

Nonlinear Cross-Diffusion with Size Exclusion

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Abstract

The aim of this paper is to investigate the mathematical properties of a continuum model for diffusion of multiple species incorporating size exclusion effects. The system for two species leads to nonlinear cross-diffusion terms with double degeneracy, which creates significant novel challenges in the analysis of the system.

We prove global existence of weak solutions and well-posedness of strong solutions close to equilibrium. We further study some asymptotics of the model, in particular we characterize the large-time behaviour of solutions.

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1 Introduction

Modelling the transport of particles with size exclusion is of high relevance in many applications nowadays, e.g. in molecular and cell biology. In the case of a single type of particle various models have been proposed at the continuum level or derived from microscopic models (CP. [23, 39]), typically leading to nonlinear drift-diffusion equations for the particle density of the form

$$\partial_t \rho = \nabla \cdot (D(\rho) \nabla (f(\rho) + V)). \quad (1.1)$$

Here D is a nonlinear mobility (diffusivity) coefficient, in most models vanishing at $\rho = 0$ and some maximum density, being positive in between. A frequently used form, which can be deduced rigorously from a microscopic model (CP. [23]), is $D(\rho) = \rho(1 - \rho)$. The function f is derived from the entropy of the system and V can describe potential forces.

Far less attention has been paid to the case of several different particles, which is however very relevant in many practical problems such as the transport of ions through narrow membrane channels (CP. [4, 24, 44]). The particles can vary by many parameters (size, diffusion coefficients, reaction to external and interaction forces) and hence need to be described by separate densities. This will give rise to systems of nonlinear drift-diffusion equations with

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strong cross-diffusion, as we shall explain in this paper. A particular example we shall discuss is a cross-diffusion system for two types of particles (called red and blue in the following) with different diffusion coefficients and different external potentials V and W :

$$\partial_t r = \nabla \cdot (D_r((1-b)\nabla r + r\nabla b + r(1-r-b)\nabla V)) \quad (1.2)$$

$$\partial_t b = \nabla \cdot (D_b((1-r)\nabla b + b\nabla r + b(1-r-b)\nabla W)). \quad (1.3)$$

Further details on the derivation of this model will be given in Section 2 (see also [5] for details and an application to the computation of ion channel conductance). We mention that for one of the densities vanishing, the model reduces to the well-studied single Fokker-Planck equation with linear diffusion and logistic mobility:

$$\partial_t \rho = \nabla \cdot (D\nabla \rho + \rho(1-\rho)\nabla V). \quad (1.4)$$

Equations of the above form feature (at least formally) the gradient flow structure

$$\partial_t \rho = \nabla \cdot \left(m(\rho) D \nabla \frac{\delta E}{\delta \rho} \right), \quad (1.5)$$

where $m(\rho)$ is a mobility function and $\frac{\delta E}{\delta \rho}$ denotes the functional derivative of an *entropy* functional E . In case of a linear mobility, models of the form (1.5) have a rigorous metric (entropy based) gradient flow structure [2, 32, 37] which produces exponential convergence to equilibrium in case the entropy is convex in a geodesics sense, CP. [36, 37, 12, 3, 10]. The metric gradient flow structure in the case of a nonlinear mobility function is much more challenging, it has been recently studied in [9, 19]. In case of a logistic mobility and with an entropy combining linear diffusion and a convex external potential, namely (1.4), such an equation can be derived as a quantum mechanical generalization of the Fokker-Planck equation, CP. [11, 8].

The mathematical analysis of such models with many species (systems) is a particular challenge for many reasons. First of all there is strong degeneracy and nonlinearity in both diffusion equations, and secondly there is no simple maximum principle. The latter can be often be used to prove a-priori bounds and subsequently existence as well as uniqueness in the case of single nonlinear diffusion equations. So far the analysis of nonlinear cross-diffusion systems has only been considered in few special cases, the most frequently studied one consisting of a triangular diffusion structure instead of full coupling. Several important examples of such equations arise from models in cell biology, CP. to mention a few [31, 26, 6], see also the review papers [27, 28] and the references therein. Such systems usually feature additional lower order coupling via reaction terms, the most celebrated example being the Keller-Segel model for chemotactic aggregation (CP. [33]). In many cases the equation for the chemical concentration is replaced by a stationary one. The presence of a logistic sensitivity only in the chemotactic part, coupled with linear diffusion with several possible modifications, has been recently considered in [17], as a generalization of similar parabolic-elliptic models previously considered in [38, 18, 35].

Models with full cross-coupling have been considered only recently (CP. [20, 13, 14, 15]), also motivated by applications in population biology (CP. [13]). Multispecies chemotaxis models considered in [29, 30] are also framed in the same context. The model we consider adds additional difficulty due to the double degeneracy of the mobility, whereas previously considered cases could rather be written with constant and hence nondegenerate mobility.

In this paper we prove the following results for the model (1.2), (1.3), in most cases with zero potential for simplicity:

- We characterize equilibria in the case with and without potential and carry out a linear stability analysis (Sections 3.3 and 3.4).
- We prove well-posedness (existence and uniqueness) for strong solutions under the condition that the initial value is close to equilibrium in H^2 . (Section 3.5)
- We prove global existence for initial values in L^2 . (Section 4)
- We prove convergence of solutions to equilibrium as $t \rightarrow \infty$. (Section 5)

Besides proving our results, we shall also discuss some technical issues which render the mathematical theory of models like (1.2), (1.3) a difficult task, see Section 6. A particular objective of the paper is however to create a basis and stimulate further research on this kind of problems, which are clearly underrepresented in the mathematics literature relative to their practical importance.

The remainder of this paper is organized as follows: In Section 2 we will discuss the formal derivation of nonlinear cross-diffusion models with size exclusion from microscopic models. We then proceed to a discussion of basic properties of solutions and equilibria in Section 3, where we also prove existence and uniqueness of strong solutions for initial values close to equilibrium. In Section 4 we verify the global existence of weak solutions by regularization and Galerkin approximations. Section 5 is devoted to asymptotics of the diffusion coefficient and the study of large-time behaviour. We finally conclude and discuss open issues in Section 6.

2 A Simple Diffusion Model with Size Exclusion

In order to motivate the investigation of the specific system of nonlinear cross-diffusion equations, we present a formal derivation from a hopping model with size exclusion in the following (for further details and extension we refer to [5]). The problem-setup is as follows:

Let \mathcal{T}_h denote an equidistant grid of mesh size h , where every grid point can be occupied by a red or blue particle. The probability of finding a red particle at location x at time t is denoted by

$$r(x, t) = P(\text{red particle is at position } x \text{ at time } t).$$

Red particles evolve according to diffusion and a scaled potential $V(x, t)$, blue ones to $W(x, t)$. The hopping rates for the red particles are then given by

$$\tilde{\Pi}_r^+ = P(\text{jump of } r \text{ from position } x \text{ to } x + h \text{ in } (t, t + \Delta t)) = \alpha_1 - \beta_1 V_x(x + h/2, t), \quad (2.1)$$

$$\tilde{\Pi}_r^- = P(\text{jump of } r \text{ from position } x \text{ to } x - h \text{ in } (t, t + \Delta t)) = \alpha_1 + \beta_1 V_x(x - h/2, t), \quad (2.2)$$

where α_1 and β_1 denote diffusion and mobility constants for the red species, α_2 and β_2 denote these coefficients for the blue species.

We assume that the diameter of every ion equals h and take into account that neighboring sites might be occupied. We realize these assumptions in the simple model:

$$\Pi_r^+ = \tilde{\Pi}_r^+ \cdot P(\text{position } x + h \text{ is at time } t \text{ not occupied}),$$

$$\Pi_r^- = \tilde{\Pi}_r^- \cdot P(\text{position } x - h \text{ is at time } t \text{ not occupied}).$$

We make the assumption that the probability of a free site is

$$P(\text{position } x \text{ is at time } t \text{ not occupied}) = 1 - r(x, t) - b(x, t),$$

which corresponds to rigorous results for single species [22]. The probability to find a particle of species r at position x at time $t + \Delta t$ is given by

$$r(x, t + \Delta t) = r(x, t)(1 - \Pi_r^+(x, t) - \Pi_r^-) + r(x + h, t)\Pi_r^-(x + h, t) + r(x - h, t)\Pi_r^-(x - h, t).$$

This means we have (suppressing the index r in Π)

$$\begin{aligned} r(x, t + \Delta t) - r(x, t) = & r(x, t)(\Pi^+(x - h, t) + \Pi^-(x + h, t) - \Pi^+(x, t) - \Pi^-(x, t)) \\ & + (r(x + h, t) - r(x, t))\Pi^-(x + h, t) + (r(x - h, t) - r(x, t))\Pi^+(x - h, t) \end{aligned}$$

and we obtain after Taylor-expansion

$$\begin{aligned} r(x, t + \Delta t) - r(x, t) = & r(x, t)(h(\Pi_x^-(x, t) - \Pi_x^+(x, t)) + \frac{h^2}{2}(\Pi_{xx}^+(x, t) + \Pi_{xx}^-(x, t))) \\ & + hr_x(\Pi^-(x + h, t) - \Pi^+(x - h, t)) + \frac{h^2}{2}r_{xx}(\Pi^-(x + h, t) + \Pi^+(x - h, t)). \end{aligned}$$

