u + \chi_{\epsilon}(x, t, \phi)) - \Delta u = 0 \quad \text{in } R^n \times [0, T], \quad (3.11)$$

$$\frac{\partial \varphi}{\partial t} + G(v)|\nabla \varphi| = 0 \quad \text{in } R^n \times [0, T], \quad (3.12)$$

$$u(x, 0) = u_0(x), \quad \varphi(x, 0) = \varphi_0(x). \quad (3.13)$$

induce two mappings  $T_1 : \phi \rightarrow u$ ,  $T_2 : v \rightarrow \varphi$  where  $v \in L^{\infty}(0, T; C^{0,1}(R^n))$  with  $v(x, 0) = u_0(x)$ ,  $\phi \in L^{\infty}(R^n \times [0, T])$  is a monotone decreasing function for almost every  $x$  with  $\Gamma_{\phi}^{-}$  being a Lipschitz graph in time-space with  $\gamma(y, \phi) \geq Cr$  if  $|y| \geq r$  as long as  $r \geq R_0$  for some absolute constants  $C$  and  $R_0$ , more precisely

$$\Gamma_{\phi}^{-} = \{(y, \gamma(y, \phi)) : |\gamma(y_1, \phi) - \gamma(y_2, \phi)| \leq (\frac{1}{C_1} + 1)|y_1 - y_2|\}.$$

To invoke the Leray-Schauder fixed point theorem for showing that  $T = T_1 \circ T_2$  has a fixed point, we need to show that  $T$  is continuous and compact.

Step 1.  $T$  is continuous.

First of all we show need to show  $\Gamma_{\varphi_k}^- \rightarrow \Gamma_{\varphi}^-$  if  $v_k \rightarrow v$ , i.e.

$$\text{dist}(\Gamma_{\varphi_k}^-, \Gamma_{\varphi}^-) \rightarrow 0 \quad \text{if } \|v_k - v\|_{L^1(0,T;L^\infty(\mathbb{R}^n))} \rightarrow 0, \quad (3.14)$$

where  $\varphi_k = T_2(v_k)$ . Observe that  $\text{dist}(\Gamma_{\varphi_k}^-, \Gamma_{\varphi}^-) \leq \text{dist}(\{\varphi_k < 0\}, \{\varphi < 0\})$ , it suffices to show

$$\text{dist}(\{T_2(v_k) < 0\}, \{T_2(v) < 0\}) \rightarrow 0, \quad \text{if } \|v_k - v\|_{L^1(0,T;L^\infty(\mathbb{R}^n))} \rightarrow 0. \quad (3.15)$$

Given by Lax formula, it follows from Theorem A.6 that the solution  $\varphi$  of (3.12) is

$$\varphi(x, t) = \text{essinf}_{y \in S_{x,t}(v,0)} \{\varphi_0(y)\}, \quad (3.16)$$

with the backward reachable set

$$S_{x,t}(v, 0) = \{y \in \mathbb{R}^n | x(0) = y, x(t) = x, |\dot{x}(s)| \leq G(v(x, s)) \text{ for } 0 < s < t\}.$$

For any point  $(x, t) \in \{\varphi < 0\}$ , there exists a Lipschitz continuous path with  $|\dot{x}(s)| \leq G(v(x, s))$  such that  $x(t) = x, x(0) = y \in \{\varphi_0 < 0\}$ . Let us construct another Lipschitz path

$$\dot{x}_k(s) = \frac{G(v_k(x, s))}{G(v(x, s))} \dot{x}(s),$$

with  $x_k(0) = y$ , and  $x_k(t) = x_k \in \{\varphi_k(t) < 0\}$ . Then using (1.3), we have

$$|x_k(t) - x(t)| \leq \frac{C_2}{C_1} L \int_0^t \|v_k - v\|_{L^\infty(\mathbb{R}^n)} ds.$$

Note that  $(x_k, t) \in \{\varphi_k < 0\}$ , thus

$$\sup_{(x,t) \in \{\varphi < 0\}} \text{dist}((x, t), \{\varphi_k < 0\}) \leq \frac{C_2}{C_1} L \int_0^t \|v_k - v\|_{L^\infty(\mathbb{R}^n)} ds.$$

Switching the positions of  $\varphi_k$  and  $\varphi$ , we argue in the same way to get

$$\sup_{(x,t) \in \{\varphi_k < 0\}} \text{dist}((x, t), \{\varphi < 0\}) \leq \frac{C_2}{C_1} L \int_0^t \|v_k - v\|_{L^\infty(\mathbb{R}^n)} ds.$$

Thus

$$\text{dist}(\{\varphi_k < 0\}, \{\varphi < 0\}) \leq \frac{2C_2 L}{C_1} \int_0^t \|v_k - v\|_{L^\infty(\mathbb{R}^n)} ds,$$

and (3.14) is true when  $\|v_k - v\|_{L^1(0,T;L^\infty(\mathbb{R}^n))} \rightarrow 0$ .

Here we show that  $\Gamma_{\varphi}^-$  is a Lipschitz continuous graph in time-space where  $\varphi$  is the solution of (3.12), thus  $\Gamma_{\varphi}^-$  is of locally finite  $n$ -dimensional Hausdorff measure.

Note that by Lax formula, we have

$$\varphi(x, t) = \text{essinf}_{y \in S_{x,t}(v,0)} \{\varphi_0(y)\} = \text{essinf}_{y \in S_{x,t}(v,\tau)} \{\varphi(y, \tau)\}, \quad (3.17)$$

where

$$S_{x,t}(v, 0) = \{y \in \mathbb{R}^n | x(0) = y, x(t) = x, |\dot{x}(s)| \leq G(v(x, s)) \text{ for } 0 < s < t\}$$

$$S_{x,t}(v, \tau) = \{y \in \mathbb{R}^n | x(\tau) = y, x(t) = x, |\dot{x}(s)| \leq G(v(x, s)) \text{ for } \tau < s < t\}$$

Given a point  $x \in \mathbb{R}^n$ , for any  $\delta > 0$ , there exists a point  $z \in \{\varphi_0 < 0\}$  such that

$$|x - z| \leq \text{dist}(x, \{\varphi_0 < 0\}) + \delta.$$

Thus by (3.17)

$$\left(x, \frac{\text{dist}(x, \{\varphi_0 < 0\}) + \delta}{C_1}\right) \in \{\varphi < 0\},$$

in that  $z \in S_{x, \tau(x)}(v, 0)$  where  $\tau(x) = \frac{\text{dist}(x, \{\varphi_0 < 0\}) + \delta}{C_1}$ . Therefore  $\gamma(x, \varphi) \leq \tau(x)$  is finite for any  $x \in \mathbb{R}^n$ .

Now we show a Lipschitz continuity estimate of  $\gamma(x, \varphi)$ . Given any two points  $(y_1, \gamma(y_1, \varphi)) \in \Gamma_\varphi^-$  and  $(y_2, \gamma(y_2, \varphi)) \in \Gamma_\varphi^-$ , assume without loss of generality  $\gamma(y_2, \varphi) \geq \gamma(y_1, \varphi)$ , for any  $\delta > 0$ , there exists a point  $z \in \{x \in \mathbb{R}^n | \varphi(x, \gamma(y_1, \varphi)) < 0\}$  such that

$$|y_2 - z| \leq (1 + \delta)|y_1 - y_2|.$$

By (3.17), we have

$$\left(y_2, \gamma(y_1, \varphi) + \frac{(1 + \delta)|y_1 - y_2|}{C_1}\right) \in \{\varphi < 0\},$$

which in turn implies

$$\gamma(y_2, \varphi) - \gamma(y_1, \varphi) \leq \frac{(1 + \delta)|y_1 - y_2|}{C_1}$$

Therefore we obtain

$$|\gamma(y_2, \varphi) - \gamma(y_1, \varphi)| \leq \frac{(1 + \delta)|y_1 - y_2|}{C_1},$$

and let  $\delta \rightarrow 0$ ,

$$|\gamma(y_2, \varphi) - \gamma(y_1, \varphi)| \leq \frac{|y_1 - y_2|}{C_1}. \quad (3.18)$$

Secondly we show for any  $p \in (1, \infty)$

$$\|T_1(\phi_k) - T_1(\phi)\|_{W_{x,t}^{2,1,p}(\mathbb{R}^n \times [0, T])} \rightarrow 0, \quad \text{if } \text{dist}(\Gamma_{\phi_k}^-, \Gamma_\phi^-) \rightarrow 0. \quad (3.19)$$

Note that  $\text{dist}(\Gamma_{\phi_k, \epsilon}^-, \Gamma_{\phi, \epsilon}^-) \leq \text{dist}(\Gamma_{\phi_k}^-, \Gamma_\phi^-)$ , and by the definition of  $\Gamma_\phi^-$ , we have

$$\begin{aligned} & \left\| \frac{\partial}{\partial t} \chi_\epsilon(\phi_k) - \frac{\partial}{\partial t} \chi_\epsilon(\phi) \right\|_{L^p(\mathbb{R}^n \times [0, T])} \\ & \leq \frac{C(n, p)}{\epsilon} (\mathcal{H}(\Gamma_\phi \cap \mathbb{R}^n \times [0, T]) + \mathcal{H}(\Gamma_{\phi_k} \cap \mathbb{R}^n \times [0, T])) \text{dist}(\Gamma_\phi, \Gamma_{\phi_k})^{\frac{1}{p}} \\ & \leq \frac{C(n, p, T, C_1)}{\epsilon} \text{dist}(\Gamma_\phi, \Gamma_{\phi_k})^{\frac{1}{p}} \end{aligned} \quad (3.20)$$

By standard  $L^p$  theory we conclude (3.19).

Given (3.14) and (3.19), we have that  $T = T_1 \circ T_2$  is continuous.

Step 2.  $T$  is compact.

It follows from Theorem B.1 with the local  $BV$  estimate (B.1) that

$$\|\varphi\|_{BV(B(0, R) \times [0, T])} \leq C(R, T, C_1, C_2, \|\phi\|_{L^\infty(\mathbb{R}^d)}) \quad (3.21)$$

By (3.18),  $\partial\{\varphi < 0\}$  is a Lipschitz graph with Lipschitz constant depending only on  $C_1$ . By the definition of  $\chi_\epsilon(x, t, \varphi)$ , (3.4) is a heat equation with  $L^\infty$  bounded right hand side. By standard  $L^p$  theory and De Giorgi estimate

$$\|u\|_{W_{x,t}^{2,1,p}(\mathbb{R}^n \times [0, T])} + \|u\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(\mathbb{R}^n \times [0, T])} \leq C(n, \epsilon) \quad (3.22)$$

Thus  $T = T_1 \circ T_2$  is compact.

By Leray-Schauder's fixed point theorem, there exists a solution  $(u^\epsilon, \varphi^\epsilon)$  to (3.4)-(3.6) with  $u^\epsilon \in W_{x,t}^{2,1,p}(\mathbb{R}^n \times (0, T]) \cap C^{\frac{1+\alpha}{2}, 1+\alpha}(\mathbb{R}^n \times [0, T])$  for any  $p \in (1, \infty)$ ,  $\alpha \in (0, 1)$ , and  $\varphi^\epsilon \in L^\infty(\mathbb{R}^n \times (0, T]) \cap BV_{loc}(\mathbb{R}^n \times (0, T])$ . This concludes Lemma 3.2.

## 4 Geometry of the Moving Front

In this section, we prove a surprising theorem in two spatial dimensions that the one-dimensional Hausdorff measure of the moving front is finite if the normal velocity  $f(x, t)$  is Hölder continuous with respect to  $x$  and is bounded from below. In other words, there is a discontinuous  $L^\infty$  solution  $\varphi$  of the following level set equation, whose level set  $\partial\{\varphi < 0\} = \partial\{\varphi > 0\}$  is of finite co-dimension one Hausdorff measure for every time  $t > 0$ .

$$\varphi_t + f(x, t)|D\varphi| = 0, \quad x \in \mathbb{R}^2, \quad t > 0, \quad (4.1)$$

$$\varphi(x, 0) = \phi(x). \quad (4.2)$$

Even when  $f(x, t)$  is Lipschitz continuous in  $x$  for every  $t$ , no explicit estimate of co-dimension one Hausdorff measure of level set was available until now.

In the case of non-isothermal growth, the normal velocity  $G(u)$  in (2.4) is as best regular as  $Lip(\log(Lip))^\beta$  ( $\beta \geq 1$ ) with respect to  $x$  as shown very likely, but not rigorously, in next section. The definition of  $Lip(\log(Lip))^\beta$  ( $\beta \geq 1$ ) is as follows. A function  $g(x)$  is said to  $Lip(\log(Lip))^\beta$  continuous with respect to  $x$  in  $\Omega$  for  $\beta \geq 1$  if there is a constant  $C$  such that for any two points  $x_1$  and  $x_2$  in  $\Omega$ ,

$$|g(x_1) - g(x_2)| \leq C |x_1 - x_2| (|\ln|x_1 - x_2|| + 1)^\beta. \quad (4.3)$$

where  $C$  is the semi-norm  $[g]_{Lip(\log(Lip))^\beta(\Omega)}$  of  $g$  in  $Lip(\log(Lip))^\beta(\Omega)$  space. The norm of  $g$  is defined as:  $\|g\|_{Lip(\log(Lip))^\beta(\Omega)} = [g]_{Lip(\log(Lip))^\beta(\Omega)} + \|g\|_{L^\infty(\Omega)}$ . New ingenious geometric concepts and techniques are introduced to study geometric properties of the moving front. It is well known that ODEs with Hölder continuous right hand side lose uniqueness. Our method derives an intrinsic estimate of co-dimension one Hausdorff measure of the moving front even when the normal velocity  $f(x, t)$  is merely Hölder continuous.

Because of the non-uniqueness of ODE solutions given Hölder continuous or  $Lip(\log(Lip))^\beta$  ( $\beta > 1$ ) continuous right hand side, the level set equation (4.1) allows discontinuity develop in finite time even if the initial phase function is smooth. As noticed before  $\varphi$  has to satisfy (4.1) in the  $L^\infty$  sense (see [13, 14, 15]). For the convenience of reader, we recall the theory of  $L^\infty$  solution of Hamilton-Jacobi equation in the appendix.

First of all we show the following Lemma on the comparison of the lengths between two enclosed convex curves. Recall that an enclosed curve  $\mathbf{C}$  is called convex if it is the boundary of a convex body  $\text{cvh}(\mathbf{C})$ , the convex hull of  $\mathbf{C}$ .

**Lemma 4.1.** *Let  $\mathbf{C}$  and  $\mathbf{D}$  be two enclosed convex curves.  $D \subset \text{cvh}(\mathbf{C})$ . Then*

$$L(\mathbf{D}) \leq L(\mathbf{C}), \quad (4.4)$$

where  $L(\mathbf{C})$  denotes the length of the curve  $\mathbf{C}$ .

*Proof:* Step 1. Any convex curve can be approximated by convex polygons in the sense that for a given convex curve  $\mathbf{C}$ , every  $\epsilon > 0$ , there is a convex polygon  $\mathbf{Y}$  such that  $\text{dist}(\mathbf{Y}, \mathbf{C}) \equiv \sup_{y \in \mathbf{Y}} \text{dist}(y, \mathbf{C}) + \sup_{x \in \mathbf{C}} \text{dist}(x, \mathbf{Y}) \leq \epsilon$  and  $|L(\mathbf{Y}) - L(\mathbf{C})| \leq \epsilon$ .

Step 2. It suffices to show the statement is true for any convex polygon  $\mathbf{D} = [A_1 \cdots A_l A_1] \subset \text{cvh}(\mathbf{C})$ , where  $A_1, \cdots, A_l$  are the vertices in the counter clockwise order.

Up to translation and rotation, without loss of generality, we assume there are at least two points of the convex polygon on the curve  $\mathbf{C}$ . Denote by  $V_1 = \{A_{i_1}, A_{i_2}, \cdots, A_{i_k}\}$  the set of vertices on the curve  $\mathbf{C}$  with  $i_1 < \cdots < i_k$ ,  $V_2 = \{A_1, \cdots, A_l\} \setminus V_1$  the set of vertices in the interior of  $\text{cvh}(\mathbf{C})$ . Fix the convex sub-polygon  $[A_{i_1} A_{i_2} \cdots A_{i_k}]$  and let us conduct so called "push up" operations.

For the sake of simplicity, considering  $A_{i_k} A_{i_1}$  as  $A_{i_k} A_{i_{k+1}}$ , we select any side  $A_{i_j} A_{i_{j+1}}$  of  $[A_{i_1} A_{i_2} \cdots A_{i_k} A_{i_1}]$  for  $j = 1, \cdots, k$  as base line of the convex sub-polygon  $[A_{i_j} A_{i_{j+1}} \cdots A_{i_{j+1}} A_{i_j}]$ . Then we move the vertices  $A_{i_{j+1}}, \cdots, A_{i_{j+1}-1}$  away from the straight line segment  $A_{i_j} A_{i_{j+1}}$  with vertices  $A_{i_j}$  and  $A_{i_{j+1}}$  toward the curve  $\mathbf{C}$  in the direction orthogonal to the line  $A_{i_j} A_{i_{j+1}}$  until some vertice attains the curve  $\mathbf{C}$ . We obtain a new convex polygon  $[A_{i_j} A'_{i_{j+1}} \cdots A'_{i_{j+1}-1} A_{i_{j+1}} A_{i_j}]$  with

$$L([A_{i_j} A'_{i_{j+1}} \cdots A'_{i_{j+1}-1} A_{i_{j+1}} A_{i_j}]) \geq L([A_{i_j} A_{i_{j+1}} \cdots A_{i_{j+1}-1} A_{i_{j+1}} A_{i_j}]).$$

Denote by  $V'_1$  the set of vertices of the polygon  $[A_{i_j}A'_{i_j+1} \cdots A'_{i_{j+1}-1}A_{i_{j+1}}A_{i_j}]$  on the curve  $\mathbf{C}$ . Then  $\#(V'_1) > \#(V_1)$ , where  $\#(V)$  denotes the cardinality of the set  $V$ . Repeat the "push up" operation, in finite steps, we obtain a "pushed up" polygon  $[A'_1 \cdots A'_l A'_1]$  with all vertices  $A'_i$  ( $i = 1, \dots, l$ ) on the curve  $\mathbf{C}$  with

$$L([A_1 A_2 \cdots A_l A_1]) \leq L([A'_1 A'_2 \cdots A'_l A'_1]) \leq L(\mathbf{C}).$$

Let  $\Omega(t) \equiv \{\varphi(\cdot, t) < 0\}$  be the reachable set driven by  $f(x, t)$  at time  $t$ . The moving front  $\Gamma(t)$  is the boundary of the reachable set  $\Omega(t)$ . We show that for every point  $x \in \Gamma(t) \equiv \partial\{\varphi(\cdot, t) < 0\}$  driven by Lipschitz continuous normal velocity  $f(x, t)$ , any characteristic path must be backward boundary characteristic path.

**Lemma 4.2.** *Suppose  $f(x, t)$  is Lipschitz continuous with respect to  $x$ . For any point  $A$  on the moving front  $\Gamma(t_2) = \partial\Omega(t_2)$  at time  $t_2 > 0$ , any characteristic path  $x : [t_1, t_2] \rightarrow \mathbb{R}^2$  with  $|\dot{x}| \leq f(x, t)$  for  $t \in [t_1, t_2]$ ,  $x(t_2) = A$ , and  $x(t_1) = B$  must be a boundary characteristic path, that is,  $|\dot{x}| = f(x, t)$  for almost all  $t \in [t_1, t_2]$  and  $x(t) \in \Gamma(t)$  for all  $t \in [t_1, t_2]$ .*

Proof: Let  $\varphi$  be the unique solution of (4.1), (4.2) for the normal velocity  $f(x, t)$ . Let  $\Omega(t) = \{\varphi(\cdot, t) < 0\}$ . Then by the Lax formula,  $\Omega(t) = \{y \in \mathbb{R}^2 \mid |\dot{x}| \leq f(x, t), x(0) \in \Omega(0), x(t) = y\}$ .

For any point  $A$  on the moving front  $\Gamma(t) = \partial\Omega$ , any path  $x : [t_1, t_2] \rightarrow \mathbb{R}^2$  with  $|\dot{x}| \leq f(x, t)$  for  $t \in [t_1, t_2]$ ,  $x(t_2) = x_A$ , and  $x(t_1) = x_B$  (where  $x_A$  and  $x_B$  are coordinates of  $A$  and  $B$ ). let us consider the flow generated by the ODE  $\dot{y} = \frac{\dot{x}}{f(x, t)} f(y, t)$ . It is straightforward to see that  $|\dot{y}| \leq f(y, t)$ . For any two paths  $y_1$  and  $y_2$  satisfying the above ODE, we have

$$\dot{y}_1 - \dot{y}_2 = \frac{\dot{x}}{f(x, t)} (f(y_1, t) - f(y_2, t)).$$

By the stability of the solution of ODE with Lipschitz continuous right hand side, the flow generated by the ODE  $\dot{y} = \frac{\dot{x}}{f(x, t)} f(y, t)$  is a bi-Lipschitz isotopy, which maps interior point to interior point, boundary point to boundary point. The so-called bi-Lipschitz isotopy is an isotopy of a bi-Lipschitz homeomorphism. Therefore  $x(t) \in \Gamma(t)$  for all  $t \in [t_1, t_2]$ , and  $|\dot{x}| = f(x, t)$  for almost all  $t \in [t_1, t_2]$ . By Lemma 3.2, two boundary characteristics intercept as the same time.