For the probabilities we have

$$\begin{aligned} \Pi_x^-(x, t) - \Pi_x^+(x, t) &= 2\beta_1 \frac{\partial}{\partial x}(V_x(1 - r - b)) + 2h\alpha_1(r_{xx} + b_{xx}) + O(h^2), \\ \Pi_{xx}^+(x, t) + \Pi_{xx}^-(x, t) &= -2\alpha_1(r_{xx} + b_{xx}) + O(h), \\ \Pi^-(x + h, t) - \Pi^+(x - h, t) &= 2\beta_1 V_x(1 - r - b) + O(h^2), \\ \Pi^-(x + h, t) + \Pi^+(x - h, t) &= (2\alpha_1 + 2\beta_1 V_{xx})(1 - r - b) + O(h), \end{aligned}$$

which yields

$$\begin{aligned} r(x, t + \Delta t) - r(x, t) = & 2h\beta_1 r \frac{\partial}{\partial x}(V_x(1 - r - b)) + 2h\beta r_x V_x(1 - r - b) + h^2 r \alpha_1(r_{xx} + b_{xx}) + h^2 r_{xx} \alpha_1(1 - r - b) = \\ & 2h\beta_1 \frac{\partial}{\partial x}(r(1 - r - b)V_x(x, t)) + h^2 \alpha_1 \frac{\partial}{\partial x}((1 - r - b)r_x + r(r_x + b_x)). \end{aligned}$$

Thus, with an appropriate scaling ($\frac{\alpha_1}{2} \approx D_1$, D_1 being the diffusion coefficient red particles) and $\Delta t = 2h^2$, then in the limit $t \rightarrow \infty$ the resulting system of continuum equations becomes:

$$\begin{aligned} \partial_t r &= D_1 \frac{\partial}{\partial x}((1 - b)r_x + rb_x + \mu_1 r(1 - r - b)\nabla V) \\ \partial_t b &= D_2 \frac{\partial}{\partial x}((1 - r)b_x + br_x - \mu_2 b(1 - r - b)\nabla W), \end{aligned}$$

where μ_1 is described by $\mu_1 = 2\beta_1/\alpha_1 h$, μ_2 is described analogously. The extension of this system of nonlinear diffusion equations to higher dimensions, with completely analogous derivation from a jump model on a lattice, is given by

$$\partial_t r = D_1 \nabla \cdot ((1 - b)\nabla r + r\nabla b + \mu_1 r(1 - r - b)\nabla V) \quad (2.3)$$

$$\partial_t b = D_2 \nabla \cdot ((1 - r)\nabla b + b\nabla r + \mu_2 b(1 - r - b)\nabla W). \quad (2.4)$$

We remark that from now on, we assume that $\mu_1 = \mu_2 = 1$. This is possible due to redefining the potentials V and W .

3 Basic Properties

In the following we discuss the basic properties of the cross-diffusion model, for simplicity restricting ourselves to the case of two particle types, called red and blue (and densities denoted by r and b) in the following. The majority of results of this section carries over to an arbitrary number with obvious modifications however. We assume that time is scaled such that the diffusion coefficient of the red species equals one, i.e. we consider the nonlinear system

$$\partial_t r = \nabla \cdot ((1-b)\nabla r + r\nabla b + r(1-r-b)\nabla V) \quad (3.1)$$

$$\partial_t b = \nabla \cdot (D((1-r)\nabla b + b\nabla r + b(1-r-b)\nabla W)) \quad (3.2)$$

in $\Omega \times (0, T)$ with no-flux boundary conditions

$$n \cdot ((1-b)\nabla r + r\nabla b + r(1-r-b)\nabla V) = 0, \quad (3.3)$$

$$n \nabla \cdot (D((1-r)\nabla b + b\nabla r + b(1-r-b)\nabla W)) = 0, \quad (3.4)$$

on $\partial\Omega \times (0, T)$, where n denotes the outward normal. This system is supplemented by initial values

$$r(\cdot, 0) = n_0, \quad b(\cdot, 0) = b_0 \quad (3.5)$$

in Ω . Throughout the whole paper we shall assume that Ω is a bounded domain in \mathbb{R}^d ($d \in \{1, 2, 3\}$) with sufficiently regular boundary. Moreover we shall not explicitly state the boundary conditions and initial values, but always attribute them to (3.1), (3.2).

In the following we shall denote the total volume density by ρ , i.e.

$$\rho(x, t) := r(x, t) + b(x, t). \quad (3.6)$$

We also mention that the equations can be rewritten in a different form using the volume density as

$$\partial_t r = \nabla \cdot ((1-\rho)\nabla r + r\nabla \rho + r(1-\rho)\nabla V) \quad (3.7)$$

$$\partial_t b = \nabla \cdot (D((1-\rho)\nabla b + b\nabla \rho + b(1-\rho)\nabla W)), \quad (3.8)$$

which can be useful to deduce bounds on ρ and understand further the structure of the model.

3.1 Formal Gradient Flow Structure

A first observation is that (3.1), (3.2) has a formal gradient flow structure. The entropy functional is given by

$$E(r, b) = \int_{\Omega} [r \log r + b \log b + (1-\rho) \log(1-\rho) + rV + bW] \, dx. \quad (3.9)$$

Then we find (noticing $\partial_r \rho = \partial_b \rho = 1$) that

$$\xi := \partial_r E = \log r - \log(1-\rho) + V, \quad (3.10)$$

$$\eta := \partial_b E = \log b - \log(1-\rho) + W. \quad (3.11)$$

Thus, the system can be written in the form

$$\partial_t \begin{pmatrix} r \\ b \end{pmatrix} = \nabla \cdot \left(\mathbf{M}(r, b) : \nabla \begin{pmatrix} \partial_r E \\ \partial_b E \end{pmatrix} \right), \quad (3.12)$$

with a nonlinear diagonal mobility tensor with diagonal elements $r(1 - \rho)$ and $Db(1 - \rho)$ respectively. This notation is the basis of a formal gradient flow structure on a manifold with an optimal transport metric. In the scalar cases such approaches have been first investigated in the case of a linear mobility leading to gradient flows in the Wasserstein metric (CP. [37], [32], [2]) and later generalized to nonlinear concave mobilities (CP. [9]). A formal derivation and discussion of such gradient flows in the case of systems can be found in [40].

We introduce a manifold

$$\mathcal{M} = \left\{ (r, b) \in L^1(\Omega)^2 \mid 0 \leq r, 0 \leq b, r + b \leq 1, \int_{\Omega} r \, dx = m_r, \int_{\Omega} b \, dx = m_b \right\}, \quad (3.13)$$

equipped with the optimal transport metric

$$d((r_1, b_1), (r_2, b_2))^2 = \inf_{R, B, U, V} \left(\frac{1}{2} \int_0^1 \int_{\Omega} (1 - R - B) [R|U|^2 + DB|V|^2] \, dx \, dt \right) \quad (3.14)$$

where the infimum is taken subject to

$$\partial_t R = \nabla \cdot (R(1 - R - B)U) \quad (3.15)$$

$$\partial_t B = \nabla \cdot (DB(1 - R - B)V) \quad (3.16)$$

$$R(t = 0) = r_1, \quad R(t = 1) = r_2, \quad (3.17)$$

$$B(t = 0) = b_1, \quad B(t = 1) = b_2. \quad (3.18)$$

Then, formally, the system (3.7), (3.8) is a gradient flow of the energy E on the manifold \mathcal{M} , i.e.,

$$\partial_t (r, b) = -\nabla_{\mathcal{M}} E(r, b). \quad (3.19)$$

In the course of this paper we shall not analyze the model as a gradient flow on the manifold \mathcal{M} in the spirit of [2], because a detailed characterization of the manifold as well as verification of properties like geodesic convexity of the entropy seem out of reach so far (CP. [40] for formal computations for such systems). However, the gradient flow structure and the associated entropy (with the possibility to define dual entropy variables) will be crucial in several proofs in the following.

3.2 Entropy Dissipation

With the entropy variables ξ and η introduced in (3.10) and (3.11), respectively, we can use (3.12) to deduce an entropy dissipation relation. For a sufficiently smooth solution of (3.7), (3.8) we find

$$\frac{d}{dt} E(r(\cdot, t), b(\cdot, t)) = - \int_{\Omega} (r(1 - \rho)|\nabla \xi|^2 + Db(1 - \rho)|\nabla \eta|^2) \, dx =: -ED. \quad (3.20)$$

Expanding ξ and η into their three terms, further a-priori estimates can be deduced from the entropy dissipation term ED .

In the case without external potentials, $V = W = 0$, further entropy functionals decreasing in time can be found. Of particular interest seems the classical logarithmic entropy

$$E_0(r, b) = \int_{\Omega} \left[r \log r + \frac{1}{D} b \log b \right] \, dx. \quad (3.21)$$

3.3 Equilibria

We start by computing stationary solutions of the nonlinear cross-diffusion model, i.e. functions r_∞ and b_∞ satisfying

$$0 = \nabla \cdot ((1 - \rho_\infty)\nabla r_\infty + r_\infty \nabla \rho_\infty + r_\infty(1 - \rho_\infty)\nabla V) \quad (3.22)$$

$$0 = \nabla \cdot (D((1 - \rho_\infty)\nabla b_\infty + b_\infty \nabla \rho_\infty + b_\infty(1 - \rho_\infty)\nabla W)), \quad (3.23)$$

where $\rho_\infty = r_\infty + b_\infty$. Similar to the entropy dissipation we can show that

$$\int_{\Omega} (1 - r_\infty - b_\infty) [r_\infty |\nabla \xi_\infty|^2 + b_\infty |\nabla \eta_\infty|^2] dx = 0,$$

with

$$\xi = \log r_\infty - \log(1 - \rho_\infty) + V, \quad \eta = \log b_\infty - \log(1 - \rho_\infty) + W.$$

Hence we conclude in particular

$$0 = \sqrt{1 - \rho_\infty} \sqrt{r_\infty} \xi_\infty = (1 - \rho_\infty)\nabla r_\infty + r_\infty \nabla \rho_\infty + r_\infty(1 - \rho_\infty)\nabla V,$$

$$0 = \sqrt{1 - \rho_\infty} \sqrt{b_\infty} \eta_\infty = (1 - \rho_\infty)\nabla b_\infty + b_\infty \nabla \rho_\infty + b_\infty(1 - \rho_\infty)\nabla W,$$

i.e. the fluxes vanish.

In the absence of external potentials, we can see that the obvious constant states are the only stationary solutions:

Proposition 3.1. *Let $m_r + m_b < |\Omega|$ and let $V = W = 0$. Then $(r_\infty, b_\infty) = \frac{1}{|\Omega|}(m_r, m_b)$ is the only stationary solution in \mathcal{M}*

Proof. We have seen above that both fluxes vanish at a stationary solution. Adding them with suitable scaling gives $\nabla \rho_\infty = 0$ almost everywhere, thus ρ_∞ is constant. Since $\rho_\infty \equiv 1$ is excluded by the condition on the total mass ($\rho_\infty \equiv 1$ implies $m_r + m_b = |\Omega|$), we further obtain from the vanishing fluxes that $\nabla r_\infty = \nabla b_\infty = 0$ almost everywhere. Consequently r_∞ and b_∞ are constant in Ω and the assertion follows from the mass constraint. \square

These constant stationary states can indeed be characterized as the unique minima of the (strictly convex) entropy, which follows from a standard computation (CP. [7]) omitted here:

Theorem 3.2. *Let $m_r + m_b < |\Omega|$ and let $V = W = 0$. Then the stationary solution $(r_\infty, b_\infty) = \frac{1}{|\Omega|}(m_r, m_b)$ is the unique minimizer of the entropy E in \mathcal{M}*

In the above results we only needed to exclude the degenerate case $m_r + m_b = |\Omega|$, which is equivalent to $\rho \equiv 1$. In this case it is obvious that any state $(r, m) \in \mathcal{M}$ is a stationary one. This corresponds also to the intuition from the stochastic particle model used for the derivation of the model. If all sites are occupied, then of course there is no chance of movement for any particle, and any such state will remain stationary.

Analogous arguments can be carried out in the case with external potential to obtain:

Theorem 3.3. *Let $m_r + m_b < |\Omega|$. Then the stationary solution*

$$(r_\infty, b_\infty) = \left(\frac{e^{u_\infty - V}}{1 + e^{u_\infty - V} + e^{v_\infty - W}}, \frac{e^{v_\infty - W}}{1 + e^{u_\infty - V} + e^{v_\infty - W}} \right)$$

with constants u_∞ and v_∞ determined uniquely from the conditions

$$\int_{\Omega} r_\infty dx = m_r \quad \int_{\Omega} b_\infty dx = m_b$$

is the unique minimizer of the entropy E in \mathcal{M} .

3.4 Linearization around Equilibrium

Theorem 3.4 (Linear Stability). *The system (3.1), (3.2) is linearly stable with respect to small perturbations $\epsilon u, \epsilon v \in L^2((0, T); H^1(\Omega))$.*

Proof. We will first prove the theorem for the case $V = W = 0$ in detail. Then, we will sketch the extension for the case with external potentials.

We make the following linearization around equilibrium states: $r = r_\infty + \epsilon u$, $b = b_\infty + \epsilon v$ and arrive at the first order linearization

$$\begin{aligned}\partial_t u &= (1 - b_\infty)\Delta u + r_\infty \Delta v \\ \partial_t v &= D(1 - r_\infty)\Delta v + Db_\infty \Delta u.\end{aligned}$$

Combining these equations to obtain an equation for $w = u + \alpha v$ which reads

$$\partial_t w = ((1 - b_\infty) + \alpha Db_\infty)\Delta u + (r_\infty + \alpha D(1 - r_\infty))\Delta v. \quad (3.24)$$

Choosing α such that

$$r_\infty + \alpha D(1 - r_\infty) = \alpha(\alpha Db_\infty + (1 - b_\infty))$$

we obtain the two heat equations

$$\partial_t w = k_{1,2}\Delta w, \quad (3.25)$$

where $k_{1,2}$ are the constants determined by (3.24). The resulting solutions for α are

$$\alpha_{1/2} = \frac{D(1 - r_\infty) - (1 - b_\infty) \pm \sqrt{((1 - b_\infty) - D(1 - r_\infty))^2 + 4Dr_\infty b_\infty}}{2Db_\infty},$$

which are both real-valued due to r_∞, b_∞ and D being larger than zero. Knowing that

$$\sqrt{((1 - b_\infty) - D(1 - r_\infty))^2 + 4Dr_\infty b_\infty} \geq |D(1 - r_\infty) - (1 - b_\infty)|,$$

is follows

$$\alpha_1 \geq 0, \quad \alpha_2 \leq 0.$$

We want to find out weather the constant k in the heat equation (3.25) is larger or less than zero.

$$k_{1/2} = \frac{1}{2}(D(1 - r_\infty) + (1 - b_\infty) \pm \sqrt{((1 - b_\infty) - D(1 - r_\infty))^2 + 4Dr_\infty b_\infty})$$

It is obvious that k_1 is always larger than zero. The condition for k_2 reads

$$1 - r_\infty - b_\infty \geq 0, \quad (3.26)$$

thus in case that $r_\infty + b_\infty = 1$, $k_2 = 0$ and we have $\partial_t w = 0$. Otherwise k_2 is always larger than zero. Thus small perturbation behave according to the heat equation, meaning that they smooth out in time. Therefore the system is stable.

To analyze the linear stability with a given external potential, we linearize the system in entropy variables (see (3.10), (3.11)), i.e. $\xi = \xi_\infty + \epsilon u$, $\eta = \eta_\infty + \epsilon v$. We obtain the first order approximation

$$\begin{aligned}\partial_\xi r \partial_t u + \partial_\eta r \partial_t v &= \nabla \cdot (r_\infty(1 - \rho_\infty)\nabla u), \\ \partial_\xi b \partial_t u + \partial_\eta b \partial_t v &= \nabla \cdot (Db_\infty(1 - \rho_\infty)\nabla v).\end{aligned}$$

This can be written as

$$H_{E^*} \begin{pmatrix} \partial_t u \\ \partial_t v \end{pmatrix} = \nabla \cdot \left(\mathbf{M}(r_\infty, b_\infty) : \begin{pmatrix} \nabla u \\ \nabla v \end{pmatrix} \right).$$

Here, by H_{E^*} we denote the Hessian of the dual entropy functional E^* and

$$\mathbf{M} := \begin{pmatrix} r_\infty(1 - \rho_\infty) & 0 \\ 0 & Db_\infty(1 - \rho_\infty) \end{pmatrix}.$$

We need to show that both these matrices are positive semidefinite. From the identity $\nabla E^*(\nabla E) = I$ we obtain

$$\nabla E^* = \begin{pmatrix} \frac{e^{\xi-V}}{1+e^{\xi-V}+e^{\eta-W}} & \\ & \frac{e^{\eta-W}}{1+e^{\eta-W}+e^{\xi-V}} \end{pmatrix}. \quad (3.27)$$