Then we need a lemma for the approximation of the moving front  $\Gamma(t) \equiv \partial\{\varphi(\cdot, t) < 0\}$  driven by Hölder continuous normal velocity  $f(x, t)$  by a sequence of the moving front  $\Gamma_\epsilon(t) \equiv \partial\{\varphi^\epsilon(\cdot, t) < 0\}$  driven by Lipschitz continuous normal velocity  $f^\epsilon(x, t)$ . Based upon Lemma 4.2, we can derive a preliminary lemma on the geometric properties of the moving front driven by Hölder continuous normal velocity. Here  $\Omega(t)$  (or  $\Gamma(t)$ ) is obtained as the approximation limit of a sequence of reachable sets (or moving fronts) driven by Lipschitz continuous normal velocity  $f^\epsilon(x, t)$  by Lemma 4.2.

**Lemma 4.3.** *Suppose that  $\partial\{\phi(x) < -1\} = \partial\{\phi(x) < 0\} = \partial\{\phi(x) > 0\} = \partial\{\phi(x) > 1\}$  is of finite co-dimension one Hausdorff measure, and  $0 < C_1 \leq f^\epsilon(x, t) \leq C_2$ ,  $\|f^\epsilon(x, t) - f(x, t)\|_{L^\infty(0, T; C^\alpha(\mathbb{R}^n))} \rightarrow 0$  as  $\epsilon \rightarrow 0$ ,  $|f^\epsilon(x_1, t) - f^\epsilon(x_2, t)| \leq \kappa |x_1 - x_2|^\alpha$ . For each  $\epsilon > 0$ ,  $f^\epsilon$  is Lipschitz continuous with respect to  $x \in \mathbb{R}^n$ . Then the unique  $L^\infty$  solution  $\varphi^\epsilon$  of*

$$\begin{aligned} \varphi_t^\epsilon + f^\epsilon(x, t)|D\varphi^\epsilon| &= 0, & x \in \mathbb{R}^n, t > 0, \\ \varphi(x, 0) &= \phi(x), \end{aligned}$$

converges to a  $L^\infty$  solution  $\varphi$  of

$$\begin{aligned} \varphi_t + f(x, t)|D\varphi| &= 0, & x \in \mathbb{R}^n, t > 0, \\ \varphi(x, 0) &= \phi(x); \end{aligned}$$

in the almost everywhere sense as  $\epsilon \rightarrow 0$ , and the moving front  $\partial\{\varphi^\epsilon < 0\}$  of  $\varphi^\epsilon$ , which is Lipschitz graph in time-space with Lipschitz constants independent of  $\epsilon$ , converge to the moving front  $\partial\{\varphi < 0\} = \partial\{\varphi > 0\}$  in the sense of (3.8). Furthermore, every boundary point has boundary characteristic path driven by  $f$  enjoys

$$|L(\mathbf{C}) - L(\overline{AB})| \leq 2^\alpha \kappa (C_2)^\alpha |t_1 - t_2|^{1+\alpha}, \quad (4.5)$$

where  $\overline{AB}$  denotes the straight line segment between the point  $A$  and the point  $B$ , and  $\mathbf{C}$  denotes the boundary characteristic path connecting  $A$  and  $B$ .

Proof: It follows from Theorem A.6 and Theorem B.1 in the appendix, and the proof of lemma 3.2 with verification of  $\varphi = \lim_{\epsilon \rightarrow 0} \varphi^\epsilon$  the  $L^\infty$  solution which is the same as the proof in section 6. Therefore at every point  $x \in \Gamma(t)$ , there is a backward boundary characteristic path.

Denote by  $\mathcal{C}$  the boundary characteristic path  $x : [t_1, t_2] \rightarrow \mathbb{R}^2$  satisfies  $|\dot{x}(t)| = f(x, t)$  with  $x(t) \in \Gamma(t)$  for  $t_1 \leq t \leq t_2$ . Let us consider another path  $y : [t_1, t_2] \rightarrow \mathbb{R}^2$  along the straight line segment  $AB$  defined as  $y(t) = y(t_1) + \int_{t_1}^t f(y(\tau), \tau) \frac{x(t_2) - x(t_1)}{|x(t_2) - x(t_1)|} d\tau$ , where  $y(t_1)$  is the coordinates of the point  $B$ . Denote by  $\mathcal{D}$  the path  $y$ . Since  $L(\mathcal{C}) + L(\mathcal{D}) \leq 2C_2(t_2 - t_1)$ ,  $|x(\tau) - y(\tau)| \leq 2C_2(t_2 - t_1)$  for  $\tau \in [t_1, t_2]$ . Thus

$$|f(y(\tau), \tau) - f(x(\tau), \tau)| \leq 2^\alpha C_2^\alpha \kappa (t_2 - t_1)^\alpha \quad \text{for } t_1 \leq \tau \leq t_2.$$

Hence

$$\begin{aligned} |L(\mathcal{C}) - L(\mathcal{D})| &= \left| \int_{t_1}^{t_2} f(y(\tau), \tau) d\tau - \int_{t_1}^{t_2} f(x(\tau), \tau) d\tau \right| \\ &\leq 2^\alpha C_2^\alpha \kappa (t_2 - t_1)^{1+\alpha} \quad \text{for } t_1 \leq \tau \leq t_2. \end{aligned}$$

Note that  $L(\mathcal{D}) \leq L(\overline{AB}) \leq L(\mathcal{C})$ . Hence (4.5) follows.

With the aid of Lemma 4.1 and Lemma 4.3, we are ready to show that the moving front driven by Hölder continuous normal velocity has finite 1-dimensional Hausdorff measure. The argument is quite lengthy due to geometric complexity. Roughly speaking, in the first step, we show that the "convex" part of moving front is bounded by the lower boundary of convex hull of the curve; we show the complement of the "convex" part, which consists of countably many "vaguely concave" parts, is locally nicely behaved; that is, each "vaguely concave" part is close to straight line in the sense that the "concavity" is small. Then we decompose each "vaguely concave" part into two or three sub-parts; the union of lower boundary of convex hull of sub-parts serves as 1st-approximation of moving front. We iteratively repeat the argument to treat each "vaguely concave" sub-part to produce  $n$ th-approximation of moving front. As the iteration number of the above procedure goes to infinity, the moving front becomes flat locally, thus completely restored as part of 1-dimensional curve, whose length is finite.

**Theorem 4.4.** *The moving front  $\Gamma(t)$ , which is the boundary of reachable set  $\Omega(t)$  driven by normal velocity  $f(x, t)$ , which is Hölder continuous with  $|f(x_1, t) - f(x_2, t)| \leq \kappa |x_1 - x_2|^\alpha$  and  $0 < C_1 \leq f(x, t) \leq C_2$ . Assume that the initial front  $\Gamma(0) = \cup_{i=1}^m \Gamma_i^0$ , where  $\Gamma_i^0$  is a curve satisfying local flattening condition: for any points  $A$  and  $B$  on  $\Gamma_i^0$  with  $L(\overline{AB}) < 1$ ,  $\sup_{x \in \overline{AB}} \text{dist}(x, \overline{AB}) \leq \xi L(\overline{AB})^{1+\beta}$ , where  $\overline{AB}$  is the connecting part of  $\Gamma_i^0$  between the point  $A$  and  $B$ . Then for any  $t > 0$ ,  $\Gamma(t)$  satisfies*

$$\mathcal{H}^1(\Gamma(t) \cap B_r(y_0)) \leq C(m, \xi, \beta, \alpha, C_1, C_2)r, \quad \text{if } r \in (0, \kappa^{-\frac{2+\alpha^2}{\alpha}}], \quad (4.6)$$

$$\mathcal{H}^1(\Gamma(t) \cap B_r(y_0)) \leq C(m, \xi, \beta, \alpha, C_1, C_2)r^2 \kappa^{\frac{2-\alpha^2}{\alpha}}, \quad \text{if } r \geq \kappa^{-\frac{2+\alpha^2}{\alpha}} \quad (4.7)$$

for any  $y_0 \in \mathbb{R}^2$ .

Proof: It suffices to establish local version (4.6) for the moving front when  $t > M \equiv \min\{(\frac{1}{16})^{\frac{1}{\alpha}} \frac{C_1}{C_2} \frac{C_1^{\frac{1}{\alpha}-1}}{\kappa^{\frac{1}{\alpha}}}, (\frac{3C_1}{32})^{\frac{2}{\alpha}} \kappa^{-1}, (\frac{C_2}{4})^{\frac{2}{\alpha}} \kappa^{-1}\}$ . Then in the case of  $t \in (0, M)$ , at large scale,  $\Gamma(t)$  is considered as small perturbation of  $\Gamma(0)$ ; in small scale, estimate of 1-dimensional Hausdorff measure of moving front is the same as the case  $t > M$ .

Case 1.  $t > M$ .

Given any point  $y \in \mathbb{R}^2$ , for any  $r \leq \min\{\kappa^{\frac{\alpha}{2}} M^{1+\frac{\alpha}{2}}, (\frac{1}{16\pi\eta})^{\frac{1}{\gamma}}\}$ , the boundary of the convex hull of  $\Gamma(t) \cap B_r(y)$  is denoted by  $\text{bch}(\Gamma(t) \cap B_r(y))$ . Here  $\eta$  and  $\gamma$  are to be defined in (4.12). By Lemma 4.1, we have

$$L(\text{bch}(\Gamma(t) \cap B_r(y))) \leq 2\pi r. \quad (4.8)$$

Note that  $\Omega(t) \cap \text{bch}(\Gamma(t) \cap B_r(y))$  is an open set relative to the convex curve  $\text{bch}(\Gamma(t) \cap B_r(y))$ , which consists of countably many disjoint open straight line segments of  $\text{bch}(\Gamma(t) \cap B_r(y))$ . The end points  $A_j, B_j$  of the  $j$ -th

open straight line segment  $\text{int}(\overline{A_j B_j})$  are on the moving front  $\Gamma(t) = \partial\Omega(t)$ . That is,  $\Omega(t) \cap \text{bch}(\Gamma(t) \cap B_r(y)) = \cup_{j=1}^{\infty} \text{int}(\overline{A_j B_j})$ . We can assign the unit normal vector  $n_{\overline{A_j B_j}}$  to each line segment  $\text{int}(\overline{A_j B_j})$  by choosing the unit orthogonal vector to  $\overline{A_j B_j}$  pointing to the interior  $\text{int}(\text{cvh}(\Gamma(t) \cap B_r(y)))$ .  $\text{bch}(\Gamma(t) \cap B_r(y)) \setminus \Omega(t)$  is closed subset of  $\text{bch}(\Gamma(t) \cap B_r(y))$ . It follows from (4.8) that

$$\sum_j^{\infty} L(\overline{A_j B_j}) + \mathcal{H}^1(\text{bch}(\Gamma(t) \cap B_r(y)) \setminus \Omega(t)) \leq 2\pi r. \quad (4.9)$$

Denote by  $\widetilde{A_j B_j}$  the set of points on the boundary  $\Gamma(t) \cap B_r(y)$ , which has one backward boundary characteristic path intersected with  $\overline{A_j B_j}$  from non-negative side. That is,  $\widetilde{A_j B_j} = \{x \in \Gamma(t) \mid \text{there is a backward boundary characteristic path } x : [\tau, t] \rightarrow \mathbb{R}^2 \text{ with } x(\tau) \in \overline{A_j B_j}, \text{ there is no subinterval } [s_1, s_2] \subset [\tau, t] \text{ such that } x(s_1) \in \text{int}(\overline{A_j B_j}) \text{ and } (x(s) - x(s_1)) \cdot n_{\overline{A_j B_j}} < 0 \text{ for } s \in (s_1, s_2)\}$ . By Jordan curve property, we have

$$\Gamma(t) \cap B_r(y) \subset (\cup_j^{\infty} \widetilde{A_j B_j}) \cup (\text{bch}(\Gamma(t) \cap B_r(y)) \setminus \Omega(t)). \quad (4.10)$$

**Claim.** For each  $j$ ,  $\widetilde{A_j B_j}$  is of finite co-dimension one Hausdorff measure with

$$\mathcal{H}^1(\widetilde{A_j B_j}) \leq e^{\frac{1}{2^{1-(\frac{3}{4})^{\frac{1}{\gamma}})}}} L(\overline{A_j B_j}). \quad (4.11)$$

Once the claim is shown, it follows from (4.9)-(4.11) that Theorem 4.4 is true for any  $r \leq \min\{\kappa^{\frac{\alpha}{2}} M^{1+\frac{\alpha}{2}}, (\frac{1}{16\pi\eta})^{\frac{1}{\gamma}}\}$  in case 1. The statement for the situation  $r \geq \min\{\kappa^{\frac{\alpha}{2}} M^{1+\frac{\alpha}{2}}, (\frac{1}{16\pi\eta})^{\frac{1}{\gamma}}\}$  is a straight-forward consequence of the situation  $r \leq \min\{\kappa^{\frac{\alpha}{2}} M^{1+\frac{\alpha}{2}}, (\frac{1}{16\pi\eta})^{\frac{1}{\gamma}}\}$  by covering argument.

Proof of Claim: For the simplicity of notation, let us denote  $A_j, B_j$  by  $A$  and  $B$ , respectively. We iteratively decompose partially concave part of  $\overline{AB}$  and use the so-called "lower boundary of convex hull" to approximate  $\overline{AB}$ .

Step 1: For any point  $C \in \overline{AB}$ , there is a backward boundary characteristic path  $x : [0, t] \rightarrow \mathbb{R}^2$  issued from  $C$  intersected with  $\overline{AB}$  from non-negative side at point  $x(\tau_1)$  at time  $\tau_1$ , and

$$\begin{aligned} \text{dist}(C, \overline{AB}) &\leq \frac{\frac{4}{3} \kappa^{\alpha} C_1 + 8\kappa C_2^{\alpha}}{2^{\frac{1+\alpha}{2}} \kappa^{\frac{\alpha(1+\alpha)}{2+\alpha}}} (L(\overline{AB}))^{\frac{1+\alpha}{2}} \equiv \eta L^{1+\gamma}(\overline{AB}) \\ L(x[\tau_1, t]) &\leq \eta L^{1+\gamma}(\overline{AB}), \end{aligned} \quad (4.12)$$

where  $\eta = \frac{\frac{4}{3} \kappa^{\alpha} C_1 + 8\kappa C_2^{\alpha}}{2^{\frac{1+\alpha}{2}} \kappa^{\frac{\alpha(1+\alpha)}{2+\alpha}}}$  and  $\gamma = \frac{\alpha}{2+\alpha}$ .

Proof of Step 1: Define

$$L(\overline{AB}) = 2l = 2\kappa^{\frac{\alpha}{2}} h^{1+\frac{\alpha}{2}}. \quad (4.13)$$

Denote by  $D$  the midpoint of  $\overline{AB}$ . Therefore

$$\Gamma(t-h) \cap B_R(x_A) = \emptyset, \quad \Gamma(t-h) \cap B_R(x_B) = \emptyset, \quad (4.14)$$

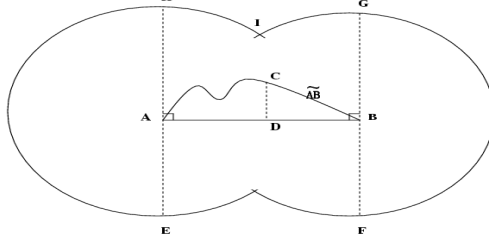
where

$$R = \int_{t-h}^t f(x_D, \tau) d\tau - 4\kappa C_2^{\alpha} h^{1+\alpha} \geq C_1 h - 4\kappa C_2^{\alpha} h^{1+\alpha} > 8l, \quad (4.15)$$

as long as  $h \leq (\frac{C_2}{4})^{\frac{2}{\alpha}} \kappa^{-1}$ ,  $\frac{C_1 h}{4} \geq 4\kappa C_2^{\alpha} h^{1+\alpha}$  and  $8\kappa^{\frac{\alpha}{2}} h^{1+\frac{\alpha}{2}} \leq \frac{3C_1}{4} h$ , i.e.,  $h \leq M$  where  $M$  is defined as above.

(4.14) follows from a contradiction argument as follows: Suppose that (4.14) is not true. Then without loss of generality, there exist a point  $Z \in \Gamma(t-h) \cap B_R(x_A)$ . Let us consider a path  $y : [t-h, t] \rightarrow \mathbb{R}^2$  starting  $Z$  along the direction of straight line segment  $ZA$  defined as  $y(\tau) = y(t-h) + \int_{t-h}^{\tau} f(y(s), s) \frac{x_A - x_Z}{|x_A - x_Z|} ds$ . By the definition of  $\Gamma(t)$ ,  $L(y[t-h, t]) \leq R$ . However,

$$L(y[t-h, t]) = \int_{t-h}^t f(y(\tau), \tau) d\tau = \int_{t-h}^t f(x_D, \tau) d\tau + \int_{t-h}^t f(y(\tau), \tau) - f(x_D, \tau) d\tau,$$



$$\begin{aligned}
&\geq \int_{t-h}^t f(x_D, \tau) d\tau - \kappa(\kappa^{\frac{\alpha}{2}} h^{1+\frac{\alpha}{2}} + C_2 h)^\alpha h, \\
&\geq \int_{t-h}^t f(x_D, \tau) d\tau - 2\kappa C_2^\alpha h^{1+\alpha} > R,
\end{aligned}$$

which contradicts the fact that  $A \in \Gamma(t)$ .

Since  $C$  is a point on  $\widetilde{AB}$ , by the definition of  $\widetilde{AB}$ , there is a backward boundary characteristic path  $x_3 : [t-h, t] \rightarrow \mathbb{R}^2$  intercepts with  $\widetilde{AB}$  at point  $Q = x(\tau_1)$  at time  $\tau_1 \in [t-h, t]$ .  $x_3[\tau_1, t]$  is backward boundary characteristic path intersected with  $\widetilde{AB}$  from non-negative side. Without loss of generality, we assume that  $x_D = (0, 0)$ ,  $x_A = (-l, 0)$ ,  $x_B = (l, 0)$ ,  $x_C = (x_1, y_1)$ ,  $x_Q = (x_0, 0)$ , where  $x_0 \in [-l, l]$ .

Note that  $x_3(t-h) \notin B_R(x_A) \cup B_R(x_B)$  by (4.14). Let  $(x, y)$ , denoted by  $P$ , be the interception point of  $x_3([t-h, t])$  with  $\partial(B_R(x_A) \cup B_R(x_B))$  at some time  $t_1 \geq t-h$ . Therefore

$$\begin{aligned}
L(x_3([t_1, t])) &\geq L(\overline{PQ}) + L(\overline{QC}) \\
&= \text{dist}((x_0, 0), (x, y)) + \text{dist}((x_0, 0), (x_1, y_1)).
\end{aligned}$$

Due to  $(x-l)^2 + y^2 = R^2$  and  $x_0 \leq l$  we obtain

$$\begin{aligned}
L^2(\overline{PQ}) &= \text{dist}^2((x_0, 0), (x, y)) \\
&= (x-x_0)^2 + y^2 \\
&= (x-l)^2 + y^2 + 2(x-l)(l-x_0) + (l-x_0)^2 \\
&\geq R^2 - 2l(l-x_0) + (l-x_0)^2 \\
&\geq R^2 - l^2 \\
&= \left( \int_{t-h}^t f(x_D, \tau) d\tau - 4\kappa C_2^\alpha h^{1+\alpha} \right)^2 - l^2,
\end{aligned}$$

if  $x \geq 0$ . If  $x \leq 0$ , we use  $(x+l)^2 + y^2 = R^2$  and  $x_0 \geq -l$  to conclude

$$\begin{aligned}
L^2(\overline{PQ}) &= \text{dist}^2((x_0, 0), (x, y)) \\
&= (x - x_0)^2 + y^2 \\
&= (x+l)^2 + y^2 + 2(x+l)(-l-x_0) + (l+x_0)^2 \\
&\geq R^2 - 2l(l+x_0) + (l+x_0)^2 \\
&\geq R^2 - l^2 \\
&= \left( \int_{t-h}^t f(x_D, \tau) d\tau - 4\kappa C_2^\alpha h^{1+\alpha} \right)^2 - l^2.
\end{aligned}$$

It follows from the above inequalities that

$$\begin{aligned}
\text{dist}(C, \overline{AB}) &\leq L(\overline{QC}) \leq L(x_3[\tau_1, t]) \leq L(x_3([t_1, t])) - L(\overline{PQ}) \\
&\leq L(x_3([t_1, t])) - \left( \left( \int_{t-h}^t f(x_D, \tau) d\tau - 4\kappa C_2^\alpha h^{1+\alpha} \right)^2 - l^2 \right)^{\frac{1}{2}}.
\end{aligned} \tag{4.16}$$

On the other hand, since  $x_3([t-h, t])$  intercepts with  $\overline{AB}$  at  $Q$ , at time  $\tau \in [t-h, t]$

$$\begin{aligned}
|x_3(\tau) - x_D| &\leq \frac{1}{2}L(\overline{AB}) + L(x_3[t-h, t]) \\
&\leq \kappa^\alpha h^{1+\alpha} + C_2 h \leq \frac{5}{4}C_2 h,
\end{aligned}$$

as long as  $h \leq \left(\frac{C_2}{4}\right)^{\frac{2}{\alpha}} \kappa^{-1}$ . And

$$\begin{aligned}
L(x_3([t_1, t])) &\leq \int_{t-h}^t f(x_3(\tau), \tau) d\tau \\
&= \int_{t-h}^t f(x_D, \tau) + f(x_3(\tau), \tau) - f(x_D, \tau) d\tau \\
&\leq \int_{t-h}^t f(x_D, \tau) d\tau + \kappa \left(\frac{5}{4}C_2\right)^\alpha h^{1+\alpha} \\
&\leq \int_{t-h}^t f(x_D, \tau) d\tau + 2\kappa C_2^\alpha h^{1+\alpha},
\end{aligned} \tag{4.17}$$

It follows from (4.16) and (4.17) that

$$\begin{aligned}
\text{dist}(C, \overline{AB}) &\leq L(x_3[\tau_1, t]) \leq R + 8\kappa C_2^\alpha h^{1+\alpha} - (R^2 - l^2)^{\frac{1}{2}} \\
&= \frac{l^2}{R + (R^2 - l^2)^{\frac{1}{2}}} + 8\kappa C_2^\alpha h^{1+\alpha} \\
&\leq \frac{\kappa^\alpha h^{2+\alpha}}{R + (R^2 - l^2)^{\frac{1}{2}}} + 8\kappa C_2^\alpha h^{1+\alpha} \\
&\leq \frac{\kappa^\alpha h^{2+\alpha}}{\frac{3}{4}C_1 h} + 8\kappa C_2^\alpha h^{1+\alpha} \\
&= \left(\frac{4}{3} \frac{\kappa^\alpha}{C_1} + 8\kappa C_2^\alpha\right) h^{1+\alpha}.
\end{aligned} \tag{4.18}$$

Hence (4.12) is shown.