It is straight forward to calculate the symmetric matrix H_{E^*} and check that all eigenvalues are non-negative. For M this is clear as $\rho_\infty < 1$. Following similar arguments as in the case without potential, we conclude that the system is linear stable. \square

3.5 Well-Posedness Close to Equilibrium

Theorem 3.5 (Existence near Equilibrium). *Consider the system (3.1), (3.2) with no-flux boundary conditions and initial data r_0, b_0 belonging to*

$$r_0, b_0 \in H^2(\Omega).$$

Furthermore, assume

$$\|r_0 - r_\infty\|_{H^2(\Omega)} \leq \epsilon, \|b_0 - b_\infty\|_{H^2(\Omega)} \leq \epsilon$$

for ϵ sufficiently small. Then, there exists a unique solution to (3.1), (3.2) in

$$B_R = \{(r, b) : \|r - r_\infty\|_X \leq R, \|b - b_\infty\|_X \leq R\},$$

with

$$X := L^\infty((0, T); H^2(\Omega)) \cap L^2((0, T); H^3(\Omega)) \cap H^1((0, T); H^1(\Omega)) \quad (3.28)$$

and where the constant R depends on ϵ only. Here, the norm on X is given by

$$\|\cdot\|_X = \sqrt{\|\cdot\|_{L^\infty((0, T); H^2(\Omega))}^2 + \|\cdot\|_{L^2((0, T); H^3(\Omega))}^2 + \|\cdot\|_{H^1((0, T); H^1(\Omega))}^2}.$$

Proof. The proof will be based on Banach's fixed point theorem. Therefore, the first step is to construct the fixed-point operator.

Construction of the fixed point operator

We consider the evolution of $r - r_\infty$ and $b - b_\infty$ which we write as

$$\begin{aligned} \partial_t(r - r_\infty) - (1 - b_\infty)\Delta r - r_\infty\Delta b = \\ \underbrace{(r - r_\infty)\Delta(b - b_\infty) - (b - b_\infty)\Delta(r - r_\infty)}_{=: F_1(r, b)}, \end{aligned} \quad (3.29)$$

$$\begin{aligned} \partial_t(b - b_\infty) - D((1 - r_\infty)\Delta b - b_\infty\Delta r) = \\ D(\underbrace{(b - b_\infty)\Delta(r - r_\infty) - (r - r_\infty)\Delta(b - b_\infty)}_{=: F_2(r, b)}). \end{aligned} \quad (3.30)$$

We want to apply the fixed-point theorem to the operator

$$G : X \times X \rightarrow X \times X$$

which gives the two solutions to (3.29), (3.30) for given r, b on the right hand side. From Section 3.4, we know that it is possible to find $\alpha_{1,2}$ such that (3.29), (3.30) can be rewritten as

$$\partial_t w_1 - k_1 \Delta w_1 = F_1 + \alpha_1 F_2, \quad (3.31)$$

$$\partial_t w_2 - k_2 \Delta w_2 = F_1 + \alpha_2 F_2, \quad (3.32)$$

where the $w_{1,2}$ are given by

$$w_1 = (r - r_\infty) + \alpha_1(b - b_\infty), \quad w_2 = (r - r_\infty) + \alpha_2(b - b_\infty). \quad (3.33)$$

Using this property, we consider the operator G to be composed of several operations. It consists of an operator $F = (F_1, F_2) : X \times X \rightarrow Y \times Y$ (where Y will be determined below), and an operator $S : Y \times Y \rightarrow Y \times Y$ which maps two elements of Y to their linear combinations with coefficient α_1 or α_2 , respectively. The operator $L = (L_1, L_2)$ provides the solutions to the equations (3.31), (3.32) for given right-hand-side. We will show that this operator maps from $Y \times Y$ to $X \times X$. Finally, we need an operator S' , defined as the inverse of S , to obtain the solution from their linear combination. With these definitions, we can write

$$G = S' \circ L \circ S \circ F. \quad (3.34)$$

In the remaining part of this proof we will ensure that the assumptions needed for Banach's fixed point theorem are satisfied by analyzing each of these operators separately.

Properties of F

Our first goal is to identify the space in which the operator $F = (F_1, F_2)$ maps. In the following we will use that H^2 is embedded into L^∞ for $n = 1, 2, 3$ (see [1, Thm. 5.4]).

Applying operator F_1 to some $r, b \in X$, we obtain

$$F_1(r, b) = \underbrace{(r - r_\infty)}_{L^\infty} \underbrace{\Delta(b - b_\infty)}_{L^2} - \underbrace{(b - b_\infty)}_{L^\infty} \underbrace{\Delta(r - r_\infty)}_{L^2} \in L^2.$$

We also note that they are of order ϵ^2 in L^2 . Differentiating again we have

$$\nabla F_1(r, b) = \nabla r \Delta b + r \nabla \Delta b - b \nabla \Delta r - \nabla b \Delta r.$$

We know $\nabla r, \nabla b \in L^2((0, T); L^\infty)$. From the definition of X we furthermore have $\Delta r, \Delta b \in L^\infty((0, T); L^2)$. Therefore, the product terms $\nabla r \Delta b$ and $\nabla b \Delta r$ above are in $L^2((0, T); L^2)$. Furthermore, as $r, b \in X$, we have

$$\nabla \Delta r, \nabla \Delta b \in L^2((0, T); L^2(\Omega)),$$

we conclude that the terms $b \nabla \Delta r$ and $r \nabla \Delta b$ are in $L^2((0, T); L^2)$ as well. Therefore we know that the operator F_1 (and thus F_2 as well) both map from X into the space

$$Y = L^\infty((0, T); L^2) \cap L^2((0, T); H^1)$$

and thus $F : X \times X \rightarrow Y \times Y$ is well defined and continuous (bilinear).

Properties of S and S'

We note that the operators

$$S : Y \times Y \rightarrow Y \times Y, \quad S' : X \times X \rightarrow X \times X$$

are bounded due to the boundedness of $\alpha_{1,2}$. We denote their norms by C_S and $C_{S'}$.

Properties of L

The operator

$$L : Y \times Y \rightarrow X \times X$$

gives a vector containing the solutions of the two heat equations (3.31), (3.32) for given right-hand-side. Employing well-known results about the heat operator (see, e.g. [34]) we conclude that

$$w_1, w_2 \in X$$

and furthermore

$$\begin{aligned} \|w_{1,2}\|_X &= (\text{ess sup}_{0 \leq t \leq T} \|w\|_{L^\infty((0,T);H^2)} + \|w\|_{L^2((0,T);H^3)} + \|w\|_{H^1((0,T);H^1)}) \\ &\leq C(\underbrace{\|F_2 + \alpha_{1,2}F_2\|_{L^2}}_{\sim \epsilon^2} + \underbrace{\|(r_0 - r_\infty, b_0 - b_\infty)\|_{H_0^1}}_{\sim \epsilon}) =: R_{1,2}(\epsilon). \end{aligned} \quad (3.35)$$

We choose $R = \max(R_1, R_2)$ the way that both w_1 and w_2 are bounded by R .

Selfmapping Property of G

Reiterating the above steps, we obtain from (3.35)

$$R \leq C(R^2 + \epsilon).$$

Replacing the less equal by an equal sign, we denote by $R = R(\epsilon)$ the smallest solution of this equation. Clearly, $R(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. From the above considerations we conclude that in the ball

$$B_R = \{(r, b) : \|r - r_\infty\|_X \leq R, \|b - b_\infty\|_X \leq R\},$$

the operator G is selfmapping.

Contractivity of G

It remains to show contractiveness. We first examine the operator F . Starting with F_1 we obtain

$$\begin{aligned} \|F_1(r_1, b_1) - F_1(r_2, b_2)\|_Y &= \|(r_1 - r_\infty)\Delta(b_1 - b_\infty) - (b_1 - b_\infty)\Delta(r_1 - r_\infty) \\ &\quad - (r_2 - r_\infty)\Delta(b_2 - b_\infty) + (b_2 - b_\infty)\Delta(r_2 - r_\infty)\|_Y \\ &\leq \|(r_1 - r_2)\Delta(b_1 - b_\infty)\|_Y + \|(r_2 - r_\infty)\Delta(b_1 - b_2)\|_Y \\ &\quad - \|(b_1 - b_2)\Delta(r_1 - r_\infty)\|_Y - \|(b_2 - b_\infty)\Delta(r_1 - r_2)\|_Y \\ &\leq 2R(\|r_1 - r_2\|_X + \|b_1 - b_2\|_X). \end{aligned}$$

From an analogous calculation for F_2 we conclude

$$\|F(r_1, b_1) - F(r_2, b_2)\|_Y \leq 4R(\|r_1 - r_2\|_X + \|b_1 - b_2\|_X).$$

Using the properties of L (namely (3.35) and its linearity) allows us to write

$$\begin{aligned} \|G(r_1, b_1) - G(r_2, b_2)\|_X &\leq \|S'\|_X \|L\|_X \|S\|_Y 4R \|r_1 - r_2\|_X + \|b_1 - b_2\|_X \\ &\leq C_S C_4 R C_{S'} (\|r_1 - r_2\|_X + \|b_1 - b_2\|_X). \end{aligned}$$

Thus we can ensure the contractiveness of G by choosing ϵ (and in turn R) such that

$$R < \frac{1}{C_S C_4 C_{S'}}.$$

The application of Banach's fixed-point theorem provides the existence of solutions (r, b) in B_R and this finishes the proof. \square

We finally mention that together with the linear stability analysis carried out above we expect exponential convergence of solutions to equilibrium for large time, if the initial value is sufficiently close to equilibrium. However, due to a gap of norms used in those results, a rigorous proof becomes difficult and is thus not attempted here. We shall instead consider the large time asymptotics of the problem with arbitrarily large initial values in Section 5 and directly prove convergence to equilibrium by entropy methods.