Step 2. Note that  $\widetilde{AB}$  may be disconnected. The orthogonal line issued from the midpoint  $D$  of  $\overline{AB}$  may or may not intersect with  $\widetilde{AB}$ .

Scenario 1. The orthogonal line issued from the midpoint  $D$  of  $\overline{AB}$  intersects with  $\widetilde{AB}$ . Let us denote by  $C$  the interception point of  $\widetilde{AB}$  and the orthogonal line to  $\overline{AB}$ , which has the least distance from the point  $D$ . Let us consider the boundary of the convex hull  $\text{bch}(\widetilde{AB} \cap \Delta ADC)$  (or  $\text{bch}(\widetilde{AB} \cap \Delta BDC)$ ) of  $\widetilde{AB} \cap \Delta ADC$  (or  $\widetilde{AB} \cap \Delta BDC$ ), where  $\Delta ADC$  denote the triangle with vertices  $A, D, C$ ;  $\Delta BDC$  denote the triangle with vertices  $B, D, C$ . The lower boundary of the convex hull of  $\widetilde{AB} \cap \Delta ADC$  (or  $\widetilde{AB} \cap \Delta BDC$ ) is defined as  $\text{lbch}(\widetilde{AB} \cap \Delta ADC) = \text{bch}(\widetilde{AB} \cap \Delta ADC) \setminus \text{int}(\overline{AC})$  if  $\text{cvh}(\widetilde{AB} \cap \Delta ADC) \neq \overline{AC}$ , otherwise  $\text{lbch}(\widetilde{AB} \cap \Delta ADC) = \overline{AC}$  (or  $\text{lbch}(\widetilde{AB} \cap \Delta BDC) = \text{bch}(\widetilde{AB} \cap \Delta BDC) \setminus \text{int}(\overline{BC})$  if  $\text{cvh}(\widetilde{AB} \cap \Delta BDC) \neq \overline{BC}$ , otherwise  $\text{lbch}(\widetilde{AB} \cap \Delta BDC) = \overline{BC}$ ). By the definition of  $C$ ,  $\text{lbch}(\widetilde{AB} \cap \Delta ADC) \cap \text{lbch}(\widetilde{AB} \cap \Delta BDC) = \{C\}$ . Then the curve  $\text{lbch}(\widetilde{AB} \cap \Delta ADC) \cup \text{lbch}(\widetilde{AB} \cap \Delta BDC)$  is a first approximation of  $\widetilde{AB}$  and is denoted by  $\widetilde{AB}^{(1)}$ . We also classify  $\widetilde{AB}^{(1)}$  as approximate boundary of  $R(1)$  class. Since the convex curve consisting of  $\text{lbch}(\widetilde{AB} \cap \Delta ADC)$  and  $\overline{AC}$  is in the convex hull of the triangle  $\Delta ADC$ ,  $L(\text{lbch}(\widetilde{AB} \cap \Delta ADC)) \leq L(\overline{AD}) + L(\overline{CD})$  by Lemma 4.1. By the same argument, we obtain  $L(\text{lbch}(\widetilde{AB} \cap \Delta BDC)) \leq L(\overline{BD}) + L(\overline{CD})$ . By step 1, we have

$$L(\widetilde{AB}^{(1)}) \leq (1 + 2\eta L^\gamma(\overline{AB}))L(\overline{AB}). \quad (4.19)$$

The curve  $\widetilde{AB}^{(1)}$  has two complementary subsets: the set  $X_{AB}$  of points coinciding with  $\widetilde{AB}$  and the set  $Z_{AB}$  of points different from  $\widetilde{AB}$ .  $X_{AB}$  is a closed set relative to  $\widetilde{AB}^{(1)}$ ,  $Z_{AB}$  is an open set relative to  $\widetilde{AB}^{(1)}$ , thus consists of countably many disjoint open straight line segments.  $Z_{AB} = \cup_{j=1}^{\infty} \text{int}(\overline{A_j^{1,1} B_j^{1,1}}) \subset \Omega(t)$ , where  $A_j^{1,1}, B_j^{1,1} \in \widetilde{AB} \cap \Delta ADC$ , or  $A_j^{1,1}, B_j^{1,1} \in \widetilde{AB} \cap \Delta BDC$ . Here the first sup-index records the number of iteration, the second sup-index denotes  $R$ - class number. Therefore by (4.19)

$$\begin{aligned} \sum_{j=1}^{\infty} L(\overline{A_j^{1,1} B_j^{1,1}}) + \mathcal{H}^1(X_{AB}) &= L(\widetilde{AB}^{(1)}) \\ &\leq (1 + 2\eta L^\gamma(\overline{AB}))L(\overline{AB}). \end{aligned} \quad (4.20)$$

By the definition of the point  $C$ , we have

$$\sup_j L(\overline{A_j^{1,1} B_j^{1,1}}) \leq \frac{1}{2}(L(\overline{AB}) + 2L(\overline{CD})) \leq \frac{(1 + 2\eta L^\gamma(\overline{AB}))}{2} L(\overline{AB}). \quad (4.21)$$

Note that for any  $x \in \widetilde{AB} \setminus X_{AB}$ , there is a backward boundary characteristic path  $\{x(\tau) : \tau \in [\gamma(z), t] \mid x(t) = x, x(\gamma(z)) = z \in \text{int}(\overline{AB})\} \subset \text{cvh}(\Gamma(t) \cap B_r(y))$ . Thus by Jordan curve property, the boundary characteristic path must intercept with one of the open straight line segments  $\overline{A_j^{1,1} B_j^{1,1}}$  for some  $j$  at time  $s$  and  $\{x(\tau) : \tau \in (s, t]\} \subset \text{cvh}(\Gamma(t) \cap B_r(y)) \setminus \text{int}(\mathcal{O}_{AB})$ , where  $\mathcal{O}_{AB}$  is the domain enclosed by  $\overline{AB}$  and  $\widetilde{AB}^{(1)}$ . We can assign the unit normal vector  $n_{\overline{A_j^{1,1} B_j^{1,1}}}$  to each line segment  $\text{int}(\overline{A_j^{1,1} B_j^{1,1}})$  by choosing the unit orthogonal vector to  $\overline{A_j^{1,1} B_j^{1,1}}$  pointing to the interior  $\text{int}(\text{cvh}(\widetilde{AB} \cap \Delta ADC))$  if  $\overline{A_j^{1,1} B_j^{1,1}} \subset \text{cvh}(\widetilde{AB} \cap \Delta ADC)$ , otherwise choosing the unit orthogonal vector to  $\overline{A_j^{1,1} B_j^{1,1}}$  pointing to the interior  $\text{int}(\text{cvh}(\widetilde{AB} \cap \Delta BDC))$ . Denote by  $\widetilde{A_j^{1,1} B_j^{1,1}}$  the set of points on boundary, which has one boundary characteristic path intersected with  $\overline{A_j^{1,1} B_j^{1,1}}$  from non-negative side. That is,  $\widetilde{A_j^{1,1} B_j^{1,1}} = \{x \in \Gamma(t) \mid \text{there is a backward boundary characteristic } x : [\tau, t] \rightarrow \mathbb{R}^2 \text{ with } x(\tau) \in \overline{A_j^{1,1} B_j^{1,1}}, \text{ there is no subinterval } [s_1, s_2] \subset [\tau, t] \text{ such that } x(s_1) \in \text{int}(\overline{A_j^{1,1} B_j^{1,1}}) \text{ and } (x(s) - x(s_1)) \cdot n_{\overline{A_j^{1,1} B_j^{1,1}}} < 0 \text{ for } s \in (s_1, s_2)\}$ . Then

$$\widetilde{AB} \subset \cup_{j=1}^{\infty} \widetilde{A_j^{1,1} B_j^{1,1}} \cup X_{AB}^0 \equiv \widetilde{AB}^1. \quad (4.22)$$

where  $X_{AB}^0$  denotes  $X_{AB}$ .

The opposite of scenario 1 is that the orthogonal line issued from the midpoint  $D$  of  $\overline{AB}$  does not intersect with  $\widetilde{AB}$ . Let us slide an orthogonal half line  $l$  to  $\overline{AB}$  from the midpoint  $D$  to the right until the

orthogonal half line  $l$  intersect with  $\widetilde{AB}$  before the orthogonal line  $l$  hits the point  $B$ , or we stop the sliding of orthogonal line  $l$  at point  $B$  even if  $l \cap \widetilde{AB} = B$ . Here  $(x - x_A) \cdot n_{AB} \geq 0$  for  $x \in l$ . Denote by  $U$  the intersection point of  $l$  and  $\widetilde{AB}$ , which has least distance from the line segment  $\overline{AB}$ .  $\overline{UV} \perp \overline{AB}$  where  $V \in \overline{AB}$ . In the same way, we slide an orthogonal half line  $l'$  to  $\overline{AB}$  from the midpoint  $D$  to the left until the orthogonal half line intersect with  $\widetilde{AB}$  before the orthogonal half line  $l'$  hits the point  $A$ , or we stop the sliding of orthogonal half line  $l'$  at point  $A$  even if  $l' \cap \widetilde{AB} = A$ . Here  $(x - x_A) \cdot n_{AB} \geq 0$  for  $x \in l'$ . Denote by  $S$  the intersection point of  $l'$  and  $\widetilde{AB}$ , which has least distance from the line segment  $\overline{AB}$ .  $\overline{ST} \perp \overline{AB}$  where  $T \in \overline{AB}$ . Let us consider the boundary of the convex hull  $\text{bch}(\widetilde{AB} \cap \Delta AST)$  (or  $\text{bch}(\widetilde{AB} \cap \Delta BUUV)$ ) of  $\widetilde{AB} \cap \Delta AST$  (or  $\widetilde{AB} \cap \Delta BUUV$ ), where  $\Delta AST$  denote the triangle with vertices  $A, S, T$ ;  $\Delta BUUV$  denote the triangle with vertices  $B, U, V$ . The lower boundary of the convex hull of  $\widetilde{AB} \cap \Delta AST$  (or  $\widetilde{AB} \cap \Delta BUUV$ ) is defined as  $\text{lbch}(\widetilde{AB} \cap \Delta AST) = \text{bch}(\widetilde{AB} \cap \Delta AST) \setminus \text{int}(\overline{AS})$  if  $\text{cvh}(\widetilde{AB} \cap \Delta AST) \neq \overline{AS}$ , otherwise  $\text{lbch}(\widetilde{AB} \cap \Delta AST) = \overline{AS}$  (or  $\text{lbch}(\widetilde{AB} \cap \Delta BUUV) = \text{bch}(\widetilde{AB} \cap \Delta BUUV) \setminus \text{int}(\overline{BV})$  if  $\text{cvh}(\widetilde{AB} \cap \Delta BUUV) \neq \overline{BU}$ , otherwise  $\text{lbch}(\widetilde{AB} \cap \Delta BUUV) = \overline{BU}$ ). Define the unit normal vector  $n_{\overline{SU}}$  to the line  $\overline{SU}$  in such way that  $n_{\overline{SU}} \cdot n_{\overline{AB}} \geq 0$ .

$$\text{Scenario 2. } 0 < L(\overline{TV}) \leq \frac{L(\overline{AB})}{2}.$$

$\text{lbch}(\widetilde{AB} \cap \Delta AST) \cup \overline{SU} \cup \text{lbch}(\widetilde{AB} \cap \Delta BUUV) = \widetilde{AB}^{(1)}$  is the first approximation of  $\widetilde{AB}$ . We classify it as approximate boundary of  $R(1)$  class. Since the convex curve consisting of  $\text{lbch}(\widetilde{AB} \cap \Delta AST)$  and  $\overline{AS}$  is in the convex hull of the triangle  $\Delta AST$ ,  $L(\text{lbch}(\widetilde{AB} \cap \Delta AST)) \leq L(\overline{AT}) + L(\overline{ST})$  by Lemma 4.1. By the same argument, we obtain  $L(\text{lbch}(\widetilde{AB} \cap \Delta BUUV)) \leq L(\overline{BV}) + L(\overline{UV})$ . On the other hand,  $L(\overline{SU}) \leq L(\overline{TV}) + |L(\overline{ST}) - L(\overline{TV})|$ . By step 1, we have

$$L(\widetilde{AB}^{(1)}) \leq (1 + 2\eta L^\gamma(\overline{AB}))L(\overline{AB}). \quad (4.23)$$

Again, the curve  $\widetilde{AB}^{(1)}$  has two complementary subsets: the set  $X_{AB}$  of points coinciding with  $\widetilde{AB}$  and the set  $Z_{AB}$  of points different from  $\widetilde{AB}$ .  $X_{AB}$  is a closed set relative to  $\widetilde{AB}^{(1)}$ ,  $Z_{AB}$  is an open set relative to  $\widetilde{AB}^{(1)}$ , thus consists of countably many disjoint open straight line segments.  $Z_{AB} = \cup_{j=1}^{\infty} \text{int}(A_j^{1,1} B_j^{1,1}) \subset \Omega(t)$ , where  $A_j^{1,1}, B_j^{1,1} \in \widetilde{AB} \cap \Delta AST$ , or  $A_j^{1,1}, B_j^{1,1} \in \widetilde{AB} \cap \Delta BUUV$ , or  $A_j^{1,1} B_j^{1,1} = \overline{SU}$ . Therefore by (4.23)

$$\sum_{j=1}^{\infty} L(\overline{A_j^{1,1} B_j^{1,1}}) + \mathcal{H}^1(X_{AB}) = L(\widetilde{AB}^{(1)}) \leq (1 + 2\eta L^\gamma(\overline{AB}))L(\overline{AB}). \quad (4.24)$$

We have

$$\sup_j L(\overline{A_j^{1,1} B_j^{1,1}}) \leq \frac{1}{2}(L(\overline{AB}) + 2 \max\{L(\overline{ST}), L(\overline{UV})\}) \leq \frac{(1 + 2\eta L^\gamma(\overline{AB}))}{2} L(\overline{AB}). \quad (4.25)$$

Note that for any  $x \in \widetilde{AB} \setminus X_{AB}$ , there is a backward boundary characteristic path  $\{x(\tau) : \tau \in [\gamma(z), t] \mid x(t) = x, x(\gamma(z)) = z \in \text{int}(\overline{AB})\} \subset \text{cvh}(\Gamma(t) \cap B_r(y))$ . Thus by Jordan curve property, the boundary characteristic path must intercept with one of the open straight line segments  $\overline{A_j^{1,1} B_j^{1,1}}$  for some  $j$  at time  $s$  and  $\{x(\tau) : \tau \in (s, t]\} \subset \text{cvh}(\Gamma(t) \cap B_r(y)) \setminus \text{int}(\mathcal{O}_{AB})$ , where  $\mathcal{O}_{AB}$  is the domain enclosed by  $\overline{AB}$  and  $\widetilde{AB}^{(1)}$ . We can assign the unit normal vector  $n_{\overline{A_j^{1,1} B_j^{1,1}}}$  to each line segment  $\text{int}(\overline{A_j^{1,1} B_j^{1,1}})$  by choosing the unit orthogonal vector to  $\overline{A_j^{1,1} B_j^{1,1}}$  pointing to the interior  $\text{int}(\text{cvh}(\widetilde{AB} \cap \Delta AST))$  if  $\overline{A_j^{1,1} B_j^{1,1}} \subset \text{cvh}(\widetilde{AB} \cap \Delta AST)$ , otherwise choosing the unit orthogonal vector to  $\overline{A_j^{1,1} B_j^{1,1}}$  pointing to the interior  $\text{int}(\text{cvh}(\widetilde{AB} \cap \Delta BUUV))$ . Denote by  $\widetilde{A_j^{1,1} B_j^{1,1}}$  the set of points on the boundary, which has one boundary characteristic path intersected with  $\overline{A_j^{1,1} B_j^{1,1}}$  from non-negative side. That is,  $\widetilde{A_j^{1,1} B_j^{1,1}} = \{x \in \Gamma(t) \mid \text{there is a backward boundary characteristic path } x : [\tau, t] \rightarrow \mathbb{R}^n \text{ with } x(\tau) \in \overline{A_j^{1,1} B_j^{1,1}}, \text{ there is no subinterval } [s_1, s_2] \subset [\tau, t] \text{ such that}$

$x(s_1) \in \text{int}(\overline{A_j^{1,1}B_j^{1,1}})$  and  $(x(s) - x(s_1)) \cdot n_{\overline{A_j^{1,1}B_j^{1,1}}} < 0$  for  $s \in (s_1, s_2)$ . Then

$$\widetilde{AB} \subset \cup_{j=1}^{\infty} \widetilde{A_j^{1,1}B_j^{1,1}} \cup X_{AB}^0 \equiv \widetilde{AB}^1. \quad (4.26)$$

where  $X_{AB}^0$  denotes  $X_{AB}$ .

Scenario 3.  $\frac{L(\overline{AB})}{2} < L(\overline{TV}) \leq L(\overline{AB})$ .

We classify  $\text{lbch}(\widetilde{AB} \cap \Delta AST) \cup \text{lbch}(\widetilde{AB} \cap \Delta BUUV)$  as approximate boundary of  $R(1)$  class, and identify  $\overline{SU}$  as approximate boundary of  $V(1)$  class. We have to handle two cases (i) either  $A = T$  or  $B = V$ ; (ii)  $T, V \in \text{int}(\overline{AB})$ . Since the argument for case (ii) applies to case (i), we only focus on case (ii). Since the convex curve consisting of  $\text{lbch}(\widetilde{AB} \cap \Delta AST)$  and  $\overline{AS}$  is in the convex hull of the triangle  $\Delta AST$ ,  $L(\text{lbch}(\widetilde{AB} \cap \Delta AST)) \leq L(\overline{AT}) + L(\overline{ST})$  by Lemma 4.1. By the same argument, we obtain  $L(\text{lbch}(\widetilde{AB} \cap \Delta BUUV)) \leq L(\overline{BV}) + L(\overline{UV})$ . On the other hand,  $L(\overline{SU}) \leq L(\overline{TV}) + |L(\overline{ST}) - L(\overline{TV})|$ . By step 1, we have

$$L(\text{lbch}(\widetilde{AB} \cap \Delta AST)) + L(\overline{SU}) + L(\text{lbch}(\widetilde{AB} \cap \Delta BUUV)) \leq (1 + 2\eta L^\gamma(\overline{AB}))L(\overline{AB}). \quad (4.27)$$

Note that for any  $x \in \widetilde{AB}$ , there is a backward boundary characteristic  $\{x(\tau) : \tau \in [\gamma(z), t] \mid x(t) = x, x(\gamma(z)) = z \in \text{int}(\overline{AB})\} \subset \text{cvh}(\Gamma(t) \cap B_r(y))$ . Thus by Jordan curve property, the boundary characteristic must intercept with one of the three curves  $\text{lbch}(\widetilde{AB} \cap \Delta AST)$ ,  $\text{lbch}(\widetilde{AB} \cap \Delta BUUV)$ , and  $\overline{SU}$ , at time  $s$ , and  $\{x(\tau) : \tau \in (s, t]\} \subset \text{cvh}(\Gamma(t) \cap B_r(y)) \setminus \text{int}(\mathcal{O}_{AB})$ , where  $\mathcal{O}_{AB}$  is the domain enclosed by  $\widetilde{AB}$  and  $\text{lbch}(\widetilde{AB} \cap \Delta AST) \cup \overline{SU} \cup \text{lbch}(\widetilde{AB} \cap \Delta BUUV)$ .

As argued in scenario 2,  $\text{lbch}(\widetilde{AB} \cap \Delta AST) \cup \text{lbch}(\widetilde{AB} \cap \Delta BUUV)$  has two complementary subsets: the set  $X_{ASUB}$  of points coinciding with  $\widetilde{AB}$  and the set  $Z_{ASUB}$  of points different from  $\widetilde{AB}$ .  $X_{ASUB}$  is a closed set relative to  $\widetilde{AB}^{(1)}$ ,  $Z_{ASUB}$  is an open set relative to  $\widetilde{AB}^{(1)}$ , thus consists of countably many disjoint open straight line segments.  $Z_{ASUB} = \cup_{j=1}^{\infty} \text{int}(A_j^{1,1}B_j^{1,1}) \subset \Omega(t)$ , where  $A_j^{1,1}, B_j^{1,1} \in \widetilde{AB} \cap \Delta AST$ , or  $A_j^{1,1}, B_j^{1,1} \in \widetilde{AB} \cap \Delta BUUV$ . Therefore

$$\sum_{j=1}^{\infty} L(\overline{A_j^{1,1}B_j^{1,1}}) + \mathcal{H}^1(X_{ASUB}) \leq L(\overline{AT}) + L(\overline{ST}) + L(\overline{UV}) + L(\overline{BV}). \quad (4.28)$$

We have

$$\begin{aligned} \sup_j L(\overline{A_j^{1,1}B_j^{1,1}}) &\leq \max\{L(\overline{ST}) + L(\overline{AS}), L(\overline{UV}) + L(\overline{BV})\} \\ &\leq \frac{(1 + 2\eta L^\gamma(\overline{AB}))}{2} L(\overline{AB}). \end{aligned} \quad (4.29)$$

We can assign the unit normal vector  $n_{\overline{A_j^{1,1}B_j^{1,1}}}$  to each line segment  $\text{int}(\overline{A_j^{1,1}B_j^{1,1}})$  by choosing the unit orthogonal vector to  $\overline{A_j^{1,1}B_j^{1,1}}$  pointing to the interior  $\text{int}(\text{cvh}(\widetilde{AB} \cap \Delta AST))$  if  $\overline{A_j^{1,1}B_j^{1,1}} \subset \text{cvh}(\widetilde{AB} \cap \Delta AST)$ , otherwise choosing the unit orthogonal vector to  $\overline{A_j^{1,1}B_j^{1,1}}$  pointing to the interior  $\text{int}(\text{cvh}(\widetilde{AB} \cap \Delta BUUV))$ . Denote by  $\widetilde{A_j^{1,1}B_j^{1,1}}$  the set of points on the boundary, which has one boundary characteristic path intersected with  $\overline{A_j^{1,1}B_j^{1,1}}$  from non-negative side. That is,  $\widetilde{A_j^{1,1}B_j^{1,1}} = \{x \in \Gamma(t) \mid \text{there is a backward boundary characteristic path } x : [\tau, t] \rightarrow \mathbb{R}^n \text{ with } x(\tau) \in \overline{A_j^{1,1}B_j^{1,1}}, \text{ there is no subinterval } [s_1, s_2] \subset [\tau, t] \text{ such that } x(s_1) \in \text{int}(\overline{A_j^{1,1}B_j^{1,1}}) \text{ and } (x(s) - x(s_1)) \cdot n_{\overline{A_j^{1,1}B_j^{1,1}}} < 0 \text{ for } s \in (s_1, s_2)\}$ .

Define  $\widetilde{SU} = \{x \in \Gamma(t) \mid \text{there is a backward boundary characteristic path } x : [\tau, t] \rightarrow \mathbb{R}^n \text{ with } x(\tau) \in \overline{SU}, \text{ there is no subinterval } [s_1, s_2] \subset [\tau, t] \text{ such that } x(s_1) \in \text{int}(\overline{SU}) \text{ and } (x(s) - x(s_1)) \cdot n_{\overline{SU}} < 0 \text{ for } s \in (s_1, s_2)\}$ . Without loss of generality, we assume that  $L(\overline{ST}) \geq L(\overline{UV})$ . By step 1 and assumption on  $L(\overline{TV})$ , we have  $\angle SUV \leq \frac{3\pi}{4}$ ,  $\angle UST \leq \frac{\pi}{2}$ . Suppose  $C \in \widetilde{SU}$ , by the virtue of Jordan curve property and the definition

of  $\widetilde{SU}$ , there is a backward boundary characteristic path intersected with  $\overline{SU}$  at the point  $R$  from non-negative side to intersect with either the half line  $\{x \in \mathbb{R}^2 \mid (x - x_S) \cdot n_{AB} = |x - x_S| \geq 0\}$  or the half line  $\{x \in \mathbb{R}^2 \mid (x - x_U) \cdot n_{AB} = |x - x_U| \geq 0\}$  at the point  $Q$ . By step 1,

$$L(\overline{RQ}) \leq \eta L(\overline{SU})^{1+\gamma}, \quad \text{and } L(\overline{RC}) \leq \eta L(\overline{SU})^{1+\gamma}.$$

Therefore either  $L(\overline{SC}) \leq L(\overline{RC}) \leq 2\eta L(\overline{SU})^{1+\gamma}$  or  $L(\overline{UC}) \leq L(\overline{QU}) + L(\overline{QC}) \leq (\sqrt{2} + 1)\eta L(\overline{SU})^{1+\gamma}$ . Hence  $\widetilde{SU} \subset B_{r_1(SU)}(S) \cup B_{r_1(SU)}(U)$ , where  $r_1(SU) = (\sqrt{2} + 1)\eta L(\overline{SU})^{1+\gamma}$ .

$$\widetilde{SU} \subset \text{cvh}(\Gamma(t) \cap B_r(y) \cap B_{r_1(SU)}(S)) \cup \text{cvh}(\Gamma(t) \cap B_r(y) \cap B_{r_1(SU)}(U)). \quad (4.30)$$

Note that  $\Omega(t) \cap \text{bch}(\Gamma(t) \cap B_r(y) \cap B_{r_1(SU)}(S)) \cup \Omega(t) \cap \text{bch}(\Gamma(t) \cap B_r(y) \cap B_{r_1(SU)}(U))$  is an open set relative to the two disjoint convex curve  $\text{bch}(\Gamma(t) \cap B_r(y) \cap B_{r_1(SU)}(S))$  and  $\text{bch}(\Gamma(t) \cap B_r(y) \cap B_{r_1(SU)}(U))$ , which consists of countably many disjoint open straight line segments of  $\text{bch}(\Gamma(t) \cap B_r(y) \cap B_{r_1(SU)}(S))$  and  $\text{bch}(\Gamma(t) \cap B_r(y) \cap B_{r_1(SU)}(U))$ . The end points  $A_j^{1,2}, B_j^{1,2}$  of the  $j$ -th open straight line segment  $\text{int}(\overline{A_j^{1,2}B_j^{1,2}})$  are on the moving front  $\Gamma(t) = \partial\Omega(t)$ . That is,  $\Omega(t) \cap \text{bch}(\Gamma(t) \cap B_r(y) \cap B_{r_1(SU)}(S)) \cup \Omega(t) \cap \text{bch}(\Gamma(t) \cap B_r(y) \cap B_{r_1(SU)}(U)) = \cup_{j=1}^{\infty} \text{int}(\overline{A_j^{1,2}B_j^{1,2}})$ . We can assign the unit normal vector  $n_{\overline{A_j^{1,2}B_j^{1,2}}}$  to each line segment  $\text{int}(\overline{A_j^{1,2}B_j^{1,2}})$  by choosing the unit orthogonal vector to  $\overline{A_j^{1,2}B_j^{1,2}}$  pointing to the interior  $\text{int}(\text{cvh}(\Gamma(t) \cap B_r(y) \cap B_{r_1(SU)}(S)))$  if  $\overline{A_j^{1,2}B_j^{1,2}} \subset \Omega(t) \cap \text{bch}(\Gamma(t) \cap B_r(y) \cap B_{r_1(SU)}(S))$ , otherwise choosing the unit orthogonal vector to  $\overline{A_j^{1,2}B_j^{1,2}}$  pointing to the interior  $\text{int}(\text{cvh}(\Gamma(t) \cap B_r(y) \cap B_{r_1(SU)}(U)))$ .  $\text{bch}(\Gamma(t) \cap B_r(y) \cap B_{r_1(SU)}(S)) \setminus \Omega(t)$  (or  $\text{bch}(\Gamma(t) \cap B_r(y) \cap B_{r_1(SU)}(U)) \setminus \Omega(t)$ ) is closed subset of  $\text{bch}(\Gamma(t) \cap B_r(y) \cap B_{r_1(SU)}(S))$  (or  $\text{bch}(\Gamma(t) \cap B_r(y) \cap B_{r_1(SU)}(U))$ ). It follows from Lemma 4.1 and (4.30) that

$$\begin{aligned} & \sum_{j=1}^{\infty} L(\overline{A_j^{1,2}B_j^{1,2}}) + \mathcal{H}^1(\text{bch}(\Gamma(t) \cap B_r(y) \cap B_{r_1(SU)}(S)) \setminus \Omega(t)) \\ & \quad + \mathcal{H}^1(\text{bch}(\Gamma(t) \cap B_r(y) \cap B_{r_1(SU)}(U)) \setminus \Omega(t)) \leq 4\pi r_1(SU) \\ & \leq \frac{\sqrt{2} + 1}{4} L(\overline{SU}) \leq \frac{\sqrt{2} + 1}{4} (L(\overline{TV}) + |L(\overline{ST}) - L(\overline{UV})|). \end{aligned} \quad (4.31)$$

Denote by  $\widetilde{A_j^{1,2}B_j^{1,2}}$  the set of points on the boundary  $\Gamma(t) \cap B_r(y) \cap (B_{r_1(SU)}(S) \cup B_{r_1(SU)}(U))$ , which has one backward boundary characteristic path intersected with  $\overline{A_j^{1,2}B_j^{1,2}}$  from non-negative side. That is,  $\widetilde{A_j^{1,2}B_j^{1,2}} = \{x \in \Gamma(t) \mid \text{there is a backward boundary characteristic path } x : [\tau, t] \rightarrow \mathbb{R}^n \text{ with } x(\tau) \in \overline{A_j^{1,2}B_j^{1,2}}, \text{ there is no subinterval } [s_1, s_2] \subset [\tau, t] \text{ such that } x(s_1) \in \text{int}(\overline{A_j^{1,2}B_j^{1,2}}) \text{ and } (x(s) - x(s_1)) \cdot n_{\overline{A_j^{1,2}B_j^{1,2}}} < 0 \text{ for } s \in (s_1, s_2)\}$ . By Jordan curve property, we have

$$\widetilde{SU} \subset \Gamma(t) \cap B_r(y) \cap (B_{r_1(SU)}(S) \cup B_{r_1(SU)}(U)) \subset (\cup_j^{\infty} \widetilde{A_j^{1,2}B_j^{1,2}}) \cup X_{SU}, \quad (4.32)$$

where  $X_{SU} = \Gamma(t) \cap (\text{bch}(\Gamma(t) \cap B_r(y) \cap B_{r_1(SU)}(S)) \cup \text{bch}(\Gamma(t) \cap B_r(y) \cap B_{r_1(SU)}(U))) \subset (\text{bch}(\Gamma(t) \cap B_r(y) \cap B_{r_1(SU)}(S)) \cup \text{bch}(\Gamma(t) \cap B_r(y) \cap B_{r_1(SU)}(U))) \setminus \Omega(t)$ .  $\cup_{j=1}^{\infty} \widetilde{A_j^{1,2}B_j^{1,2}} \cup X_{SU}$  is classified as approximate boundary of  $R(2)$  class. Therefore we can define the first order of approximation  $\widetilde{AB}^{(1)}$  of  $\widetilde{AB}$  as follows.  $\widetilde{AB}^{(1)} = (\cup_{j=1}^{\infty} \overline{A_j^{1,2}B_j^{1,2}}) \cup X_{SU} \cup \text{lbch}(\widetilde{AB} \cap \Delta AST) \cup \text{lbch}(\widetilde{AB} \cap \Delta BUV)$ . By Jordan curve property, we have

$$\widetilde{AB} \subset \cup_{j=1}^{\infty} \widetilde{A_j^{1,1}B_j^{1,1}} \cup X_{ASUB} \cup \cup_{j=1}^{\infty} \widetilde{A_j^{1,2}B_j^{1,2}} \cup X_{SU} = \cup_{i=1}^2 \cup_{j=1}^{\infty} \widetilde{A_j^{1,i}B_j^{1,i}} \cup X_{AB}^0 \equiv \widetilde{AB}^1, \quad (4.33)$$

where  $X_{AB}^0 = X_{AB} = X_{ASUB} \cup X_{SU}$ . By (4.28), (4.29), and (4.31), we obtain

$$\begin{aligned} & \sup_{1 \leq j < \infty, 1 \leq i \leq 2} L(\overline{A_j^{1,i}B_j^{1,i}}) \leq \frac{(1 + 2\eta L^\gamma(\overline{AB}))}{2} L(\overline{AB}) \\ & \sum_{i=1}^2 \sum_{j=1}^{\infty} L(\overline{A_j^{1,i}B_j^{1,i}}) + \mathcal{H}^1(X_{AB}^0) \leq (1 + 2\eta L^\gamma(\overline{AB})) L(\overline{AB}). \end{aligned} \quad (4.34)$$

For the convenience of notation, let  $r \equiv r_0$ . Therefore

$$r_1 \leq \frac{\sqrt{2} + 1}{16\pi} r_0. \quad (4.35)$$

Step 3. In  $k$ th iteration for  $k \geq 2$ , we repeat the argument in step 2 to study  $A_j^{k-1, i} \widetilde{B_j^{k-1, i}}$  where  $i = 1, 2, \dots, k$ . We obtain the first approximation of  $A_j^{k-1, i} \widetilde{B_j^{k-1, i}}$  and is denoted by  $A_j^{k-1, i} \widetilde{B_j^{k-1, i}}^{(1)}$ . By the above argument in step 2, we get

$$L(A_j^{k-1, i} \widetilde{B_j^{k-1, i}}^{(1)}) \leq (1 + 2\eta L^\gamma(\overline{A_j^{k-1, i} B_j^{k-1, i}})) L(\overline{A_j^{k-1, i} B_j^{k-1, i}}). \quad (4.36)$$

Based upon the above construction, we can refine the  $k-1$ th approximation. The  $k$ th approximation of  $\widetilde{AB}$  is  $\widetilde{AB}^{(k)} = \cup_{i=1}^k (\cup_{j=1}^\infty A_j^{k-1, i} \widetilde{B_j^{k-1, i}}^{(1)}) \cup (\cup_{i=0}^{k-1} X_{AB}^i)$ .

As argued in step 2,  $A_j^{k-1, i} \widetilde{B_j^{k-1, i}}^{(1)}$  has two complementary subsets: the set  $X_{A_j^{k-1, i} B_j^{k-1, i}}$  which is identified as in step 2 according to scenario 1, 2, and 3; and the set  $Z_{A_j^{k-1, i} B_j^{k-1, i}}$  of points different from  $A_j^{k-1, i} \widetilde{B_j^{k-1, i}}^{(1)}$ .  $X_{A_j^{k-1, i} B_j^{k-1, i}}$  is a closed set relative to  $A_j^{k-1, i} \widetilde{B_j^{k-1, i}}^{(1)}$ ,  $Z_{A_j^{k-1, i} B_j^{k-1, i}} \subset \Omega(t)$  is an open set relative to  $A_j^{k-1, i} \widetilde{B_j^{k-1, i}}^{(1)}$ , and it consists of countably many disjoint open straight line segments. Therefore  $Z_{A_j^{k-1, i} B_j^{k-1, i}} = \cup_{m=i}^{i+1} \cup_{l=1}^\infty \text{int}(\overline{A_{j,l}^{k,m} B_{j,l}^{k,m}})$ , where  $\overline{A_{j,l}^{k,i} B_{j,l}^{k,i}}$  is resulted in scenario 1, or 2 as discussed in step 2;  $\overline{A_{j,l}^{k,i+1} B_{j,l}^{k,i+1}}$  is resulted in scenario 3 as discussed in step 2. Therefore

$$\begin{aligned} \sum_{m=i}^{i+1} \sum_{l=1}^\infty L(\overline{A_{j,l}^{k,m} B_{j,l}^{k,m}}) + \mathcal{H}^1(X_{A_j^{k-1, i} B_j^{k-1, i}}) &= L(A_j^{k-1, i} \widetilde{B_j^{k-1, i}}^{(1)}) \\ &\leq (1 + 2\eta L^\gamma(\overline{A_j^{k-1, i} B_j^{k-1, i}})) L(\overline{A_j^{k-1, i} B_j^{k-1, i}}). \end{aligned} \quad (4.37)$$

Furthermore, we have

$$\sup_{1 \leq l < \infty} \sup_{i \leq m \leq i+1} L(\overline{A_{j,l}^{k,m} B_{j,l}^{k,m}}) \leq \frac{(1 + 2\eta L^\gamma(\overline{A_j^{k-1, i} B_j^{k-1, i}}))}{2} L(\overline{A_j^{k-1, i} B_j^{k-1, i}}). \quad (4.38)$$

As discussed in Step 2, we can assign the unit normal vector  $n_{\overline{A_{j,l}^{k,m} B_{j,l}^{k,m}}}$  to each line segment  $\text{int}(\overline{A_{j,l}^{k,m} B_{j,l}^{k,m}})$  corresponding to scenario 1, 2, and 3 for  $m = i, i+1$ . Let  $\overline{A_{j,l}^{k,m} B_{j,l}^{k,m}} = \{x \in \Gamma(t) \mid \text{there is a backward boundary characteristic path } x : [\tau, t] \rightarrow \mathbb{R}^n \text{ with } x(\tau) \in \overline{A_{j,l}^{k,m} B_{j,l}^{k,m}}, \text{ there is no subinterval } [s_1, s_2] \subset [\tau, t] \text{ such that } x(s_1) \in \text{int}(\overline{A_{j,l}^{k,m} B_{j,l}^{k,m}}) \text{ and } (x(s) - x(s_1)) \cdot n_{\overline{A_{j,l}^{k,m} B_{j,l}^{k,m}}} < 0 \text{ for } s \in (s_1, s_2)\}$ . Note that  $A_j^{k-1, i} \widetilde{B_j^{k-1, i}} \subset \cap_{m=0}^i B_{r_m}(y_m)$  with  $r_m \leq \frac{\sqrt{2}+1}{16\pi} r_{m-1}$  for  $m = 1, 2, \dots, i$  by (4.35). By the definition of  $A_j^{k-1, i} \widetilde{B_j^{k-1, i}}$ , we have

$$A_j^{k-1, i} \widetilde{B_j^{k-1, i}} \subset \cup_{m=i}^{i+1} \cup_{l=1}^\infty \overline{A_{j,l}^{k,m} B_{j,l}^{k,m}} \cup X_{A_j^{k-1, i} B_j^{k-1, i}}, \quad (4.39)$$

Define the  $k$ th approximation  $\widetilde{AB}^{(k)}$  of  $\widetilde{AB}$  as follows.  $\widetilde{AB}^{(k)} = \cup_{i=1}^{k+1} \cup_{m=i}^{i+1} \cup_{j=1}^\infty \cup_{l=1}^\infty \overline{A_{j,l}^{k,m} B_{j,l}^{k,m}} \cup (\cup_{i=1}^k \cup_{j=1}^\infty X_{A_j^{k-1, i} B_j^{k-1, i}}) \cup (\cup_{i=0}^{k-2} X_{AB}^i)$ .