4 Global Existence

In the following we show the global existence of weak solutions to (3.7), (3.8) in the absence of external potentials, i.e. we assume $V = W = 0$ throughout this section.

Theorem 4.1 (Global Existence). *There exists a weak solution for*

$$\partial_t r = \operatorname{div}((1 - \rho)\nabla r + r\nabla\rho) \quad (4.1)$$

$$\partial_t b = \operatorname{div}((1 - \rho)\nabla b + b\nabla\rho) \quad (4.2)$$

in $(L^2((0, T); L^2) \cap H^1((0, T); H^{-1}))^2$, such that

$$\rho, \sqrt{1 - \rho r}, \sqrt{1 - \rho b} \in L^2((0, T); H^1),$$

and furthermore

$$0 \leq r, \quad 0 \leq b, \quad b + r \leq 1 \quad \text{almost everywhere.}$$

Proof. We consider the system in entropy variables, i.e. we employ transformations

$$u = \frac{\partial E}{\partial r} = \log \frac{r}{1 - \rho}$$

$$v = \frac{\partial E}{\partial b} = \log \frac{b}{1 - \rho}$$

and thus

$$r = \frac{e^u}{1 + e^u + e^v}, \quad b = \frac{e^v}{1 + e^u + e^v}. \quad (4.3)$$

In these variables, the system is of the form

$$\partial_t \left(\frac{e^u}{1 + e^u + e^v} \right) = \operatorname{div} \left(\frac{e^u}{(1 + e^u + e^v)^2} \nabla u \right) \quad (4.4)$$

$$\partial_t \left(\frac{e^v}{1 + e^u + e^v} \right) = \operatorname{div} \left(\frac{e^v}{(1 + e^u + e^v)^2} \nabla v \right) \quad (4.5)$$

or

$$a(u, v)u_t + b(u, v)v_t = \operatorname{div}(A(u, v)\nabla u), \quad (4.6)$$

$$b(u, v)u_t + c(u, v)v_t = \operatorname{div}(B(u, v)\nabla v) \quad (4.7)$$

with

$$a(u, v) = \frac{e^u(1 + e^v)}{(1 + e^u + e^v)^2}, \quad b(u, v) = \frac{-e^u e^v}{(1 + e^u + e^v)^2}, \quad c(u, v) = \frac{e^v(1 + e^u)}{(1 + e^u + e^v)^2}.$$

Apparently, $A(u, v) \geq 0$ and $B(u, v) \geq 0$, and the matrix

$$M := \begin{pmatrix} a(u, v) & b(u, v) \\ b(u, v) & c(u, v) \end{pmatrix} \quad (4.8)$$

is positive semi-definite, because $a \geq |b|$ and $c \geq |b|$. The proof will consist of different steps.

4.1 Existence for the approximated system

We will first proof global in time existence for an approximated system with

$$M_\epsilon := M + \epsilon I, \quad (4.9)$$

where I is the two dimensional identity matrix. For $\epsilon > 0$ the matrix M_ϵ is positive definite and furthermore we approximate the diffusion coefficients as

$$A_\epsilon(u, v) := A(u, v) + \epsilon > 0, \quad B_\epsilon(u, v) := B(u, v) + \epsilon > 0. \quad (4.10)$$

Next, we introduce the following Galerkin approximation. For some fixed $n \in \mathbb{N}$ we choose $\mathcal{V}_n \subset H^1$ such that the n dimensional Galerkin space \mathcal{V}_n contains the constant functions for all n and furthermore $\cup_n \mathcal{V}_n$ is dense in $W^{1,\infty}$. We now try to find solutions $u^{\epsilon,n}$ and $v^{\epsilon,n}$ to the weak formulation of the approximated system given by

$$\int_{\Omega} (a_\epsilon(u^{\epsilon,n}, v^{\epsilon,n})u_t^{\epsilon,n}\phi + b(u^{\epsilon,n}, v^{\epsilon,n})v_t^{\epsilon,n}\phi) dx = - \int_{\Omega} A_\epsilon(u^{\epsilon,n}, v^{\epsilon,n})\nabla u^{\epsilon,n} \cdot \nabla \phi dx, \quad (4.11a)$$

$$\int_{\Omega} (b(u^{\epsilon,n}, v^{\epsilon,n})u_t^{\epsilon,n}\psi + c_\epsilon(u^{\epsilon,n}, v^{\epsilon,n})v_t^{\epsilon,n}\psi) dx = - \int_{\Omega} B_\epsilon(u^{\epsilon,n}, v^{\epsilon,n})\nabla v^{\epsilon,n} \cdot \nabla \psi dx \quad (4.11b)$$

for all $\phi, \psi \in \mathcal{V}_n$. Choosing an orthonormal basis φ_j of \mathcal{V}_n , we can express

$$u^{\epsilon,n} = \sum_{j=1}^n \alpha_j^{\epsilon,n}(t)\varphi_j(x),$$

$$v^{\epsilon,n} = \sum_{j=1}^n \beta_j^{\epsilon,n}(t)\varphi_j(x),$$

where the coefficients $\alpha_j^{\epsilon,n}(t)$ and $\beta_j^{\epsilon,n}(t)$ need to be determined. From (4.11a), (4.11b) we know that the $\alpha_j^{\epsilon,n}(t)$ fulfil the system of ordinary differential equations

$$\begin{aligned} \sum_{j,k} \int_{\Omega} \tilde{a}_\epsilon(\alpha^{\epsilon,n}, \beta^{\epsilon,n}, x) \frac{d\alpha_j^{\epsilon,n}(t)}{dt} \varphi_j(x)\varphi_k(x) dx + \sum_{j,k} \int_{\Omega} \tilde{b}(\alpha^{\epsilon,n}, \beta^{\epsilon,n}, x) \frac{d\beta_j^{\epsilon,n}(t)}{dt} \varphi_j(x)\varphi_k(x) dx = \\ - \sum_{j,k} \int_{\Omega} \tilde{A}_\epsilon(\alpha^{\epsilon,n}, \beta^{\epsilon,n}, x) \alpha_j^{\epsilon,n} \nabla \varphi_j \nabla \varphi_k dx, \\ \sum_{j,k} \int_{\Omega} \tilde{b}(\alpha^{\epsilon,n}, \beta^{\epsilon,n}, x) \frac{d\alpha_j^{\epsilon,n}(t)}{dt} \varphi_j(x)\varphi_k(x) dx + \sum_{j,k} \int_{\Omega} \tilde{c}_\epsilon(\alpha^{\epsilon,n}, \beta^{\epsilon,n}, x) \frac{d\beta_j^{\epsilon,n}(t)}{dt} \varphi_j(x)\varphi_k(x) dx = \\ - \sum_{j,k} \int_{\Omega} \tilde{B}_\epsilon(\alpha^{\epsilon,n}, \beta^{\epsilon,n}, x) \beta_j^{\epsilon,n} \nabla \varphi_j \nabla \varphi_k dx. \end{aligned}$$

Defining

$$\gamma^{\epsilon,n}(t) = (\alpha_1^{\epsilon,n}, \dots, \alpha_n^{\epsilon,n}, \beta_1^{\epsilon,n}, \dots, \beta_n^{\epsilon,n}) \quad (4.13)$$

we obtain the following ODE:

$$\tilde{M}(\gamma^{\epsilon,n}(t)) \frac{d}{dt} \gamma^{\epsilon,n}(t) = R(\gamma^{\epsilon,n}(t)).$$

Due to \tilde{M} being positive definite, which follows from the properties of M_ϵ , there exists the inverse \tilde{M}^{-1} . Thus the existence of a solution to (4.1) follows from standard ODE theory via the Picard-Lindelöf theorem ([21]), since it is easy to verify that R and \tilde{M} are Lipschitz continuous.