For the sake of simplicity, let us re-numerate the open straight line segments  $\overline{A_{j,l}^{k,m} B_{j,l}^{k,m}}$  (boundary "segments"  $\overline{A_{j,l}^{k,m} B_{j,l}^{k,m}}$ ) for  $1 \leq m \leq k+1$ , where  $l = 1, \dots, ; j = 1, \dots,$ , as  $\overline{A_j^{k,m} B_j^{k,m}}$  ( $A_j^{k,m} B_j^{k,m}$ ) for

$1 \leq m \leq k+1$  for fixed  $k$ . Denote  $\cup_{i=1}^k \cup_{j=1}^{\infty} X_{A_j^{k-1,i} B_j^{k-1,i}}$  by  $X_{AB}^{k-1}$ . Thus by the virtue of (4.36)-(4.38), we obtain that the  $k$ th family of open straight line segments and  $k-1$ th family of open straight line segments enjoy the following inequality

$$\begin{aligned} & \sum_{m=1}^{k+1} \sum_{j=1}^{\infty} L(\overline{A_j^{k,m} B_j^{k,m}}) + \mathcal{H}^1(X_{AB}^{k-1}) \\ & \leq \left[ \sup_{1 \leq m \leq k} \sup_{1 \leq j < \infty} (1 + 2\eta L^\gamma(\overline{A_j^{k-1,m} B_j^{k-1,m}})) \right] \sum_{m=1}^k \sum_{j=1}^{\infty} L(\overline{A_j^{k-1,m} B_j^{k-1,m}}). \end{aligned} \quad (4.40)$$

Furthermore, by (4.40)

$$\begin{aligned} L(\widetilde{AB}^{(k)}) &= \sum_{m=1}^{k+1} \sum_{j=1}^{\infty} L(\overline{A_j^{k,m} B_j^{k,m}}) + \sum_{j=0}^{k-1} \mathcal{H}^1(X_{AB}^j) \\ &\leq \left[ \sup_{1 \leq m \leq k} \sup_{1 \leq j < \infty} (1 + 2\eta L^\gamma(\overline{A_j^{k-1,m} B_j^{k-1,m}})) \right] \sum_{m=1}^k \sum_{j=1}^{\infty} L(\overline{A_j^{k-1,m} B_j^{k-1,m}}) + \sum_{j=0}^{k-2} \mathcal{H}^1(X_{AB}^j) \\ &\leq \left[ \sup_{1 \leq m \leq k} \sup_{1 \leq j < \infty} (1 + 2\eta L^\gamma(\overline{A_j^{k-1,m} B_j^{k-1,m}})) \right] L(\widetilde{AB}^{(k-1)}) \\ &\leq \prod_{i=2}^k \left[ \sup_{1 \leq m \leq i} \sup_{1 \leq j < \infty} (1 + 2\eta L^\gamma(\overline{A_j^{i-1,m} B_j^{i-1,m}})) \right] L(\widetilde{AB}^{(1)}), \end{aligned} \quad (4.41)$$

$$\begin{aligned} \sup_{1 \leq m \leq k+1} \sup_{1 \leq j < \infty} L(\overline{A_j^{k,m} B_j^{k,m}}) &\leq \prod_{i=2}^k \sup_{1 \leq m \leq i} \sup_j \frac{(1 + 2\eta L^\gamma(\overline{A_j^{i-1,m} B_j^{i-1,m}}))}{2} L(\widetilde{AB}^{(1)}) \\ &\leq \prod_{i=2}^k \sup_{1 \leq m \leq i} \sup_{1 \leq j < \infty} \frac{(1 + 2\eta L^\gamma(\overline{A_j^{i-1,m} B_j^{i-1,m}}))}{2} \frac{(1 + 2\eta L^\gamma(\overline{AB}))}{2} L(\overline{AB}), \end{aligned} \quad (4.42)$$

where  $k \geq 2$ .

$$\widetilde{AB} \subset \cup_{m=1}^{k+1} \cup_{j=1}^{\infty} \widetilde{A_j^{k,m} B_j^{k,m}} \cup (\cup_{i=0}^{k-1} X_{AB}^i) \equiv \widetilde{AB}^k, \quad \widetilde{AB}^{k-1} \subset \widetilde{AB}^k. \quad (4.43)$$

Let  $a_k = L(\widetilde{AB}^{(k)})$  and  $b_k = \sup_m \sup_j L(\overline{A_j^{k,m} B_j^{k,m}})$ . (4.41) and (4.42) can be rewritten as

$$a_k \leq \prod_{i=1}^k (1 + 2\eta b_i^\gamma) a_0, \quad (4.44)$$

$$b_k \leq \prod_{i=1}^k \frac{1}{2} (1 + 2\eta b_i^\gamma) a_0, \quad (4.45)$$

where  $k \geq 1$  and  $a_0 = b_0 = L(\overline{AB})$ . It easily follows from (4.13) that as long as  $a_0 \leq \min\{2\kappa^{\frac{\alpha}{2}} M^{1+\frac{\alpha}{2}}, (\frac{1}{4\eta})^{\frac{1}{\gamma}}\}$ , i.e.,  $L(\overline{AB}) \leq \min\{2\kappa^{\frac{\alpha}{2}} M^{1+\frac{\alpha}{2}}, (\frac{1}{4\eta})^{\frac{1}{\gamma}}\}$ , then  $b_k \leq (\frac{1}{4\eta})^{\frac{1}{\gamma}}$  for any  $k \in \mathbb{N}$ . Hence  $b_k \leq (\frac{3}{4})^{k-1} (\frac{1}{4\eta})^{\frac{1}{\gamma}}$ . Thus,  $\ln(a_k) \leq \sum_{i=0}^{k-1} \ln(1+2\eta b_i^\gamma) + \ln(a_0) \leq \sum_{i=0}^{k-1} 2\eta b_i^\gamma + \ln(a_0) \leq \ln(a_0) + \frac{1}{2(1-(\frac{3}{4})^{\frac{1}{\gamma}})}$ . Therefore  $\lim_{k \rightarrow \infty} L(\widetilde{AB}^{(k)}) \leq$

$\frac{1}{e^{2(1-(\frac{3}{4})^{\frac{1}{\gamma}})}} L(\overline{AB})$ . Note that for any given  $k_1, k_2 \geq K$  with  $K \in \mathbb{N}$ ,  $\text{dist}(\widetilde{AB}^{(k_1)}, \widetilde{AB}^{(k_2)}) \leq \sum_{j=K}^{\infty} \eta b_j^{1+\gamma} \leq \frac{1}{1-(\frac{3}{4})^{1+\gamma}} (\frac{3}{4})^{K(1+\gamma)}$ . Define  $\widetilde{AB}^{(\infty)} = \lim_{k \rightarrow \infty} \widetilde{AB}^{(k)} \equiv \{y \in \mathbb{R}^2 \mid y = \lim_{k \rightarrow \infty} x_k \text{ where } x_k \in \widetilde{AB}^{(k)}\}$ .

Observe that for each  $k \in \mathbb{N}$ ,  $m = 1, \dots, k+1$ ,  $A_j^{k,m}, B_j^{k,m} \in \widetilde{AB}^{(k)}$ , and  $\text{dist}(\overline{A_j^{k,m} B_j^{k,m}}, \overline{A_j^{k,m} B_j^{k,m}}) \leq 2\eta L^{1+\gamma}(\overline{A_j^{k,m} B_j^{k,m}}) \leq 2\eta (\frac{3}{4})^{\gamma k} L(\overline{AB})$ .  $\text{dist}(\widetilde{AB}^{(k)}, \widetilde{AB}^k) \leq 2\eta (\frac{3}{4})^{\gamma k} L(\overline{AB})$ . Therefore  $\widetilde{AB}^\infty = \lim_{k \rightarrow \infty} \widetilde{AB}^k \equiv$

$\{y \in \mathbb{R}^2 \mid y = \lim_{k \rightarrow \infty} x_k \text{ where } x_k \in \widetilde{AB}^k\}$  and  $\widetilde{AB}^\infty = \widetilde{AB}^{(\infty)}$ . Obviously  $\widetilde{AB} \subset \widetilde{AB}^\infty$ . By the definition of the Hausdorff measure [21],  $\mathcal{H}^1(\widetilde{AB}) \leq \lim_{k \rightarrow \infty} L(\widetilde{AB}^{(k)}) \leq e^{\frac{1}{2(1-(\frac{3}{4})^{\frac{1}{\gamma}})}} L(\widetilde{AB})$ . This concludes the claim.

Case 2.  $t < M$ .

We repeat using the concept of "boundary of convex hull" to give a primitive estimate of  $\Gamma(t) \cap B_r(y)$  where  $r \leq M_1 \equiv \frac{1}{8} \min\{M, (\frac{1}{2\xi})^{\frac{1}{\beta}}\}$  and  $y \in \mathbb{R}^2$ . The boundary of the convex hull of  $\Gamma(t) \cap B_r(y)$  is denoted by  $\text{bch}(\Gamma(t) \cap B_r(y))$ . By Lemma 4.1, we have

$$L(\text{bch}(\Gamma(t) \cap B_r(y))) \leq 2\pi r. \quad (4.46)$$

Note that  $\Omega(t) \cap \text{bch}(\Gamma(t) \cap B_r(y))$  is an open set relative to the convex curve  $\text{bch}(\Gamma(t) \cap B_r(y))$ , which consists of countably many disjoint open straight line segments of  $\text{bch}(\Gamma(t) \cap B_r(y))$ . The end points  $A_j, B_j$  of the  $j$ -th open straight line segment  $\text{int}(\overline{A_j B_j})$  are on the moving front  $\Gamma(t) = \partial\Omega(t)$ . That is  $\Omega(t) \cap \text{bch}(\Gamma(t) \cap B_r(y)) = \cup_j^\infty \text{int}(\overline{A_j B_j})$ . Moreover  $\text{bch}(\Gamma(t) \cap B_r(y)) \setminus \Omega(t)$  is closed subset of  $\text{bch}(\Gamma(t) \cap B_r(y))$ . It follows from (4.46) that

$$\sum_j^\infty L(\overline{A_j B_j}) + \mathcal{H}^1(\text{bch}(\Gamma(t) \cap B_r(y)) \setminus \Omega(t)) \leq 2\pi r. \quad (4.47)$$

Since every point on  $\Gamma(t)$  is connected to  $\Gamma(0)$  by a boundary characteristic path, which intercepts with at least one straight line segment by Lemma 4.3. Denote by  $\widetilde{A_j B_j}$  the set of points on the boundary, which has one backward boundary characteristic path intercepted with  $\overline{A_j B_j}$  from one side as defined in Case 1, then

$$\Gamma(t) \cap B_r(y) \subset (\cup_j^\infty \widetilde{A_j B_j}) \cup (\text{bch}(\Gamma(t) \cap B_r(y)) \setminus \Omega(t)). \quad (4.48)$$

There are four scenarios to estimate  $L(\widetilde{A_j B_j})$ :

- (i)  $L(A_j B_j) \leq \kappa^{\frac{\alpha}{2}} t^{1+\frac{\alpha}{2}}$ ,
- (ii)  $\kappa^{\frac{\alpha}{2}} t^{1+\frac{\alpha}{2}} \leq L(A_j B_j) \leq \frac{C_1 t}{12}$ ;
- (iii)  $\frac{C_1 t}{12} \leq L(A_j B_j) \leq 8C_2 t$ ,
- (iv)  $8C_2 t \leq L(A_j B_j) \leq M_1$ .

In scenario (i), we argue in the same way as in Case 1

$$L(\widetilde{A_j B_j}) \leq e^{\frac{1}{2(1-(\frac{3}{4})^{\frac{1}{\gamma}})}} L(\overline{A_j B_j}). \quad (4.49)$$

In scenario (ii), repeating the argument in step 1 of Case 1, we obtain that the convex hull  $\text{cvh}(\widetilde{A_j B_j})$  of  $\widetilde{A_j B_j}$  is on one side of the straight line segment  $\overline{A_j B_j}$ . For any point  $C \in \widetilde{A_j B_j}$ , there is a backward boundary characteristic path  $x : [0, t] \rightarrow \mathbb{R}^2$  issued from  $C$  intersected with  $\overline{A_j B_j}$  from non-negative side as  $x(\tau_j)$  at time  $\tau_j$  and

$$L(x[\tau_j, t]) \leq \frac{L^2(\overline{A_j B_j})}{3C_1 t} + 8\kappa C_2^\alpha (L(\overline{A_j B_j}))^{\frac{1+\alpha}{1+\frac{\alpha}{2}}}. \quad (4.50)$$

Then we repeat the argument in step 2 of Case 1 for a finite number of approximation  $\widetilde{A_j B_j}^{(k)}$  of  $\widetilde{A_j B_j}$  until the length  $L(A_i^k B_i^k)$  of any elementary open straight line segment  $A_i^k B_i^k$  in  $\widetilde{A_j B_j}^{(k)}$  is less than  $\kappa^{\frac{\alpha}{2}} t^{1+\frac{\alpha}{2}}$ , by (4.50) and (4.49), we get

$$L(\widetilde{A_j B_j}) \leq e^{\frac{1}{1-(\frac{3}{4})^{\frac{1}{\gamma}}}} L(\overline{A_j B_j}). \quad (4.51)$$

In scenario (iii), by (4.51), using a covering argument, we have

$$L(\widetilde{A_j B_j}) \leq C e^{\frac{1}{1-(\frac{3}{4})^{\frac{1}{\gamma}}}} L(\overline{A_j B_j}). \quad (4.52)$$

where  $C = C(\frac{C_2}{C_1})$  is a constant.

In scenario (iv), by the assumption in Theorem 4.4, we obtain that the convex hull  $\widetilde{\text{cvh}}(\widetilde{A_j B_j})$  of  $\widetilde{A_j B_j}$  is on one side of the straight line segment  $\overline{A_j B_j}$ . For any point  $C \in \widetilde{A_j B_j}$ , there is a backward boundary characteristic path  $x : [0, t] \rightarrow \mathbb{R}^2$  issued from  $C$  intersected with  $\overline{A_j B_j}$  from non-negative side as  $x(\tau_j)$  at time  $\tau_j$  and

$$L(x[\tau_j, t]) \leq 2C_2 t + \xi(L(\overline{A_j B_j}))^{1+\beta}. \quad (4.53)$$

Repeating the argument in step 2 of Case 1 for a finite number of approximations  $\widetilde{A_j B_j}^{(k)}$  of  $\widetilde{A_j B_j}$  until the length  $L(A_i^k B_i^k)$  of any elementary open straight line segment  $A_i^k B_i^k$  in  $\widetilde{A_j B_j}^{(k)}$  is less than  $8C_2 t$ , by (4.53) and (4.52), we have

$$L(\widetilde{A_j B_j}) \leq C e^{\frac{1}{2^{1-(\frac{3}{4})^\beta}}} e^{\frac{1}{1-(\frac{3}{4})^\beta}} L(\overline{A_j B_j}), \quad (4.54)$$

where  $C = C(\frac{C_2}{C_1})$  is a constant. The situation  $r \geq M_1$  follows by using a covering argument, and Theorem 4.4 is concluded.

**Remark 4.5.** In the above proof, the so-called Jordan curve property is crucial in our argument. It seems nontrivial to show a three-dimensional version of Theorem 4.4.

For the purpose of a priori regularity estimate of temperature, we have to weaken the assumption on the Lipschitz constant. From now on, let us assume that the velocity function  $f(x, t)$  satisfies:

$$|f(x_1, t) - f(x_2, t)| \leq \kappa(t) |x_1 - x_2|^\alpha, \quad \text{where } \kappa \in L^p(0, T), \quad (4.55)$$

Denote by  $\mathcal{K} = \|\kappa\|_{L^p(0, T)}$  for  $p > 1$ . Repeating the proof of Lemma 4.2, Lemma 4.3, Theorem 4.4 by slightly changing the indices, we have the following statements:

**Lemma 4.6.** *Suppose that  $\partial\{\phi(x) < -1\} = \partial\{\phi(x) < 0\} = \partial\{\phi(x) > 0\} = \partial\{\phi(x) > 1\}$  is of finite co-dimension one Hausdorff measure, and  $0 < C_1 \leq f^\epsilon(x, t) \leq C_2$ ,  $\|f^\epsilon(x, t) - f(x, t)\|_{L^p(0, T; C^\alpha(\mathbb{R}^n))} \rightarrow 0$  as  $\epsilon \rightarrow 0$ ,  $|f^\epsilon(x_1, t) - f^\epsilon(x_2, t)| \leq \kappa(t) |x_1 - x_2|^\alpha$ , where  $\kappa \in L^p(0, T)$ . For each  $\epsilon > 0$ ,  $f^\epsilon$  is Lipschitz continuous with respect to  $x \in \mathbb{R}^n$ . Then the unique  $L^\infty$  solution  $\varphi^\epsilon$  of*

$$\begin{aligned} \varphi_t^\epsilon + f^\epsilon(x, t)|D\varphi^\epsilon| &= 0, & x \in \mathbb{R}^n, t > 0, \\ \varphi(x, 0) &= \phi(x), \end{aligned}$$

converge to a  $L^\infty$  solution  $\varphi$  of

$$\begin{aligned} \varphi_t + f(x, t)|D\varphi| &= 0, & x \in \mathbb{R}^n, t > 0, \\ \varphi(x, 0) &= \phi(x); \end{aligned}$$

in the almost everywhere sense by the definition of  $L^\infty$  solution as  $\epsilon \rightarrow 0$ , and the moving front  $\partial\{\varphi^\epsilon < 0\}$  of  $\varphi^\epsilon$ , which is a Lipschitz graph in time-space with Lipschitz constants independent of  $\epsilon$ , converge to the moving front  $\partial\{\varphi < 0\} = \partial\{\varphi > 0\}$  in the sense of (3.8).

**Lemma 4.7.** *Assume that  $|f(x_1, t) - f(x_2, t)| \leq \kappa(t) |x_1 - x_2|^\alpha$ , where  $\kappa \in L^p(0, T)$ . Denote  $\|\kappa\|_{L^p(0, T)}$  by  $\mathcal{K}$ . For any point  $A$  on the moving front  $\Gamma(t_2) = \partial\Omega(t_2)$  at time  $t_2 > 0$ , any characteristic path  $x : [t_1, t_2] \rightarrow \mathbb{R}^2$  with  $|\dot{x}| \leq f(x, t)$  for  $t \in [t_1, t_2] \subset [0, T]$ ,  $x(t_2) = A$ , and  $x(t_1) = B$  must be a boundary characteristic path, i.e.  $|\dot{x}| = f(x, t)$  for almost all  $t \in [t_1, t_2]$  and  $x(t) \in \Gamma(t)$  for all  $t \in [t_1, t_2]$ . Furthermore,*

$$|L(\mathcal{C}) - L(\overline{AB})| \leq 2^\alpha \mathcal{K} (C_2)^\alpha |t_1 - t_2|^{1+\alpha-\frac{1}{p}}, \quad (4.56)$$

where  $\overline{AB}$  denotes the straight line segment between the point  $A$  and the point  $B$ , and  $\mathcal{C}$  denotes the boundary characteristic path connecting  $A$  and  $B$ .

**Theorem 4.8.** *Let the moving front  $\Gamma(t)$  be the boundary of reachable set  $\Omega(t)$  driven by normal velocity  $f(x, t)$ , which is Hölder continuous in space with  $|f(x_1, t) - f(x_2, t)| \leq \kappa(t) |x_1 - x_2|^\alpha$ ,  $0 < C_1 \leq f(x, t) \leq C_2$ , and  $\kappa \in L^p(0, T)$  for sufficiently large  $p > 1$  such that  $\frac{1}{p} \leq \frac{\alpha}{2}$ . Denote  $\|\kappa\|_{L^p(0, T)}$  by  $\mathcal{K}$ . Assume that the*

initial front  $\Gamma(0) = \cup_{i=1}^m \Gamma_i^0$ , where  $\Gamma_i^0$  is the curve satisfying a local flattening condition: for any points  $A$  and  $B$  on  $\Gamma_i^0$  with  $L(\overline{AB}) < 1$ ,  $\sup_{x \in \overline{AB}} \text{dist}(x, \overline{AB}) \leq \xi L(\overline{AB})^{1+\beta}$ , where  $\overline{AB}$  is the connecting part of  $\Gamma_i^0$  between the point  $A$  and  $B$ . Then for any  $t > 0$ , for any  $y_0 \in \mathbb{R}^2$ ,  $\Gamma(t)$  satisfies

$$\mathcal{H}^1(\Gamma(t) \cap B_r(y_0)) \leq C(m, \xi, \beta, \alpha, C_1, C_2)r, \quad \text{if } r \in (0, \mathcal{K}^{\frac{-2+\alpha^2}{\alpha(1+\frac{1}{p})-\frac{2}{p}}}]]; \quad (4.57)$$

$$\mathcal{H}^1(\Gamma(t) \cap B_r(y_0)) \leq C(m, \xi, \beta, \alpha, C_1, C_2)r^2 \mathcal{K}^{\frac{2-\alpha^2}{\alpha(1+\frac{1}{p})-\frac{2}{p}}}, \quad (4.58)$$

if  $r \geq \mathcal{K}^{\frac{-2+\alpha^2}{\alpha(1+\frac{1}{p})-\frac{2}{p}}}$ .

## 5 A-Priori Estimates of the Temperature

In this section, we show the Hölder regularity estimate independent of  $\epsilon$  for the temperature  $u^\epsilon$  of (3.4)-(3.6) with respect to  $x$ . Based upon Theorem 4.8 and Lemma 3.2, using a solution formula, we employ a decomposition argument to derive the desired estimate for  $u^\epsilon(x, t)$  and  $\nabla u^\epsilon(x, t)$ . Because of finite propagation speed of the moving front, the decomposition is conducted in such a way that the contribution from the parts far away from the point  $(x, t)$  is bounded by a coarse estimate of the moving front based upon Lemma 3.2, that of the parts close to  $(x, t)$  is estimated with the aid of Theorem 4.4. In our argument, finite speed of propagation of the moving front plays an essential role.

**Theorem 5.1.** *Assume for the initial front  $\Gamma(0) = \cup_{i=1}^m \Gamma_i^0$ , where  $\Gamma_i^0$  is the curve satisfying a local flattening condition: for any points  $A$  and  $B$  on  $\Gamma_i^0$  with  $L(\overline{AB}) < 1$ ,  $\sup_{x \in \overline{AB}} \text{dist}(x, \overline{AB}) \leq \xi L(\overline{AB})^{1+\beta}$ , where  $\overline{AB}$  is the connecting part of  $\Gamma_i^0$  between the point  $A$  and  $B$ .  $\|u_0^\epsilon\|_{C^{0,1}(\mathbb{R}^2)} \leq C$ . Then the temperature  $u^\epsilon$  of (3.4)-(3.6) enjoys*

$$\|u^\epsilon\|_{L^\infty(\mathbb{R}^2 \times [0, T])} \leq C, \quad (5.1)$$

$$|u^\epsilon(x, t_1) - u^\epsilon(x, t_2)| \leq C(\gamma)|t_1 - t_2|^\gamma, \quad \text{for any } \gamma \in (0, \frac{1}{2}) \quad (5.2)$$

$$|u^\epsilon(x_1, t) - u^\epsilon(x_2, t)| \leq C(\alpha) |x_1 - x_2|^\alpha \quad \text{for any } \alpha \in (0, 1). \quad (5.3)$$

for any  $x, x_1, x_2 \in \mathbb{R}^2$  and  $t_1, t_2 > 0$ .

Proof: By the representation formula for the solution of the heat equation, for any  $(x, t) \in \mathbb{R}^2 \times [0, T]$ ,

$$\begin{aligned} u^\epsilon(x, t) &= \int_{\mathbb{R}^2} \frac{1}{4\pi t} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy \\ &+ \int_0^t \int_{\mathbb{R}^2} \frac{1}{4\pi(t-s)} e^{-\frac{|x-y|^2}{4(t-s)}} (-\partial_t \chi_\epsilon(x, t, \phi^\epsilon)) dy ds \equiv u_1^\epsilon + u_2^\epsilon \end{aligned} \quad (5.4)$$

It is easy to check that

$$\|u_1^\epsilon\|_{L^\infty(\mathbb{R}^2 \times [0, T])} \leq \|u_0\|_{L^\infty(\mathbb{R}^2)}, \quad (5.5)$$

$$\|\nabla u_1^\epsilon\|_{L^\infty(\mathbb{R}^2 \times [0, T])} \leq \|\nabla u_0\|_{L^\infty(\mathbb{R}^2)}. \quad (5.6)$$

By the definition of  $\chi_\epsilon$ , we have

$$0 \leq u_2^\epsilon(x, t) = \iint_{\Gamma_{\varphi^\epsilon, \epsilon}^-} \frac{1}{4\pi(t-s)} e^{-\frac{|x-y|^2}{4(t-s)}} \frac{1}{\epsilon} dy ds. \quad (5.7)$$

Thus

$$\partial_{x_i} u_2^\epsilon(x, t) = \iint_{\Gamma_{\varphi^\epsilon, \epsilon}^-} \frac{(y_i - x_i)}{8\pi\epsilon(t-s)^2} e^{-\frac{|x-y|^2}{4(t-s)}} dy ds$$

$$= \sum_{k=-([\log_2 t]+1)}^{\infty} \iint_{\Gamma_{\varphi^\epsilon, \epsilon}^-(k)} \frac{(y_i - x_i)}{8\epsilon\pi(t-s)^2} e^{-\frac{|x-y|^2}{4(t-s)}} dy ds \quad (5.8)$$

where  $\Gamma_{\varphi^\epsilon, \epsilon}^-(k) = \Gamma_{\varphi^\epsilon, \epsilon}^- \cap \mathbb{R}^2 \times [t - 2^{-k}, t - 2^{-k-1}]$  and  $\Gamma_{\varphi^\epsilon, \epsilon}^-$  is defined as in (3.9).

To show (5.3), there are two steps. In Step 1, we show that for  $\epsilon < 1$

$$|\nabla u^\epsilon(x, t)| \leq C(Z^{\frac{7}{8}} + Z^{\frac{1}{4}} \log_2(Z+1) + \log_2(Z+1) + |\log_2 \delta| + 1) \quad (5.9)$$

where  $Z = [u^\epsilon]_{L^p(0, T; C^{0, \alpha_0}(\mathbb{R}^2))}$  for  $p$  and  $\alpha_0$  to be determined later, and  $\text{dist}(x, \partial\Gamma_{\varphi^\epsilon, \epsilon}^-(t)) = \delta > 0$  for  $t \in [0, T]$ . Without loss of generality, we may assume that  $\delta \leq 1$ . Otherwise, by an interior estimate of the parabolic equation,  $|\nabla u| \leq C$  where  $C$  is an absolute constant. In Step 2, exploiting (5.9), we conclude that  $\|u^\epsilon\|_{L^\infty(0, T; C^{0, \alpha}(\mathbb{R}^2))} < C(m, \xi, \beta, C_1, C_2, L, \alpha)$ .

Step 1: There are three cases to discuss: (I)  $(x, t) \in \{\chi_\epsilon(x, t, \varphi^\epsilon) = -1\}$ , (II)  $(x, t) \in \{\varphi^\epsilon > 0\} = \{\chi_\epsilon(x, t, \varphi^\epsilon) = 1\}$ , (III)  $(x, t) \in \{-1 < \chi_\epsilon(x, t, \varphi^\epsilon) < 1\}$ .

Let us focus on the case (I):  $(x, t) \in \{\chi_\epsilon(x, t, \varphi^\epsilon) = -1\}$ . In (5.8), we have to estimate each term in the following situations: (i)  $k \leq \log_2(Z^{\frac{7}{4}} + 1) + 1$ , (ii)  $\log_2(Z^{\frac{7}{4}} + 1) + 1 \leq k \leq \log_2(Z^{\frac{5}{2}} + 1) + 1$ , (iii)  $\log_2(Z^{\frac{5}{2}} + 1) + 1 \leq k < \infty$ .

For  $k \leq \log_2(Z^{\frac{7}{4}} + 1) + 1$ , by Lemma 3.2 and the definition of  $\Gamma_{\varphi^\epsilon, \epsilon}^-$ , we have

$$\mathcal{H}^3(\Gamma_{\varphi^\epsilon, \epsilon}^-(k) \cap A_j) \leq C\epsilon j 2^{-k} ((C_1)^{-2} + 1)^{\frac{1}{2}}, \quad (5.10)$$

where  $A_j = \{(y, s) \in \mathbb{R}^2 \times \mathbb{R}^+ \mid (j-1)(t-s)^{\frac{1}{2}} \leq |y-x| \leq j(t-s)^{\frac{1}{2}}, s < t\}$  and  $C$  is an absolute constant. Thus by the virtue of (5.10)

$$\begin{aligned} & \left| \iint_{\Gamma_{\varphi^\epsilon, \epsilon}^-(k)} \frac{(y_i - x_i)}{8\epsilon\pi(t-s)^2} e^{-\frac{|x-y|^2}{4(t-s)}} dy ds \right| \\ & \leq \iint_{\Gamma_{\varphi^\epsilon, \epsilon}^-(k)} \frac{|y_i - x_i|}{8\epsilon\pi(t-s)^2} e^{-\frac{|x-y|^2}{4(t-s)}} dy ds \\ & = \sum_{j=1}^{\infty} \iint_{\Gamma_{\varphi^\epsilon, \epsilon}^-(k) \cap A_j} \frac{|y_i - x_i|}{8\epsilon\pi(t-s)^2} e^{-\frac{|x-y|^2}{4(t-s)}} dy ds \\ & \leq \sum_{j=1}^{\infty} \frac{j 2^{-\frac{k}{2}}}{8\pi\epsilon 2^{-2(k+1)}} e^{-\frac{(j-1)^2}{4}} C\epsilon j 2^{-k} ((C_1)^{-2} + 1)^{\frac{1}{2}} \\ & \leq C(C_1, C_2) \sum_{j=1}^{\infty} j^2 e^{-\frac{(j-1)^2}{4}} \leq C(C_1, C_2) 2^{\frac{k}{2}}, \end{aligned} \quad (5.11)$$

Consequently

$$\begin{aligned} & \sum_{k=-([\log_2 t]-1)}^{[\log_2(Z^{\frac{7}{4}}+1)+1]} \left| \iint_{\Gamma_{\varphi^\epsilon, \epsilon}^-(k)} \frac{y_i - x_i}{8\epsilon\pi(t-s)^2} e^{-\frac{|x-y|^2}{4(t-s)}} dy ds \right| \\ & \leq \sum_{k=-([\log_2 t]-1)}^{[\log_2(Z^{\frac{7}{4}}+1)+1]} C(C_1, C_2) 2^{\frac{k}{2}} \leq C(C_1, C_2) 2^{\frac{[\log_2(Z^{\frac{7}{4}}+1)+1]}{2}} \\ & \leq C(C_1, C_2) (Z^{\frac{7}{8}} + 1). \end{aligned} \quad (5.12)$$

Note that by Theorem 4.8, selecting an  $\alpha_0 < 1$  and large  $p > 1$  so that  $\frac{2-\alpha_0^2}{\alpha_0(1+\frac{1}{p})-\frac{2}{p}} = \frac{9}{8}$  in (4.57) and (4.58), we have for  $s \in (0, t)$ ,

$$\mathcal{H}^1(\Gamma_{\varphi^\epsilon}^-(s) \cap B_r(0)) \leq C(m, \xi, \beta, C_1, C_2)r, \quad \text{for } r \leq Z^{-\frac{9}{8}}, \quad (5.13)$$

$$\mathcal{H}^1(\Gamma_{\varphi^\epsilon}^-(s) \cap B_r(0)) \leq C(m, \xi, \beta, C_1, C_2)r^2 Z^{\frac{9}{8}}, \quad \text{for } r \geq Z^{-\frac{9}{8}}. \quad (5.14)$$

Since  $C_1 \leq G(u^\epsilon) \leq C_2$ , by the definition of  $\Gamma_{\varphi^\epsilon, \epsilon}^-(s)$ , we have

$$\Gamma_{\varphi^\epsilon, \epsilon}^-(s) = \cup_{\tau \in [s-2\epsilon, s]} \Gamma_{\varphi^\epsilon}^-(\tau) \quad \text{and} \quad \text{dist}(\Gamma_{\varphi^\epsilon, \epsilon}^-(s), \Gamma_{\varphi^\epsilon}^-(s)) \leq C_2\epsilon. \quad (5.15)$$

Thus for any  $k$ , by (5.14), (5.15), and Lemma 3.2,

$$\mathcal{H}^3(\Gamma_{\varphi^\epsilon, \epsilon}^-(k) \cap A_j) \leq C\epsilon 2^{-k} C(m, \xi, \beta, C_1, C_2) [(j2^{-\frac{k}{2}})^2 Z^{\frac{9}{8}} + j2^{-\frac{k}{2}}], \quad \text{for } j \leq [2^{\frac{k}{2}} Z^{-\frac{3}{4}}], \quad (5.16)$$

$$\mathcal{H}^3(\Gamma_{\varphi^\epsilon, \epsilon}^-(k) \cap A_j) \leq C\epsilon j 2^{-k} ((C_1)^{-2} + 1)^{\frac{1}{2}}, \quad \text{for } j \geq [2^{\frac{k}{2}} Z^{-\frac{3}{4}}]. \quad (5.17)$$

Here  $A_j$  is defined as above. For  $\log_2(Z^{\frac{7}{4}} + 1) + 1 \leq k \leq \log_2(Z^{\frac{5}{2}} + 1) + 1$ , using of (5.16) and (5.17), we have

$$\begin{aligned} & \left| \iint_{\Gamma_{\varphi^\epsilon, \epsilon}^-(k)} \frac{(y_i - x_i)}{8\epsilon\pi(t-s)^2} e^{-\frac{|x-y|^2}{4(t-s)}} dy ds \right| \\ & \leq \iint_{\Gamma_{\varphi^\epsilon, \epsilon}^-(k)} \frac{|y_i - x_i|}{8\epsilon\pi(t-s)^2} e^{-\frac{|x-y|^2}{4(t-s)}} dy ds \\ & = \left( \sum_{j=1}^{[2^{\frac{k}{2}} Z^{-\frac{3}{4}}]} + \sum_{j=[2^{\frac{k}{2}} Z^{-\frac{3}{4}}]+1}^{\infty} \right) \iint_{\Gamma_{\varphi^\epsilon, \epsilon}^-(k) \cap A_j} \frac{|y_i - x_i|}{8\epsilon\pi(t-s)^2} e^{-\frac{|x-y|^2}{4(t-s)}} dy ds \\ & \leq \left( \sum_{j=1}^{[2^{\frac{k}{2}} Z^{-\frac{3}{4}}]} + \sum_{j=[2^{\frac{k}{2}} Z^{-\frac{3}{4}}]+1}^{\infty} \right) \frac{j2^{-\frac{k}{2}}}{8\pi\epsilon 2^{-2(k+1)}} e^{-\frac{(j-1)^2}{4}} \mathcal{H}^3(\Gamma_{\varphi^\epsilon, \epsilon}^-(k) \cap A_j) \\ & \leq \sum_{j=1}^{[2^{\frac{k}{2}} Z^{-\frac{3}{4}}]} \frac{j2^{-\frac{k}{2}}}{8\pi\epsilon 2^{-2(k+1)}} e^{-\frac{(j-1)^2}{4}} C(m, \xi, \beta, C_1, C_2) \epsilon j 2^{-\frac{3k}{2}} (1 + j2^{-\frac{k}{2}} Z^{\frac{9}{8}}) \\ & \quad + \sum_{j=[2^{\frac{k}{2}} Z^{-\frac{3}{4}}]+1}^{\infty} \frac{j2^{-\frac{k}{2}}}{8\pi\epsilon 2^{-2(k+1)}} e^{-\frac{(j-1)^2}{4}} C\epsilon j 2^{-k} ((C_1)^{-2} + 1)^{\frac{1}{2}} \\ & \leq C(2^{-\frac{k}{2}} Z^{\frac{9}{8}} \sum_{j=1}^{[2^{\frac{k}{2}} Z^{-\frac{3}{4}}]} j^3 e^{-\frac{(j-1)^2}{4}} + \sum_{j=1}^{[2^{\frac{k}{2}} Z^{-\frac{3}{4}}]} j^2 e^{-\frac{(j-1)^2}{4}} + 2^{\frac{k}{2}} \sum_{j=[2^{\frac{k}{2}} Z^{-\frac{3}{4}}]+1}^{\infty} j e^{-\frac{(j-1)^2}{4}}) \\ & \leq C(Z^{\frac{1}{4}} + 1), \end{aligned} \quad (5.18)$$

where the last inequality is due to the fact that  $2^k \geq Z^{\frac{7}{4}}$  and  $2^{\frac{k}{2}} Z^{-\frac{3}{4}} \geq 2^{\frac{k}{14}}$ . Thus by (5.18)

$$\begin{aligned} & \sum_{k=[\log_2(Z^{\frac{5}{2}}+1)]+1}^{[\log_2(Z^{\frac{5}{2}}+1)]+1} \left| \iint_{\Gamma_{\varphi^\epsilon, \epsilon}^-(k)} \frac{y_i - x_i}{8\epsilon\pi(t-s)^2} e^{-\frac{|x-y|^2}{4(t-s)}} dy ds \right| \\ & \leq C(m, \xi, \beta, C_1, C_2) (Z^{\frac{1}{4}} + 1) \log_2(Z + 1) \end{aligned} \quad (5.19)$$

for  $(x, t) \in \{\varphi^\epsilon < 0\}$ .

Let us focus on situation (iii). Without loss of generality, let us assume that  $\log_2(Z^{\frac{5}{2}} + 1) + 1 \leq -\log_2(t - \gamma(x, \varphi^\epsilon) - 2\epsilon)$ . Otherwise, the argument is much simpler. Since  $G(u^\epsilon) \geq C_1$ ,  $\delta > C_1(t - \gamma(x, \varphi^\epsilon) - 2\epsilon)$ . For  $k \in [\log_2(Z^{\frac{5}{2}} + 1) + 1, -4\log_2(t - \gamma(x, \varphi^\epsilon) - 2\epsilon)]$ , by (5.13), (5.15), and Lemma 3.2, we have

$$\mathcal{H}^3(\Gamma_{\varphi^\epsilon, \epsilon}^-(k) \cap A_j) \leq C(m, \xi, \beta, C_1, C_2, L) j 2^{-\frac{3k}{2}} \epsilon, \quad \text{for } j \leq [2^{\frac{k}{2}} Z^{-\frac{9}{8}}], \quad (5.20)$$

$$\mathcal{H}^3(\Gamma_{\varphi^\epsilon, \epsilon}^-(k) \cap A_j) \leq C\epsilon j 2^{-k} ((C_1)^{-2} + 1)^{\frac{1}{2}}, \quad \text{for } j \geq [2^{\frac{k}{2}} Z^{-\frac{9}{8}}]. \quad (5.21)$$

where  $A_j = \{(y, s) \in \mathbb{R}^2 \times \mathbb{R}^+ \mid (j-1)(t-s)^{\frac{1}{2}} \leq |y-x| \leq j(t-s)^{\frac{1}{2}}, s < t\}$ . By the virtue of (5.20), (5.21), we have

$$\left| \iint_{\Gamma_{\varphi^\epsilon, \epsilon}^-(k)} \frac{(y_i - x_i)}{8\epsilon\pi(t-s)^2} e^{-\frac{|x-y|^2}{4(t-s)}} dy ds \right|$$

$$\begin{aligned}
& \leq \iint_{\Gamma_{\varphi^\epsilon, \epsilon}^-(k)} \frac{|y_i - x_i|}{8\epsilon\pi(t-s)^2} e^{-\frac{|x-y|^2}{4(t-s)}} dy ds \\
& = \left( \sum_{j=1}^{\lfloor 2^{\frac{k}{2}} Z^{-\frac{9}{8}} \rfloor} + \sum_{j=\lfloor 2^{\frac{k}{2}} Z^{-\frac{9}{8}} \rfloor + 1}^{\infty} \right) \iint_{\Gamma_{\varphi^\epsilon, \epsilon}^-(k) \cap A_j} \frac{|y_i - x_i|}{8\epsilon\pi(t-s)^2} e^{-\frac{|x-y|^2}{4(t-s)}} dy ds \\
& \leq \left( \sum_{j=1}^{\lfloor 2^{\frac{k}{2}} Z^{-\frac{9}{8}} \rfloor} + \sum_{j=\lfloor 2^{\frac{k}{2}} Z^{-\frac{9}{8}} \rfloor + 1}^{\infty} \right) \frac{j2^{-\frac{k}{2}}}{8\pi\epsilon 2^{-2(k+1)}} e^{-\frac{(j-1)^2}{4}} \mathcal{H}^3(\Gamma_{\varphi^\epsilon, \epsilon}^-(k) \cap A_j) \\
& \leq \sum_{j=1}^{\lfloor 2^{\frac{k}{2}} Z^{-\frac{9}{8}} \rfloor} \frac{j2^{-\frac{k}{2}}}{8\pi\epsilon 2^{-2(k+1)}} e^{-\frac{(j-1)^2}{4}} C(m, \xi, \beta, C_1, C_2) \epsilon j 2^{-\frac{3k}{2}} \\
& \quad + \sum_{j=\lfloor 2^{\frac{k}{2}} Z^{-\frac{9}{8}} \rfloor + 1}^{\infty} \frac{j2^{-\frac{k}{2}}}{8\pi\epsilon 2^{-2(k+1)}} e^{-\frac{(j-1)^2}{4}} C \epsilon j 2^{-k} ((C_1)^{-2} + 1)^{\frac{1}{2}} \\
& \leq C \left( \sum_{j=1}^{\lfloor 2^{\frac{k}{2}} Z^{-\frac{9}{8}} \rfloor} j^2 e^{-\frac{(j-1)^2}{4}} + 2^{\frac{k}{2}} \sum_{j=\lfloor 2^{\frac{k}{2}} Z^{-\frac{9}{8}} \rfloor + 1}^{\infty} j^2 e^{-\frac{(j-1)^2}{4}} \right) \\
& \leq C(m, \xi, \beta, C_1, C_2), \tag{5.22}
\end{aligned}$$

where the last inequality is ensured by the fact that  $2^k \geq Z^{\frac{5}{2}}$  and  $2^{\frac{k}{2}} Z^{-\frac{9}{8}} \geq 2^{\frac{k}{20}}$ . Thus by (5.22)

$$\begin{aligned}
& \sum_{k=\lfloor \log_2(Z^{\frac{5}{2}}+1)+1 \rfloor}^{\lfloor -4\log_2(t-\gamma(x, \varphi^\epsilon)-2\epsilon) \rfloor + 1} \left| \iint_{\Gamma_{\varphi^\epsilon, \epsilon}^-(k)} \frac{y_i - x_i}{8\epsilon\pi(t-s)^2} e^{-\frac{|x-y|^2}{4(t-s)}} dy ds \right| \\
& \leq C(m, \xi, \beta, C_1, C_2) (|\log_2 \delta| + 1). \tag{5.23}
\end{aligned}$$

Observe that, due to the fact that  $G(u^\epsilon) \geq C_1$ , we have

$$\bar{C}_x \cap \Gamma_{\varphi^\epsilon, \epsilon}^- = \emptyset, \tag{5.24}$$

where  $\bar{C}_x = \{(y, s) \in \mathbb{R}^2 \times \mathbb{R}^+ \mid |y - x| < C_1(s - \gamma(x, \varphi^\epsilon) - 2\epsilon)\}$ . For  $k \geq \lfloor 4\log_2(t - \gamma(x, \varphi^\epsilon) - 2\epsilon) \rfloor$ , by (5.24)

$$\mathcal{H}^3(\Gamma_{\varphi^\epsilon, \epsilon}^-(k) \cap A_j) = 0, \quad \text{if } j < J(k) \tag{5.25}$$

where  $J(k) = \lfloor \frac{C_1(-2^{-k} + t - \gamma(x, \varphi^\epsilon) - 2\epsilon)}{2^{-\frac{k}{2}}} \rfloor \geq C_1 2^{\frac{k}{4}} - 1$  and  $A_j$  is defined as above.

Repeating the same argument as above, we obtain

$$\begin{aligned}
& \left| \iint_{\Gamma_{\varphi^\epsilon, \epsilon}^-(k)} \frac{(y_i - x_i)}{8\epsilon\pi(t-s)^2} e^{-\frac{|x-y|^2}{4(t-s)}} dy ds \right| \\
& \leq \iint_{\Gamma_{\varphi^\epsilon, \epsilon}^-(k)} \frac{|y_i - x_i|}{8\epsilon\pi(t-s)^2} e^{-\frac{|x-y|^2}{4(t-s)}} dy ds \\
& \leq \sum_{j=J(k)}^{\infty} \iint_{\Gamma_{\varphi^\epsilon, \epsilon}^-(k) \cap A_j} \frac{|y_i - x_i|}{8\epsilon\pi(t-s)^2} e^{-\frac{|x-y|^2}{4(t-s)}} dy ds \\
& \leq \sum_{j=J(k)}^{\infty} \frac{j2^{-\frac{k}{2}}}{8\pi\epsilon 2^{-2(k+1)}} e^{-\frac{(j-1)^2}{4}} C \epsilon j 2^{-k} ((C_1)^{-2} + 1)^{\frac{1}{2}} \\
& \leq C((C_1)^{-2} + 1)^{\frac{1}{2}} 2^{\frac{k}{2}} \sum_{j=J(k)}^{\infty} j^2 e^{-\frac{(j-1)^2}{4}} \\
& \leq C((C_1)^{-2} + 1)^{\frac{1}{2}} 2^k e^{-\frac{(C_1 2^{\frac{k}{4}} - 1)^2}{4}} \leq C(C_1, C_2) 2^{-k}, \tag{5.26}
\end{aligned}$$

Hence

$$\begin{aligned} & \sum_{k=-4 \log_2(t-\gamma(x, \varphi^\epsilon)-2\epsilon)+1}^{\infty} \left| \iint_{\Gamma_{\phi^\epsilon, \epsilon}^-(k)} \frac{y_i - x_i}{8\epsilon\pi(t-s)^2} e^{-\frac{|x-y|^2}{4(t-s)}} dy ds \right| \\ & \leq C(C_1, C_2) \sum_{k=-4 \log_2 \delta}^{\infty} 2^{-k} \leq C(C_1, C_2) \delta^4. \end{aligned} \quad (5.27)$$

With (5.12), (5.19), (5.23), and (5.27) at hand, we obtain

$$|\nabla u^\epsilon(x, t)| \leq C(m, \xi, \beta, C_1, C_2) (Z^{\frac{7}{8}} + (Z^{\frac{1}{4}} + 1) \log_2(Z + 1) + |\log_2 \delta| + 1), \quad (5.28)$$

where  $\text{dist}(x, \partial\Gamma_{\varphi^\epsilon, \epsilon}^-(t) \setminus \Gamma_{\varphi^\epsilon}^-(t)) = \text{dist}(x, \Gamma_{\varphi^\epsilon}^-(t - 2\epsilon)) = \delta \in (0, 1)$  for  $(x, t) \in \{\chi_\epsilon(x, t, \varphi^\epsilon) = -1\}$ .