4.2 Passing to the Limit - Preliminaries

Due to $\frac{\partial E}{\partial r} = u$, the convex conjugate E^* of E fulfils $\frac{\partial E^*}{\partial u} = r$. For the approximated system we have the dual entropy

$$\tilde{E}_\epsilon(u, v) = E^*(r(u, v), b(u, v)) + \frac{\epsilon}{2} \int_{\Omega} u^2 dx + \frac{\epsilon}{2} \int_{\Omega} v^2 dx$$

with

$$\tilde{E}_\epsilon''(u, v) = \begin{pmatrix} a_\epsilon & b \\ b & c_\epsilon \end{pmatrix}$$

We examine the relative entropy or Bregman distance between (u, v) and constant equilibria (u_∞, v_∞) :

$$\begin{aligned} & \frac{d}{dt} \left(\overbrace{\tilde{E}_\epsilon(u_\infty, v_\infty) - \tilde{E}_\epsilon(u^{\epsilon, n}, v^{\epsilon, n}) - \tilde{E}'_\epsilon(u^{\epsilon, n}, v^{\epsilon, n}) \cdot (u_\infty - u^{\epsilon, n}, v_\infty - v^{\epsilon, n})^T}^{D_\epsilon} \right) \\ &= (\partial_t u^{\epsilon, n}, \partial_t v^{\epsilon, n})^T \cdot (\tilde{E}_\epsilon''(u^{\epsilon, n}, v^{\epsilon, n}) (u_\infty - u^{\epsilon, n}, v_\infty - v^{\epsilon, n})^T) \end{aligned}$$

For the right-hand side we can insert the weak formulation (4.11a), (4.11b) with test functions

$$\phi = u_\infty - u^{\epsilon, n} \in \mathcal{V}_n, \quad (4.14)$$

$$\psi = v_\infty - v^{\epsilon, n} \in \mathcal{V}_n, \quad (4.15)$$

thus

$$\begin{aligned} \frac{d}{dt} D_\epsilon &= - \int_{\Omega} A_\epsilon(u^{\epsilon, n}, v^{\epsilon, n}) \nabla u^{\epsilon, n} \cdot (\nabla u^{\epsilon, n} - \nabla u_\infty) dx \\ &\quad - \int_{\Omega} B_\epsilon(u^{\epsilon, n}, v^{\epsilon, n}) \nabla v^{\epsilon, n} \cdot (\nabla v^{\epsilon, n} - \nabla v_\infty) dx. \end{aligned}$$

Integrating this expression leads to

$$D_\epsilon(t) + \int_0^T \int_{\Omega} A_\epsilon |\nabla u^{\epsilon, n}|^2 dx dt + \int_0^T \int_{\Omega} B_\epsilon |\nabla v^{\epsilon, n}|^2 dx dt \leq D_\epsilon(0) =: C.$$

As $A_\epsilon \geq \epsilon$, $B_\epsilon \geq \epsilon$ we immediately conclude

$$\epsilon \int_0^T \int_{\Omega} |\nabla u^{\epsilon, n}|^2 dx dt + \epsilon \int_0^T \int_{\Omega} |\nabla v^{\epsilon, n}|^2 dx dt \leq C \quad (4.16)$$

and furthermore

$$\frac{\epsilon}{2} \int_{\Omega} (u^{\epsilon, n})^2 + (v^{\epsilon, n})^2 dx \leq C \quad (4.17)$$

for almost every $t \in (0, T)$. Thus $u^{\epsilon, n}$ and $v^{\epsilon, n}$ are uniformly bounded in the space

$$L^\infty((0, T); L^2) \cap L^2((0, T); H^1)$$

Next we consider

$$\epsilon \leq A_\epsilon(u, v) = \epsilon + \frac{e^u}{(1 + e^u + e^v)^2} \leq \epsilon + 1$$

which leads to

$$A_\epsilon(u^{\epsilon, n}, v^{\epsilon, n}) \nabla u^{\epsilon, n} \in L^2((0, T); L^2)$$

and therefore

$$\operatorname{div}(A_\epsilon(u^{\epsilon, n}, v^{\epsilon, n}) \nabla u^{\epsilon, n}) \in L^2((0, T); H^{-1}).$$

In the same way we obtain

$$\operatorname{div}(B_\epsilon(u^{\epsilon, n}, v^{\epsilon, n}) \nabla v^{\epsilon, n}) \in L^2((0, T); H^{-1}).$$

Using these results, we can conclude that the function

$$F_\epsilon(u^{\epsilon, n}, v^{\epsilon, n}) := \left(\epsilon u^{\epsilon, n} + \frac{e^{u^{\epsilon, n}}}{1 + e^{u^{\epsilon, n}} + e^{v^{\epsilon, n}}}, \epsilon v^{\epsilon, n} + \frac{e^{v^{\epsilon, n}}}{1 + e^{u^{\epsilon, n}} + e^{v^{\epsilon, n}}} \right) \quad (4.18)$$

has the following regularity properties:

$$\partial_t F_\epsilon(u^{\epsilon, n}, v^{\epsilon, n}) \in (L^2((0, T); H^{-1}))^2$$

and

$$\nabla F_\epsilon(u^{\epsilon, n}, v^{\epsilon, n}) \in (L^2((0, T); L^2))^2.$$

Thus

$$F_\epsilon(u^{\epsilon, n}, v^{\epsilon, n}) \in (L^2((0, T); H^1))^2 \cap (H^1((0, T); H^{-1}))^2. \quad (4.19)$$

Employing the Lemma of Aubin-Lions ([41], III Prop. 1.3) we conclude the existence of a subsequence

$$\{F_\epsilon(u^{\epsilon, n_j}, v^{\epsilon, n_j})\}_{n_j}$$

strongly converging in $(L^2((0, T); L^2))^2$. As F_ϵ is strictly monotone and thus invertible, we find strongly converging subsequences u^{ϵ, n_j} and v^{ϵ, n_j} as well. Furthermore, due to (4.17) there exists a weakly converging subsequence in $L^2((0, T); H^1)$ and a weakly converging subsequence of $F_\epsilon(u^{\epsilon, n}, v^{\epsilon, n})$ in $H^1((0, T); H^{-1})$.

4.3 Passing to the limit $n \rightarrow \infty$

Next we use the above considerations, more precisely

$$u^{\epsilon, n} \rightarrow u^\epsilon \quad v^{\epsilon, n} \rightarrow v^\epsilon \quad \text{in } L^2$$

to pass to the limit $n \rightarrow \infty$. We first consider $\phi, \psi \in W^{1, \infty}$ and a sequence of test functions $\phi_n, \psi_n \in \mathcal{V}_n$ such that

$$\phi_n \rightarrow \phi, \quad \psi_n \rightarrow \psi \quad \text{as } n \rightarrow \infty \text{ in } W^{1, \infty} \hookrightarrow H^1.$$

We know

$$\partial_t F_\epsilon(u^{\epsilon,n}, v^{\epsilon,n}) \rightharpoonup \partial_t F_\epsilon(u^\epsilon, v^\epsilon) \quad \text{in } H^{-1}, \quad (4.20)$$

therefore

$$\langle \partial_t F_\epsilon^1(u^{\epsilon,n}, v^{\epsilon,n}), \phi_n \rangle \rightarrow \langle \partial_t F^1(u^\epsilon, v^\epsilon), \phi \rangle \quad (4.21)$$

for almost all t . Furthermore

$$\begin{aligned} A_\epsilon(u^{\epsilon,n}, v^{\epsilon,n}) &\rightarrow A_\epsilon(u^\epsilon, v^\epsilon) \text{ in } L^2, \\ \nabla u^{\epsilon,n} &\rightharpoonup \nabla u^\epsilon \text{ in } L^2 \end{aligned}$$

and thus

$$\langle A_\epsilon(u^{\epsilon,n}, v^{\epsilon,n}) \nabla u^{\epsilon,n}, \nabla \phi_n \rangle \rightarrow \langle A_\epsilon(u^\epsilon, v^\epsilon) \nabla u^\epsilon, \nabla \phi \rangle.$$

From this we finally conclude that the limit functions u^ϵ , v^ϵ are solutions of the weak formulations (4.11a), (4.11b) for all test functions in $W^{1,\infty}$. Note that by density of $W^{1,\infty}$ test functions in H^1 can be approximated by sequences $\phi_m \in W^{1,\infty}$. As

$$A_\epsilon(u^\epsilon, v^\epsilon) \in L^\infty,$$

it follows

$$A_\epsilon \nabla u^\epsilon \in L^2$$

and

$$\langle A_\epsilon \nabla u^\epsilon, \phi_m \rangle \rightarrow \langle A_\epsilon \nabla u^\epsilon, \phi \rangle,$$

Applying this argument also to all other terms, we finally find that the weak formulation is also fulfilled for test functions in H^1 . Furthermore, the entropy estimates are retained in the limit process. Due to convergence in L^2 , we find

$$\liminf_{n \rightarrow \infty} \int_0^T \int_\Omega (A_\epsilon |\nabla u^{\epsilon,n}|^2 + B_\epsilon |\nabla v^{\epsilon,n}|^2) \geq \int_0^T \int_\Omega (A_\epsilon |\nabla u^\epsilon|^2 + B_\epsilon |\nabla v^\epsilon|^2) \, dxdt$$