In case (II), the proof is the same as in case (I).

In case (III) simply observe that

$$\iint_{C(x, t, \delta)} \frac{(y_i - x_i)}{8\pi\epsilon(t-s)^2} e^{-\frac{|x-y|^2}{4(t-s)}} dy ds = 0, \quad (5.29)$$

where  $C(x, t, \delta) = \{(y, s) \in \mathbb{R}^2 \times \mathbb{R} + \|y - x\| \leq C_1(t - s), t - s \leq \frac{C_1}{(C_1^2 + 1)^{\frac{1}{2}}} \delta\}$ . Thus we can invoke again the same argument in case (I) to establish (5.9).

Step 2. For any  $(x, t) \in \Gamma_{\varphi^\epsilon, \epsilon}^-$ , let  $\mu(x, t) = \text{dist}(x, \Gamma_{\varphi^\epsilon}^-(t))$  and  $\eta(x, t) = \text{dist}(x, \partial\Gamma_{\varphi^\epsilon, \epsilon}^- \setminus \Gamma_{\varphi^\epsilon}^-) = \text{dist}(x, \Gamma_{\varphi^\epsilon}^-(t - 2\epsilon))$ . Given the fact that  $C_1 \leq G(u^\epsilon) \leq C_2$ , we have

$$C_1(t - \gamma(x, \varphi^\epsilon)) \leq \mu(x, t) \leq C_2(t - \gamma(x, \varphi^\epsilon)), \quad (5.30)$$

$$C_1(2\epsilon + \gamma(x, \varphi^\epsilon) - t) \leq \eta \leq C_2(2\epsilon + \gamma(x, \varphi^\epsilon) - t), \quad (5.31)$$

It follows from (5.9), the definition of  $\Gamma_{\varphi^\epsilon, \epsilon}^-$ , (5.30), (5.31), Lemma 3.2, and decay of  $\nabla u^\epsilon$  at infinity that

$$\|\nabla u^\epsilon(x, t)\|_{L^q(\mathbb{R}^2 \times [0, T])} \leq C(Z^{\frac{7}{8}} + Z^{\frac{1}{4}} \log_2(Z + 1) + \log_2(Z + 1) + 1), \quad (5.32)$$

where  $C = C(m, \xi, \beta, C_1, C_2, q)$  for all  $\epsilon$ . Therefore we choose  $q_0$  sufficiently large so that  $q_0 > p$  and  $1 - \frac{n}{q_0} \geq \alpha_0$  in (5.32). By the Sobolev imbedding theorem, we obtain

$$\|\nabla u^\epsilon\|_{L^{q_0}(\mathbb{R}^2 \times (0, T))} \leq C(m, \xi, \beta, C_1, C_2, q_0). \quad (5.33)$$

It follows (5.33) that (4.57), (4.58) is true for the moving front  $\partial\{\varphi^\epsilon(\cdot, t) > 0\}$  for every  $t$ . Repeating the above decomposition argument, we can obtain

$$|\nabla u^\epsilon(x, t)| \leq C(m, \xi, \beta, C_1, C_2) (|\log_2 \delta| + 1) \quad (5.34)$$

where  $\delta = \text{dist}(x, \partial\Gamma_{\varphi^\epsilon, \epsilon}^-(t)) > 0$ . By (5.33), (4.57), (4.58), (5.30), and (5.32), we obtain

$$\|\nabla u^\epsilon\|_{L^\infty(0, T; L^q(\mathbb{R}^2))} \leq C(m, \xi, \beta, C_1, C_2, q), \quad (5.35)$$

for large  $q$ . Again by the Sobolev imbedding theorem, for  $\alpha = 1 - \frac{2}{q}$ ,

$$\|u^\epsilon\|_{L^\infty(0, T; C^\alpha(\mathbb{R}^2))} \leq C(m, \xi, \beta, C_1, C_2, \alpha). \quad (5.36)$$

Equations (5.1), (5.2) are a straightforward consequence of (5.36) by the same decomposition argument.

## 6 Passage to The Limit

It follows from Theorem 5.1, Lemma 3.2, that the a priori  $C^{\alpha, \frac{\alpha}{2}}$  estimate of temperature  $u^\epsilon$ , up to a subsequence, ensure the approximate temperature  $u^\epsilon$  of (3.4)-(3.6) converge to a solution  $u$  of (3.1) in the distribution sense as  $\epsilon \rightarrow 0$  where  $\varphi = \lim_{k \rightarrow \infty} \varphi^{\epsilon_k}$  a.e. by uniform local  $BV$  estimate of  $\varphi^\epsilon$  and  $\{\varphi^{\epsilon_k}\} \equiv \{\varphi_k\}_{k=1}^\infty$  is a subsequence of  $\varphi^\epsilon$ . Once we verify  $\varphi$  is  $L^\infty$  solution of (3.2), we conclude the Main Theorem 2.1.

Before we show the passage to the limit for the phase function, let us address the so-called interior topological cone property of forward reachable domain and backward reachable domain.

Applying the concept and property of boundary characteristic path in Lemma 4.2, we obtain that for all  $\tau \in [t_1, t)$ , the family of backward reachable domain  $\Omega(x, t, \tau, f) = \{x(\tau) | x(t) = x, |\dot{x}(s)| \leq f(x(s), s), \tau \leq s \leq t\}$  driven by Lipschitz continuous normal velocity  $f(x, t)$  with boundary  $\Gamma(x, t, \tau, f) = \partial\Omega(x, t, \tau, f)$  has interior topological cone  $\mathbf{C}(y, C_1, C_2, t - \tau) \subset \Omega(x, t, \tau, f)$  at boundary point  $y$ , where  $\mathbf{C}(y, C_1, C_2, t - \tau) = \{z \in \mathbb{R}^2 | |z - x(s)| \leq C_1(s - \tau), x(\tau) = y, |\dot{x}(s)| = f(x(s), s), C_1|s_1 - s_2| \leq |x(s_1) - x(s_2)| \leq C_2|s_1 - s_2|\}$ , for  $s, s_1, s_2 \in (\tau, t)$ . It is obvious that at boundary point  $y$ , for  $r \in (0, C_1(t - \tau))$

$$|\mathbf{C}(y, C_1, C_2, t - \tau) \cap B_r(y)| \geq C(C_1, C_2)|B_r(y)|, \quad (6.1)$$

for  $C(C_1, C_2) \in (0, 1)$  independent of  $f$ . On the other hand,  $y \in B_r(y) \subset \Omega(x, t, \tau, f)$  for  $r \leq \text{dist}(y, \Gamma(x, t, \tau, f))$ . We conclude that for any  $y \in \Omega(x, t, t_1, f)$  and  $r \in (0, \frac{C_1(t-t_1)}{2})$ ,

$$|\Omega(x, t, t_1, f) \cap B_r(y)| \leq C(C_1, C_2)|B_r(y)|. \quad (6.2)$$

The same argument leads to the same density estimate (6.2) for forward reachable domain  $\Omega(x, t, t_1, f) = \{x(t_1) | x(t) = x, |\dot{x}(s)| \leq f(x(s), s), t \leq s \leq t_1\}$  where  $r \in (0, \frac{C_1(t_1-t)}{2})$ .

To verify that  $\varphi$  is a  $L^\infty$  sub-solution of (3.2) (see Appendix A), it suffices to show that for every Lebesgue point  $(x, t)$  of  $\varphi$  satisfying  $\varphi(x, t) = \lim_{k \rightarrow \infty} \varphi_k(x, t)$ , for any given  $p \in C(\mathbb{R}^2 \times [0, T])$  and a.e.  $\tau \in (0, t)$ ,

$$\sup_{y \in \hat{S}_{x,t}(f, \tau)} \{S(\varphi(y, \tau)) + \int_\tau^t (\langle \dot{x}(s), p \rangle - f(x, s)|p|) ds\} \geq \varphi(x, t), \quad (6.3)$$

where  $f(x, t) = G(u(x, t))$  and  $\hat{S}_{x,t}(f, \tau) = \{x(\tau) | x(t) = x, |\dot{x}| \leq C_2(1 + |x|)\}$ . Note that,

$$\varphi_k(x, t) = \text{essinf}_{y \in S_{x,t}(f_k, \tau)} \{\varphi_k(y, \tau)\} = \text{essinf}_{y \in S_{x,t}(f_k, \tau)} \{\varphi_k(y, \tau)\} + \int_\tau^t (\langle \dot{x}_k(s), p \rangle - f_k(x_k, s)|p|) ds,$$

where  $f_k(x, t) = G(u^{\epsilon_k}(x, t))$ ,  $S_{x,t}(f_k, \tau) = \{x(\tau) | x(t) = x, |\dot{x}| \leq f_k(x, s)\}$  and  $\dot{x}_k(s) = \frac{p}{|p|} f_k(x_k(s))$ ,  $x_k(\tau) \in S_{x,t}(f_k, \tau) \subset \hat{S}_{x,t}(f, \tau)$ . Therefore up to a subsequence, there is  $y_0 = \lim_{k \rightarrow \infty} x_k(\tau) \in \mathbb{R}^2$  such that

$$\varphi_k(x, t) \leq S(\varphi_k(x_k(\tau), \tau)) + \int_\tau^t (\langle \dot{x}_k(s), p \rangle - f(x_k, s)|p|) ds + C(p)(t - \tau)\|f_k - f\|_{L^\infty(\mathbb{R}^2 \times [0, t])}.$$

Note that  $\varphi_k(y, \tau) \geq \varphi_k(x, t)$  a.e. in  $\Omega(x, t, \tau, f_k) \cap B_r(x_k(\tau))$  for any  $r \in (0, \frac{C_1(t-\tau)}{2})$ . As  $k \rightarrow \infty$ , (6.3) is concluded by the virtue of (6.2). By the same argument with the aid of (6.2) for forward reachable domain, one can easily verify  $\varphi$  is also super-solution of (3.2).

## 7 Conclusions

In this paper we have developed a new concept of weak solutions for a non-isothermal phase-change model of polymeric materials, which allows to treat in particular the growth and impingement of multiple crystals in a sound mathematical way. As a major ingredient of an existence proof we have derived results on the geometric properties of the front and the Hölder continuity of the temperature.

The important physical question is the stability of the moving front. In the manufacturing process, the interface between crystal (solid phase) and melt (liquid phase) should be well controlled so that high-quality polymers can be produced. The lately discovered scaling law of crystallization of polymer is different than that of metal solidification (see [39]). The study of the moving front for metal solidification was extensive, though a rigorous mathematical justification is completely open. We will address the stability of non-isothermal moving front for polymeric materials in the forthcoming work [37].

## A Uniqueness of Discontinuous solution of Hamilton-Jacobi equation

At first we recall the definition of  $L^\infty$  solutions of Hamilton-Jacobi equations.

$$\varphi_t + H(x, t, \varphi, D\varphi) = 0, \quad x \in R^d, \quad t > 0, \quad (\text{A.1})$$

$$\varphi(x, 0) = \phi(x). \quad (\text{A.2})$$

As usual, we consider the Hamiltonians satisfying:

- (I).  $H(x, t, z, p)$  is continuous in  $(x, t, z, p)$  and increasing in  $z$ ;
- (II).  $|H(x, t, z, p_1) - H(x, t, z, p_2)| \leq C_0(1 + |x|)|p_1 - p_2|$ , and  $|H(x, t, z, 0)| \leq C_0(1 + |x| + |z|)$ , for all  $t \in (0, T]$ ;
- (III).  $|H(x, t, z_1, p) - H(x, t, z_2, p)| \leq C_0(1 + |x| + |p|)|z_1 - z_2|$ .

Define the essential infimum and supremum of an  $L^\infty_{loc}(R^d)$  function  $v(x)$  at every point  $x \in R^d$ :

$$I(v)(x) := \sup_{A \in S_x} \text{ess inf}_{y \in A} v(y), \quad S(v)(x) := \inf_{A \in S_x} \text{ess sup}_{y \in A} v(y),$$

where  $S_x = \{A \subset R^d \text{ measurable} \mid \lim_{r \rightarrow 0} \frac{m^d(A \cap B(x, r))}{m^d(B(x, r))} = 1\}$ . It is clear that  $I(v)(x)$  and  $S(v)(x)$  are well-defined at every point  $x \in R^d$  and  $I(v)(x) = S(v)(x)$  a.e.

The definitions of the winning and losing profit functions introduced in [13] are stated as follows:

**Definition A.1.** Fix  $\tau \in [0, T]$  and  $p(x, t) \in C(R^d \times [0, T]; R^d)$ . Given a measurable function  $v(x)$  and a position (or value) function  $f(x, t)$ , define the winning and losing profit functions:

$$\Lambda_-^v(x, t, (\tau, f, p)) = \inf\{I(v)(x(\tau)) - z(\tau) \mid (x(\cdot), z(\cdot)) \in \text{Sol}(t, f(x, t), p)\}, \quad (\text{A.3})$$

$$\Lambda_+^v(x, t, (\tau, f, p)) = \sup\{S(v)(x(\tau)) - z(\tau) \mid (x(\cdot), z(\cdot)) \in \text{Sol}(t, f(t, x), p)\}, \quad (\text{A.4})$$

where  $\text{Sol}(t, f(t, x), p)$  denotes the set of solutions:  $(x(\cdot), z(\cdot)) : [\tau, t] \rightarrow R^d \times R$ , for  $t \geq \tau$ , of the characteristic inclusions:  $(\dot{x}(\cdot), \dot{z}(\cdot)) \in E(x, t, z, p)$  satisfying the conditions:  $x(t) = x$ ,  $z(t) = f(x, t)$ , where

$$E(x, t, z, p) := \{(h, g) \in R^d \times R \mid |h| \leq C_0(1 + |x|), g = \langle h, p \rangle - H(x, t, z, p)\}.$$

We now recall the definitions of  $L^\infty$  super-solutions and  $L^\infty$  sub-solutions for the Cauchy problem (A.1)-(A.2) in terms of profit functions in [13, 15] based upon the theme of differential games.

**Definition A.2.** A locally bounded measurable function  $w(x, t) : R^d \times [0, T] \rightarrow R$  is called a super-solution (sub-solution) if  $w(x, t)$  satisfies the following conditions:

- (i) For any  $p(x, t) \in C(R^d \times [0, T]; R^d)$ ,

$$\Lambda_-^w(x, t, (0, w, p)) \leq 0 \quad (\Lambda_+^w(x, t, (0, w, p)) \geq 0), \quad (\text{A.5})$$

for almost all  $(x, t) \in R^d \times [0, T]$ . Additionally, for almost every  $t \in [0, T]$ , (A.5) holds for almost all  $x \in R^d$ .

- (ii) The semigroup property: For almost every  $\tau \in [0, T]$ ,

$$\Lambda_-^{w(x, \tau)}(x, t, (\tau, w, p)) \leq 0 \quad (\Lambda_+^{w(x, \tau)}(x, t, (\tau, w, p)) \geq 0), \quad (\text{A.6})$$

for almost all  $(x, t) \in R^d \times [\tau, T]$ . Additionally, for almost every  $t \in [\tau, T]$ , (A.6) holds for almost all  $x \in R^d$ .

We say that  $\varphi(x, t)$  is a solution of the Cauchy problem (A.1)-(A.2) if  $\varphi(x, t)$  is a super-solution and also a subsolution.

The global existence of  $L^\infty$  solutions of Hamilton-Jacobi equations (A.1)-(A.2) was established for general Lipschitz Hamiltonians for arbitrary large  $L^\infty$  initial data in [13] under the assumptions (I)-(III) and (IV) as follows.

- (IV).  $|H(x_1, t, z, p) - H(x_2, t, z, p)| \leq \lambda(L_0)(1 + |p|)|x_1 - x_2|$ , for  $|x_1|, |x_2| \leq L_0$ .

It was also shown that  $L^\infty$  solutions consist with viscosity solutions introduced by Crandall-Lions [16], [27].

Uniqueness in the almost everywhere sense of  $L^\infty$  solution for convex Hamiltonian  $H = H(D\varphi)$  was shown in [15]. As shown in Lemma 4.3, the existence of  $L^\infty$  solutions is also true for  $H(x, t, z, p) = f(x, t)|p|$  when  $f(x, t)$  is merely Hölder continuous with respect to  $x$ .

For the purpose of this paper, we have to show the uniqueness of  $L^\infty$  solution of the following Cauchy problem:

$$\varphi_t + f(x, t)|D\varphi| = 0, \quad x \in \mathbb{R}^d, \quad t > 0, \quad (\text{A.7})$$

$$\varphi(x, 0) = \phi(x), \quad (\text{A.8})$$

where

$$0 < C_1 \leq f(x, t) \leq C_2 \quad (\text{A.9})$$

and

$$|f(x_1, t) - f(x_2, t)| \leq M|x_1 - x_2|. \quad (\text{A.10})$$

We first introduce the definition of a.e.-continuity.

**Definition A.3.** A measurable function  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  is said to be almost everywhere continuous if there is a set  $\Gamma$  satisfying  $m^d(\Gamma) = 0$  such that  $g(x)$  is continuous at every point  $x \in \mathbb{R}^d \setminus \Gamma$ , i.e.

$$\lim_{y \rightarrow x} g(y) = g(x),$$

for any  $x \in \mathbb{R}^d \setminus \Gamma$ .

In what follows, we will also use the following notations:

$$g^*(x) = \limsup_{y \rightarrow x} g(y), \quad g_*(x) = \liminf_{y \rightarrow x} g(y),$$

i.e.  $g^*(x)$  is the upper envelope of  $g(x)$  and  $g_*(x)$  the lower envelope of  $g(x)$ . Moreover, we define

$$g^{**}(x) = \limsup_{y \rightarrow x, y \in \mathbb{R}^d \setminus \Gamma} g(y), \quad g_{**}(x) = \liminf_{y \rightarrow x, y \in \mathbb{R}^d \setminus \Gamma} g(y).$$

Let us recall the comparison principle of  $L^\infty$  solutions [15].

**Lemma A.4.** Assume that  $\phi^\pm(x)$  are almost everywhere continuous and  $\phi(x)$  is continuous with  $\pm(\phi^\pm(x) - \phi(x)) \geq 0$  almost everywhere. Let  $\varphi^\pm(x, t)$  be  $L^\infty$  super(sub)-solutions of (A.7) and (A.8) with initial data  $\phi^\pm(x)$ , and  $\varphi(x, t)$  the continuous solution with initial data  $\phi(x)$  for  $0 \leq t \leq T < \infty$ . Then  $\pm(\varphi^\pm(x, t) - \varphi(x, t)) \geq 0$  almost everywhere.

Before we prove the uniqueness of  $L^\infty$  solution in the almost everywhere sense, we first introduce a lemma, which indicates the existence of continuous approximations to an almost everywhere continuous function.

**Lemma A.5.** Let  $\phi(x)$  be an almost everywhere continuous function. Then there are a monotone increasing sequence of continuous functions  $\{\phi_k^-\}_{k=1}^\infty$  from below and a monotone decreasing sequence  $\{\phi_k^+\}_{k=1}^\infty$  of continuous functions from above such that

$$\mp(\phi_k^\mp(x) - \phi(x)) \geq 0, \quad \text{a.e.} \quad \lim_{k \rightarrow \infty} m(\{|\phi_k^\mp - \phi| > \frac{1}{k}\} \cap B(0, r)) = 0 \quad (\text{A.11})$$

for  $r > 0$  and, for  $x \in \Gamma$  (the set of discontinuity points of  $\phi$ ),

$$\lim_{r \rightarrow 0} (\phi^\mp)_*(x, r) = \phi_{**}(x), \quad \lim_{r \rightarrow 0} (\phi^\mp)^*(x, r) = \phi^{**}(x) \quad (\text{A.12})$$

where

$$(\phi^\mp)_*(x, r) = \lim_{k \rightarrow \infty} \inf_{y \in B(x, r)} \phi_k^\mp(y), \quad (\phi^\mp)^*(x, r) = \lim_{k \rightarrow \infty} \sup_{y \in B(x, r)} \phi_k^\mp(y),$$

and  $B(x, r) = \{y \in \mathbb{R}^d \mid |y - x| < r\}$ .

**Proof.** If  $\phi(x)$  has less than quadratic growth at infinity and  $m(\bar{\Gamma}) = 0$ , this lemma can be simply shown by taking the well-known sup-convolution and inf-convolution of  $\phi(x)$ :

$$\phi_k^+(x) = \sup_{y \in \mathbb{R}^d \setminus \Gamma} \{\phi(y) - k^4|x - y|^2\}, \quad \phi_k^-(x) = \inf_{y \in \mathbb{R}^d \setminus \Gamma} \{\phi(y) + k^4|x - y|^2\}.$$

For the general case, see [15].

Now we prove the following uniqueness theorem in the measure sense.

**Theorem A.6.** *Suppose that  $\phi(x)$  is almost everywhere continuous. Then the  $L^\infty$  solution of (A.7) and (A.8) is unique a.e. and the solution is determined by the Lax formula:*

$$\varphi(x, t) = \inf_{y \in S_{x,t}} \{\phi(y)\}, \quad (\text{A.13})$$

where  $S_{x,t} = \{y \in \mathbb{R}^d | x(t) = x, x(0) = y, |\dot{x}(\tau)| \leq f(x, \tau) \text{ for } \tau \in [0, t]\}$ .

**Proof.** By Lemma A.5, we can construct two sequences of continuous functions  $\{\phi_k^\pm\}_{k=1}^\infty$  satisfying (A.11) and (A.12). Let  $\{\varphi_k^\pm\}_{k=1}^\infty$  be the sequences of continuous viscosity solutions to (A.7) and (A.8) with initial data  $\{\phi_k^\pm\}_{k=1}^\infty$ . By Lemma A.4, for every  $k$ ,

$$\varphi_k^-(x, t) \leq \varphi_*(x, t) \leq \varphi(x, t) \leq \varphi^*(x, t) \leq \varphi_k^+(x, t), \quad \text{a.e.} \quad (\text{A.14})$$

where  $\varphi(x, t)$  is the  $L^\infty$  solution with initial data  $\phi(x)$ .

It is clear that  $\{\varphi_k^\pm\}_{k=1}^\infty$  are monotone decreasing and increasing sequences of continuous functions, respectively. Denote

$$\bar{\varphi}(x, t) = \lim_{k \rightarrow \infty} \varphi_k^+(x, t), \quad \underline{\varphi}(x, t) = \lim_{k \rightarrow \infty} \varphi_k^-(x, t).$$

Then these functions  $\bar{\varphi}(x, t)$  and  $\underline{\varphi}(x, t)$  are measurable in  $\mathbb{R}^d \times \mathbb{R}^+$  and in  $\mathbb{R}^d \times \{t\}$  for every  $t > 0$ , and

$$\underline{\varphi}(x, t) \leq \bar{\varphi}(x, t).$$

From (A.14), we have

$$\underline{\varphi}(x, t) \leq \varphi_*(x, t) \leq \varphi(x, t) \leq \varphi^*(x, t) \leq \bar{\varphi}(x, t), \quad \text{a.e.} \quad (\text{A.15})$$

Theorem A.6 is established once we show that, for any given time

$$\underline{\varphi}(x, t) = \bar{\varphi}(x, t), \quad \text{a.e. for } (x, t) \in \mathbb{R}^n \times \mathbb{R}^+.$$

Suppose the latter is not true, then there exist  $\delta > 0$  and  $t_0 > 0$  such that

$$m^{d+1}(A) > 0,$$

where  $A = \{(x, t) \in \mathbb{R}^d \times [0, T] | \underline{\varphi}(x, t_0) < \bar{\varphi}(x, t_0) - \delta\}$  is a measurable set in  $\mathbb{R}^d \times \{t_0\}$ . Therefore, there exists  $R > 0$  such that  $m^{d+1}(A(R)) > 0$  with  $A(R) = A \cap B(0, R)$ . Then there exists a Lebesgue point  $(t_0, x_0)$  for both  $\bar{\varphi}(x, t)$  and  $\underline{\varphi}(x, t)$  satisfying  $(x_0, t_0) \in A(R)$ . That is,

$$\lim_{r \rightarrow 0} \frac{m^{d+1}(\{|\underline{\varphi}(x, t) - \underline{\varphi}(x_0, t_0)| > \epsilon\} \cap B((x_0, t_0), r))}{m^{d+1}(B((x_0, t_0), r))} = 0, \quad (\text{A.16})$$

$$\lim_{r \rightarrow 0} \frac{m^{d+1}(\{|\bar{\varphi}(x, t) - \bar{\varphi}(x_0, t_0)| > \epsilon\} \cap B((x_0, t_0), r))}{m^{d+1}(B((x_0, t_0), r))} = 0, \quad (\text{A.17})$$

for every  $\epsilon > 0$ . Hence

$$\lim_{r \rightarrow 0} \frac{m^{d+1}(\{\bar{\varphi}(x, t) > \underline{\varphi}(x, t) + \delta\} \cap B((x_0, t_0), r))}{m^{d+1}(B((x_0, t_0), r))} = 1. \quad (\text{A.18})$$

By the well-known Lax formula for continuous viscosity solutions, we have

$$\varphi_k^\pm(x_0, t_0) = \inf_{y \in S_{x_0, t_0}} \{\phi_k^\pm(y)\},$$

where

$$S_{x_0, t_0}(0) = \{y \in R^n | x(0) = y, x(t_0) = x_0, |\dot{x}(s)| \leq f(x, s) \text{ for } s \in (0, t_0)\}$$

There exists a sequence  $\{y_k^-\}_{k=1}^\infty \subset B(x_0, C_2 t_0)$  satisfying

$$\varphi_k^-(x_0, t_0) \leq \phi_k^-(y_k^-) \leq \varphi_k^-(x_0, t_0) + \frac{1}{k}. \quad (\text{A.19})$$

Thus there is a subsequence of  $\{y_k^-\}_{k=1}^\infty$  (still denoted by  $\{y_k^-\}_{k=1}^\infty$ ) such that

$$\lim_{k \rightarrow \infty} y_k^- = \bar{y}, \quad \liminf_{k \rightarrow \infty} \phi_k^-(y_k^-) \geq \phi_{**}(\bar{y}). \quad (\text{A.20})$$

Claim: For any point  $(x, t) \in C^+(x_0, t_0) = \{(y, s) | |y - x_0| \leq \frac{C_1}{4}(s - t_0), s > t_0\}$ ,  $\bar{\varphi}(x, t) \leq \phi_{**}(\bar{y})$ .

This can be seen as follows: For a given point  $(x, t) \in C^+(x_0, t_0)$ , the ball  $B(x_0, \frac{C_1}{2}(t - t_0))$  is a subset of the backward reachable set  $S_{x, t}(t_0) = \{y \in R^n | x(t_0) = y, x(t) = x, |\dot{x}(s)| \leq f(x, s) \text{ for } s \in (t_0, t)\}$  under the assumption (A.9). For a fixed  $k$ , there is a Lipschitz continuous path  $x_k : [0, t_0] \rightarrow R^n$  satisfying  $|\dot{x}_k(\tau)| \leq f(x_k(\tau), \tau)$  with  $x_k(0) = y_k^-$  and  $x_k(t_0) = x_0$ . Let us consider the following Lipschitz flow issued from  $B(x_0, \frac{C_1}{2}(t - t_0))$ :

$$\dot{x}_{k, z}(\tau) = \frac{f(x_{k, z}(\tau), \tau)}{f(x_k(\tau), \tau)} \dot{x}_k(\tau), \quad (\text{A.21})$$

with  $x_{k, z}(t_0) = z \in B(x_0, \frac{C_1}{2}(t - t_0))$  for  $t > t_0$ . Obviously  $|\dot{x}_{k, z}(\tau)| \leq f(x_{k, z}(\tau), \tau)$ . Denote  $y_k(z) = x_{k, z}(0)$ . Then by assumption (A.10) and the Gronwall inequality, the mapping  $y_k : z \rightarrow y_k(z)$  is a bi-Lipschitz homeomorphism, where  $z \in B(x_0, \frac{C_1}{2}(t - t_0))$ , i.e.

$$e^{-M(t-t_0)} |z_1 - z_2| \leq |y_k(z_1) - y_k(z_2)| \leq e^{M(t-t_0)} |z_1 - z_2|, \quad (\text{A.22})$$

for any  $z_1, z_2 \in B(x_0, \frac{C_1}{2}(t - t_0))$ . Therefore by (A.20), (A.21), and (A.22), there is an integer  $K$  such that as long as  $k \geq K$ ,

$$\bar{y} \in B(\bar{y}, \frac{C_1}{2}(t - t_0)e^{-M(t-t_0)}) \subset y_k(B(x_0, \frac{C_1}{2}(t - t_0))) \subset S_{x, t}.$$

Hence,  $\varphi_k^+(x, t) \leq \phi_{**}(\bar{y}) + \frac{1}{k}$  for  $(x, t) \in C^+(x_0, t_0)$  and  $k \geq K$ . Thus the claim is proven.

However, by virtue of (A.19), the definition of  $\underline{\varphi}$ , the definition of  $C^+(x_0, t_0)$ , the claim is a contradiction to (A.18). Therefore Theorem A.6 is shown.

**Remark A.7.** The uniqueness theorem A.6 is still true if the Lipschitz continuity (A.10) of  $f$  with respect to  $x$  is replaced by  $Lip(\log(Lip))$  continuity as follows:

$$|f(x_1, t) - f(x_2, t)| \leq M|x_1 - x_2| |\ln|x_1 - x_2||. \quad (\text{A.23})$$

## B BV Estimates for Hamilton-Jacobi Equations with Coercive Hamiltonians

Here we provide a local BV estimate for Hamilton-Jacobi equations with coercive Hamiltonians for the readers' convenience as a general version of BV regularity was shown in [15].

**Theorem B.1.** *Let  $\varphi(x, t)$  be the unique  $L^\infty$  solution of (A.7)–(A.8) with locally bounded initial data. Then*

$$\|\varphi\|_{BV(B(0, R) \times [0, T])} \leq C(R, T, C_1, C_2, \|\phi\|_{L^\infty(R^d)}) \quad (\text{B.1})$$

**Proof.** As shown in the proof of Theorem A.6, there always exists a sequence of continuous viscosity solutions  $\varphi_k$  such that  $\lim_{k \rightarrow \infty} \varphi_k = \varphi$  in the almost everywhere sense with continuous initial data  $\lim_{k \rightarrow \infty} \varphi_k = \varphi$ , a.e. It suffices to show the local BV estimate for a continuous solution  $\varphi$  as long as the local BV bound of  $\varphi$  depends only on the local  $L^\infty$  bound of  $\varphi$ . Without loss of generality, we assume that initial data  $\varphi$  is Lipschitz continuous with compact support in  $B(0, R)$ .

Consider the following approximate viscous equation:

$$\varphi_t^\epsilon - \epsilon \Delta \varphi^\epsilon + f(x, t) |D\varphi^\epsilon| = 0, \quad \text{in } R^d \times [0, T], \quad (\text{B.2})$$

$$\varphi^\epsilon(x, 0) = \phi(x), \quad \text{in } R^d. \quad (\text{B.3})$$

The existence of a smooth solution of (B.2)–(B.3) can be established easily by the Leray-Schauder fixed point theorem. We now derive some basic estimates of the solution. It follows from the maximum principle that

$$\|\varphi^\epsilon\|_{L^\infty(R^d \times [0, T])} \leq \|\phi\|_{L^\infty(R^d)} \equiv C_5. \quad (\text{B.4})$$

Then we have the spatial decay estimate of  $\varphi^\epsilon(x, t)$ :

$$|\varphi^\epsilon(x, t)| \leq w_\epsilon(x, t) \equiv C_5 \exp\left(\frac{1}{\sqrt{\epsilon}}(C_\epsilon t + R - |x|)\right), \quad \text{for } (x, t) \in \mathcal{C}_\epsilon, \quad (\text{B.5})$$

where  $C_\epsilon = C_2 + 2\sqrt{\epsilon}$  and  $\mathcal{C}_\epsilon = \{(x, t) \mid |x| \geq C_\epsilon t + R\}$ .

This fact can be proven as follows: Note that, on  $\partial\mathcal{C}_\epsilon$ ,

$$|\varphi^\epsilon(x, 0)| = 0 \leq w_\epsilon(x, 0), \quad \text{in } \mathbb{R}^d \setminus B(0, R),$$

$$|\varphi^\epsilon(x, t)| \leq C_5 \leq w_\epsilon(x, t), \quad \text{on } |x| = R + C_\epsilon t.$$

In addition we have

$$|\varphi_t^\epsilon - \epsilon \Delta \varphi^\epsilon| = f(x, t) |D\varphi^\epsilon| \leq C_2 |D\varphi^\epsilon|, \quad \text{in } \mathcal{C}_\epsilon,$$

$$\begin{aligned} w_t^\epsilon - \epsilon \Delta w^\epsilon - C_2 |Dw^\epsilon| &= \frac{1}{\sqrt{\epsilon}} w^\epsilon \{C_\epsilon - C_2 - \sqrt{\epsilon} - (n-1)\epsilon|x|^{-1}\} \\ &\geq \frac{1}{\sqrt{\epsilon}} w^\epsilon \{C_\epsilon - C_2 - 2\sqrt{\epsilon}\} = 0. \end{aligned}$$

Thus the inequality (B.5) follows from the maximum principle. Similarly, we differentiate (B.2), repeat the above procedure, and conclude that  $\lim_{|x| \rightarrow \infty} |D\varphi^\epsilon| = 0$ .

Integrating the equation (B.2) over  $R^d \times [0, T]$ , we have

$$\int_{R^d} \varphi^\epsilon(x, T) dx + \int_0^T \int_{R^d} f(x, t) |D\varphi^\epsilon| dx dt = \int_{R^d} \phi(x) dx.$$

It follows from (A.9), (B.4), and (B.5) that

$$\int_0^T \int_{R^d} |D\varphi^\epsilon| dx dt \leq C(R, T, C_1, \|\phi\|_{L^\infty}). \quad (\text{B.6})$$

Let  $\epsilon \rightarrow 0$ ,  $\varphi^\epsilon \rightarrow \varphi$  in  $Lip([R^d \times [0, T]])$  where  $\varphi(x, t)$  is the unique continuous viscosity solution of (A.7)–(A.8). It follows from (B.6) that

$$\int_0^T \int_{R^d} |D\varphi(x, t)| dx dt \leq C(R, T, C_1, \|\phi\|_{L^\infty}). \quad (\text{B.7})$$

Since (A.7) holds for  $\varphi(x, t)$  at almost everywhere  $(x, t) \in R^d \times [0, T]$ . Thus, it follows from (B.7) and (A.9) that

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^d} |\varphi_t(x, t)| dx dt &= \int_0^T \int_{\mathbb{R}^d} f(x, t) |D\varphi(x, t)| dx dt \\ &\leq C(R, T, C_1, C_2, \|\phi\|_{L^\infty}). \end{aligned} \quad (\text{B.8})$$

Therefore,

$$\|\varphi\|_{BV(B(0, R) \times [0, T])} \leq C(R, T, C_1, C_2, \|\phi\|_{L^\infty}). \quad (\text{B.9})$$

Thus (B.1) is proven.

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## References

- [1] Avrami, M., *Kinetics of phase change I-III*, J. Chem. Phys. **7**(1939), 1103-1112; **8**(1940), 212-224; **9**(1941), 177-184.
- [2] Alvarez, O.; Cardaliaguet, P.; Monneau, R.; *Existence and uniqueness for dislocation dynamics with nonnegative velocity*, Interfaces Free Bound. **7** (2005) 415-434.
- [3] Barles, G, Soner, H.M., and Souganidis, P.E.; *Front propagation and phase-field theory*, SIAM J. Cont. Optim. **31** (1993), 439-469.
- [4] Barles, G.; Souganidis, P.E. *A new approach to front propagation problems: theory and applications*, Arch. Rational Mech. Anal. **141** (1998), no. 3, 237-296.
- [5] Barles, G.; Ley, O. *Nonlocal first-order Hamilton-Jacobi equations modelling dislocations dynamics*, Comm. Partial Differential Equations **31** (2006) 1191-1208.
- [6] Burger, M., *Growth and impingement in polymer melts*, in: Colli, P., et. al., eds., Free Boundary Problems, Birkhaeuser, Basel, 2003, 65-74.
- [7] Burger, M. *Growth of multiple crystals in polymer melts*, European J. Appl. Math. **15** (2004), 347-363.
- [8] Burger, M., and Capasso, V., *Mathematical modelling and simulation of non-isothermal crystallization of polymers*, Math. Models and Meth. in Appl. Sciences **11** (2001), 1029-1054.
- [9] Burger, M., Capasso, V., and Eder, G., *Modelling crystallization of polymers in temperature fields*, ZAMM **82** (2002), 51-63.
- [10] Burger, M., Capasso, V., Salani, C., *Modelling multi-dimensional crystallization of polymers in interaction with heat transfer*, Nonlinear Analysis B, Real World Applications **3** (2002), 139-160.
- [11] Caffarelli, L.A.; Evans, L.C., *Continuity of the temperature in the two-phase Stefan problem*, Arch. Rational Mech. Anal. **81** (1983), no. 3, 199-220.
- [12] Capasso, V., Salani, C., *Stochastic birth-and-growth processes modelling crystallization of polymers in a spatially heterogenous temperature field*, Nonlinear Analysis, Real World Applications **1** (2000), 485-498.
- [13] Chen, G.-Q., Su, B., *Discontinuous solutions in  $L^\infty$  for Hamilton-Jacobi equations*, Chinese Ann. Math. Ser. B **21** (2000), no. 2, 165-186.
- [14] Chen, G.-Q., Su, B., *On global discontinuous solutions of Hamilton-Jacobi equations*, C. R. Math. Acad. Sci. Paris **334** (2002), no. 2, 113-118.
- [15] Chen, G.-Q., Su, B., *Discontinuous solutions for Hamilton-Jacobi equations: uniqueness and regularity*, Discrete Contin. Dyn. Syst. **9** (2003), no. 1, 167-192.

- [16] Crandall, M.G., Lions, P.L., *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. **277** (1983), 1-42.
- [17] DiBenedetto, E., *Continuity of weak solutions to certain singular parabolic equations*, Ann. Mat. Pura Appl. (4) **130** (1982), 131-176.
- [18] Eder, G., *Crystallization kinetic equations incorporating surface and bulk nucleation*, ZAMM **76** (1996), S4, 489-492.
- [19] Eder, G., *Fundamentals of structure formation in crystallizing polymers*, in K.Hatada, T.Kitayama, O.Vogl, eds., *Macromolecular Design of polymeric Materials* (M.Dekker, New York, 1997), 761-782.
- [20] Evans, V. R., *The laws of expanding circles and spheres in relations to the lateral growth rate of surface films and the grain-size of metals*, Trans. Faraday Soc. **41** (1945), 365-374.
- [21] Evans, L.C., Gariepy, R. F., *Measure Theory and fine properties of functions. Studies in Advanced Mathematics*. CRC Press, Boca Raton, FL, 1992.
- [22] Friedman, A., *Variational Principles and Free-Boundary Problems*, Wiley and Sons, Inc., New York, 1982.
- [23] Friedman, A., Velazquez, J.L.; *A free boundary problem associated to crystallization of polymers*, Indiana Univ. Math. Journal **50** (2001), 1609-1650.
- [24] Kolmogorov, A.N., *Statistical theory of crystallization of metals* Bull. Acad. Sci. USSR, Math.Ser. **1**(1937), 355-359.
- [25] Ladyzhenskaya, O.A., Solonnikov, V.A., Ural'ceva, N.N., *Linear and Quasilinear Equations of Parabolic Type*. Translations of Mathematical Monographs, Vol. 23 American Mathematical Society, Providence, R.I. 1967.
- [26] Ley, O.; *Lower-bound gradient estimates for first-order Hamilton-Jacobi equations and applications to the regularity of propagating fronts*, Adv. Differential Equations **6** (2001) 547-576.
- [27] Lions, P.L., *Generalized Solutions of Hamilton-Jacobi Equations*, Pitman, Boston, London, Melbourne, 1982.
- [28] Meirmanov, A.M., *The Stefan Problem*, De Gruyter, Berlin, 1992.
- [29] Micheletti, A., Burger, M., *Stochastic and deterministic simulation of nonisothermal crystallization of polymers*, J.Math.Chem. **30** (2001), 169-193.
- [30] A.Micheletti, V.Capasso, *The stochastic geometry of polymer crystallization processes*, Stoch. Anal. Appl. **15** (1997), 355-373.
- [31] B.Monasse, J.M.Haudin, *Thermal dependence of nucleation and growth rate in polypropylene by non-isothermal calorimetry*, Colloid & Polymer Sci. **264** (1986), 117-122.
- [32] Osher, S., Sethian, J.A., *Fronts propagating with curvature-dependent speed: algorithms based on Hamilton-Jacobi formulations*, J. Comp. Phys. **79** (1988), 12-49.
- [33] E.Ratajski, H.Janeschitz-Kriegl, *How to determine high growth speeds in polymer crystallization*, Colloid Polym. Sci. **274**, (1996), 938 - 951.
- [34] Sacks, P.E., *Continuity of solutions of a singular parabolic equation*, Nonlinear Anal. **7** (1983), no. 4, 387-409.
- [35] Schulze, G.E.W., Naujeck, T.R., *A growing 2D spherulite and calculus of variations*, Colloid & Polymer Science **269** (1991), 689-703.
- [36] Su, B.; Burger, M.; *Global weak solutions of non-isothermal front propagation problem*, Electron. Res. Announc. Amer. Math. Soc. **13** (2007) 46-52.

- [37] Su, B., Trivedi, R., *in preparation*.
- [38] Taylor, J.E., Cahn, J.W., Handwerker, C.A., *Geometric models of crystal growth*, Acta metall. mater. **40** (1992), 1443-1472.
- [39] Trivedi, R., Teng, J., *in preparation*.
- [40] Visintin, A., *Models of Phase Transitions*, Birkhaeuser, Boston, 1996.
- [41] Ziemer, W. P. *Interior and boundary continuity of weak solutions of degenerate parabolic equations*, Trans. Amer. Math. Soc. **271** (1982), no. 2, 733-748.