The sequence $D_\epsilon(t)$ is lower-semi continuous with respect to n , thus we still have the bound

$$D_\epsilon(t) + \int_0^T \int_\Omega (A_\epsilon |\nabla u^\epsilon|^2 + B_\epsilon |\nabla v^\epsilon|^2) \, dxdt \leq C.$$

4.4 Passing to the limit $\epsilon \rightarrow 0$

Using the entropy dissipation, i.e.

$$\begin{aligned} &E(r^\epsilon(t), b^\epsilon(t)) + \frac{\epsilon}{2} \int_\Omega u^\epsilon(t)^2 \, dx + \frac{\epsilon}{2} \int_\Omega v^\epsilon(t)^2 \, dx \\ &+ \int_0^T \int_\Omega (|\nabla \sqrt{1 - \rho^\epsilon}|^2 + (1 - \rho^\epsilon) |\nabla r^\epsilon|^2 + (1 - \rho^\epsilon) |\nabla b^\epsilon|^2) \, dxdt \\ &+ \epsilon \int_0^T \int_\Omega (|\nabla u^\epsilon|^2 + |\nabla v^\epsilon|^2) \, dxdt \\ &\leq C. \end{aligned}$$

we have the following a priori estimates

$$\begin{aligned} \sqrt{\epsilon}\partial_t u^\epsilon &\text{ bounded in } L^2((0, T); H^{-1}), \\ \sqrt{\epsilon}u^\epsilon &\text{ bounded in } L^2((0, T); H^1) \end{aligned}$$

and thus

$$\epsilon\partial_t u^\epsilon \rightarrow 0 \text{ in } L^2((0, T); H^{-1}), \quad (4.22)$$

$$\epsilon\Delta u^\epsilon \rightarrow 0 \text{ in } L^2((0, T); H^{-1}). \quad (4.23)$$

Moreover we obviously have

$$0 \leq r^\epsilon = \frac{e^{u^\epsilon}}{1 + e^{u^\epsilon} + e^{v^\epsilon}}, \quad 0 \leq b^\epsilon = \frac{e^{v^\epsilon}}{1 + e^{u^\epsilon} + e^{v^\epsilon}}$$

and

$$1 \geq r^\epsilon + b^\epsilon = 1 - \frac{1}{1 + e^{u^\epsilon} + e^{v^\epsilon}}$$

almost everywhere, which remains true for the weak limits. Next we estimate the H^{-1} norm of $\tilde{r}^\epsilon := r^\epsilon + \epsilon u^\epsilon$. To do this we first solve

$$\begin{cases} \partial_t \tilde{r}^\epsilon = \Delta \psi, \\ \psi|_{\partial\Omega} = 0. \end{cases}$$

Now we have

$$\begin{aligned} \|\nabla\psi\|_{L^2((0,T);L^2)}^2 &= \int_0^T \int_\Omega |\nabla\psi|^2 \, dxdt = - \int_0^T \int_\Omega \psi \Delta\psi \, dxdt + \int \int_{\partial\Omega} \psi \nabla\psi \cdot n \, dsdt \\ &= \int_0^T \int_\Omega \nabla\psi((1 - \rho^\epsilon)\nabla r^\epsilon + r^\epsilon\nabla\rho^\epsilon + \epsilon\nabla u^\epsilon) \, dxdt \\ &\quad - \int_0^T \int_{\partial\Omega} \psi((1 - \rho^\epsilon)\nabla r^\epsilon + r^\epsilon\nabla\rho^\epsilon + \epsilon\nabla u^\epsilon) \cdot n \, dsdt \\ &\leq \|\nabla\psi\|_{L^2((0,T);L^2)} \|((1 - \rho^\epsilon)\nabla r^\epsilon + r^\epsilon\nabla\rho^\epsilon + \epsilon u^\epsilon)\|_{L^2((0,T);L^2)} \\ &\leq \|\nabla\psi\|_{L^2((0,T);L^2)} \left(\underbrace{\|\sqrt{1 - \rho^\epsilon}\|_{L^2((0,T);L^\infty)}}_{\leq 1} \|\sqrt{1 - \rho^\epsilon}\nabla r^\epsilon\|_{L^2((0,T);L^2)} \right. \\ &\quad \left. + \underbrace{\|r^\epsilon\|_{L^2((0,T);L^\infty)}}_{\leq 1} \|\nabla\rho^\epsilon\|_{L^2((0,T);L^2)} + \|\epsilon\nabla u^\epsilon\|_{L^2((0,T);L^2)} \right) \\ &\leq C\|\nabla\psi\|_{L^2((0,T);L^2)}. \end{aligned}$$

From the boundedness of $\|\nabla\psi\|_{L^2((0,T);L^2)}$ we conclude

$$\|\Delta\psi\|_{L^2((0,T);H^{-1})} \leq \tilde{C}$$

and thus

$$\|\partial_t \tilde{r}^\epsilon\|_{L^2((0,T);H^{-1})} \leq \tilde{C}. \quad (4.24)$$

This implies the existence of a weakly converging subsequence (which we again call $\partial_t \tilde{r}^\epsilon$) such that $\partial_t \tilde{r}^\epsilon \rightharpoonup \tilde{q}$. However, as we know that

$$\partial_t r^\epsilon = \partial_t \tilde{r}^\epsilon - \epsilon \partial_t u^\epsilon$$

we conclude that $q = \lim_{\epsilon \rightarrow 0} \partial_t r^\epsilon$. Furthermore, as r^ϵ is bounded in L^∞ ($0 \leq r^\epsilon \leq 1$) there is also a sequence converging weakly-* to some r in L^∞ . Next we note that due to $\nabla \sqrt{1 - \rho^\epsilon} \in L^2$ (which follows from the entropy estimate) we have

$$\nabla(\sqrt{1 - \rho^\epsilon r^\epsilon}) = \nabla \sqrt{1 - \rho^\epsilon} r^\epsilon + \sqrt{1 - \rho^\epsilon} \nabla r^\epsilon \in L^2, \quad (4.25)$$

and therefore the existence a converging subsequence of

$$\sqrt{1 - \rho^\epsilon r^\epsilon} \text{ in } L^2((0, T); H^1). \quad (4.26)$$

To finally pass to the limit $\epsilon \rightarrow 0$, we transform u, v back into the original variables, i.e.

$$r^\epsilon = \frac{e^{u^\epsilon}}{1 + e^{u^\epsilon} + e^{v^\epsilon}}, \quad b^\epsilon = \frac{e^{v^\epsilon}}{1 + e^{u^\epsilon} + e^{v^\epsilon}}.$$

and consider the equations (4.1), (4.2) but with the ϵ -terms still in entropy variables:

$$\begin{aligned} \partial_t r^\epsilon + \epsilon \partial_t u^\epsilon &= \operatorname{div}((1 - \rho^\epsilon) \nabla r^\epsilon + r^\epsilon \nabla \rho^\epsilon + \epsilon \nabla u^\epsilon) \\ \partial_t b^\epsilon + \epsilon \partial_t v^\epsilon &= \operatorname{div}((1 - \rho^\epsilon) \nabla b^\epsilon + r^\epsilon \nabla \rho^\epsilon + \epsilon \nabla v^\epsilon). \end{aligned}$$

This system can be rewritten as

$$\begin{aligned} \partial_t r^\epsilon + \epsilon \partial_t u^\epsilon &= \operatorname{div} \left(\sqrt{1 - \rho^\epsilon} \nabla(\sqrt{1 - \rho^\epsilon} r^\epsilon) - 3 \sqrt{1 - \rho^\epsilon} r^\epsilon \nabla \sqrt{1 - \rho^\epsilon} + \epsilon \nabla u^\epsilon \right) \\ \partial_t b^\epsilon + \epsilon \partial_t v^\epsilon &= \operatorname{div} \left(\sqrt{1 - \rho^\epsilon} \nabla(\sqrt{1 - \rho^\epsilon} b^\epsilon) - 3 \sqrt{1 - \rho^\epsilon} b^\epsilon \nabla \sqrt{1 - \rho^\epsilon} + \epsilon \nabla v^\epsilon \right). \end{aligned}$$

We notice that the left-hand-sides are in $L^2((0, T); H^{-1})$. $1 - \rho^\epsilon$ is bounded in $L^2((0, T); H^1) \cap H^1((0, T); H^{-1})$. Applying the lemma of Aubin-Lions, we know that there exists a strongly convergent subsequence $\{1 - \rho_k^\epsilon\}_n$ in L^2 . Thus, there exists a strongly convergent subsequence of $\sqrt{1 - \rho^\epsilon}$ in $L^4 \hookrightarrow L^2$. This means we know

$$\begin{aligned} \sqrt{1 - \rho^{\epsilon_k}} &\rightarrow \sqrt{1 - \rho} \\ r^{\epsilon_k} &\rightharpoonup r \end{aligned}$$

and therefore

$$\sqrt{1 - \rho^{\epsilon_k} r^{\epsilon_k}} \rightharpoonup \sqrt{1 - \rho r}.$$

From the above consideration we also know

$$\nabla(\sqrt{1 - \rho^{\epsilon_k} r^{\epsilon_k}}) \rightharpoonup \nabla(\sqrt{1 - \rho r}),$$

which implies the weak convergence of the first term on the right hand side, i.e.

$$\sqrt{1 - \rho^{\epsilon_k}} \nabla(\sqrt{1 - \rho^{\epsilon_k} r^{\epsilon_k}}) \rightharpoonup \sqrt{1 - \rho} \nabla(\sqrt{1 - \rho r}).$$

We know that $\nabla \sqrt{1 - \rho^\epsilon}$ is bounded in L^2 , accordingly there exists a subsequence with

$$\nabla \sqrt{1 - \rho^{\epsilon_k}} \rightharpoonup \nabla \sqrt{1 - \rho}$$

in L^2 . Thus for the convergence of the second term on the right and side it only remains to show that

$$\sqrt{1 - \rho^\epsilon r^{\epsilon_k}} \rightarrow \sqrt{1 - \rho r}. \quad (4.27)$$

To do so we want to apply the Kolmogorov-Riesz-Theorem (see [25, Thm. 5]) and thus estimate

$$\begin{aligned} & \int_0^T \int_\Omega (\sqrt{1 - \rho^\epsilon r^\epsilon}(x+h, t+\tau) - \sqrt{1 - \rho^\epsilon r^\epsilon}(x, t))^2 dx dt \\ & \leq \underbrace{\int_0^T \int_\Omega (\sqrt{1 - \rho^\epsilon r^\epsilon}(x+h, t+\tau) - \sqrt{1 - \rho^\epsilon r^\epsilon}(x, t+\tau))^2 dx dt}_I \\ & \quad + \underbrace{\int_0^T \int_\Omega (\sqrt{1 - \rho^\epsilon r^\epsilon}(x, t+\tau) - \sqrt{1 - \rho^\epsilon r^\epsilon}(x, t))^2 dx dt}_{II}. \end{aligned}$$

For the first term we use the identity

$$\begin{aligned} \sqrt{1 - \rho^\epsilon r^\epsilon}(x+h, t+\tau) - \sqrt{1 - \rho^\epsilon r^\epsilon}(x, t+\tau) &= \\ &= \int_x^{x+h} \nabla(\sqrt{1 - \rho^\epsilon r^\epsilon}(s, t+\tau)) ds \leq h \nabla(\sqrt{1 - \rho^\epsilon r^\epsilon})(\xi, t+\tau), \end{aligned}$$

where $x \leq \xi \leq x+h$. Because $\nabla(\sqrt{1 - \rho^\epsilon r^\epsilon})$ is bounded in L^2 , we know that $I \leq hC$. For the second term we apply

$$\begin{aligned} (II) & \leq \int_0^T \int_\Omega \underbrace{r^\epsilon(x, t+\tau)(\sqrt{1 - \rho^\epsilon(x, t+\tau)} - \sqrt{1 - \rho^\epsilon(x, t)})}_{IIa} dx dt \\ & \quad + \underbrace{\int_0^T \int_\Omega \underbrace{\sqrt{1 - \rho^\epsilon(x, t)}}_{q(x, t)} (r^\epsilon(x, t+\tau) - r^\epsilon(x, t))}_{IIb} dx dt. \end{aligned}$$

We apply Cauchy-Schwarz' inequality for the first term and obtain

$$\begin{aligned} IIa &= \int_0^T \left\langle r^\epsilon(x, t+\tau), \int_t^{t+\tau} \nabla \sqrt{1 - \rho^\epsilon(x, s)} ds \right\rangle dt \\ & \leq \int_0^T \|r^\epsilon\|_{L^2} \tau \|\nabla \sqrt{1 - \rho^\epsilon}\|_{L^2} dt \\ & \leq \tau C. \end{aligned}$$

For the second term we get

$$\begin{aligned}
IIb &= \int_0^T \int_t^{t+\tau} \langle \partial_t r^\epsilon(x, s), q(x, t) \rangle ds dt \\
&\leq \int_0^T \int_t^{t+\tau} \|\partial_t r^\epsilon\|_{H^{-1}} \|q\|_{H^1} ds dt \\
&\leq \int_0^T \|\partial_t r^\epsilon\|_{H^{-1}} \sqrt{\tau} dt \|q\|_{L^2(0, T; H^1)} \\
&\leq \sqrt{T} \sqrt{\tau} \|\partial_t r^\epsilon\|_{L^2(0, T; H^{-1})} \|q\|_{L^2(0, T; H^1)} \\
&\leq C \sqrt{\tau}.
\end{aligned}$$

We apply the same estimates for the second equation. We find that the regularized system converges to system (3.7), (3.8), and thus r and b are solutions of system (3.7), (3.8). \square

5 Asymptotics

In the following we study some special cases and asymptotics in the cross-diffusion model (3.1), (3.2). For simplicity we will put $V = W = 0$ in most arguments.

5.1 Similar Diffusion Scales

The probably simplest case is the one of two types of particles having equal diffusion coefficients, i.e. $D = 1$ in our scaling. For vanishing external potentials the particles are completely indistinguishable in our model and hence one would expect that they can be described well by the total density $\rho = r + b$, which follows a simple diffusion law. Indeed by adding the equations for r and b we immediately obtain the linear diffusion equation

$$\partial_t \rho = \Delta \rho, \tag{5.1}$$

hence the behaviour of the total density can be well understood from well-known results about the heat equation. In particular, ρ is unique, satisfies a maximum principle, and converges exponentially to the constant solution $m_r + m_b$ in any L^p -norm. Due to the simple behaviour of ρ one can then show further properties of the densities r and b in a second step. For given ρ , (3.1) and (3.2) are linear Fokker-Planck equations, for which the uniqueness of solutions can be shown in a standard way. Moreover, since ρ will converge to a constant in the supremum norm, its maximal value will uniformly be bounded away from one after finite time if $m_r + m_b < 1$. Thus, the entropy dissipation together with usual log-Sobolev inequalities can be used to verify the exponential rate of convergence of r and b to m_r and m_b , respectively.

5.2 Very Different Diffusion Scales

Another interesting case is the asymptotic when the diffusion time scales are very different, i.e., $D \rightarrow \infty$. In this case one expects a quasi-stationary problem for b , equation (3.8) becomes

$$\nabla \cdot ((1 - \rho) \nabla b + b \nabla \rho + b(1 - \rho) \nabla W) = 0.$$

As in the computation of equilibria, we can argue that the unique solution for b , given r and W , solves

$$\log b - \log(1 - \rho) + W = c, \quad (5.2)$$

for some constant c to be determined from the mass constraint. With $\gamma := e^c$ we can compute

$$b = (1 - r) \frac{\gamma e^{-W}}{1 + \gamma e^{-W}}. \quad (5.3)$$

This relation can again be inserted into (3.1) to obtain a single equation for r :

$$\partial_t r = \nabla \cdot \left(\frac{1}{1 + \gamma e^{-W}} \nabla r - r \frac{(1 - r) \gamma e^{-W}}{(1 + \gamma e^{-W})^2} \nabla W + r \frac{1 - r}{1 + \gamma e^{-W}} \nabla V \right). \quad (5.4)$$

The resulting equation thus corresponds to a problem

$$\partial_t r = \nabla \cdot (D_{eff} \nabla r + D_{eff} r (1 - r) \nabla V_{eff}) \quad (5.5)$$

with effective variable diffusion coefficient $D_{eff} = \frac{1}{1 + \gamma e^{-W}}$ and effective potential

$$V_{eff} = V + \log(1 + \gamma e^{-W}).$$

In the absence of external potentials, this asymptotic case clearly leads to a linear diffusion equation for r ,

$$\partial_t r = D_{eff} \Delta r. \quad (5.6)$$

The constant diffusion coefficient $D_{eff} = \frac{1}{1 + \gamma}$ can be computed explicitly in terms of the masses in this case. From $b = (1 - r) \frac{\gamma}{1 + \gamma}$ we obtain after integration

$$m_b(1 + \gamma) = (|\Omega| - m_r) \gamma, \quad \text{i.e.} \quad \gamma = \frac{m_b}{|\Omega| - m_r - m_b},$$

hence

$$D_{eff} = \frac{|\Omega| - m_r - m_b}{|\Omega| - m_r} \leq 1. \quad (5.7)$$

We remark reduced diffusion coefficient we derive in this case corresponds very well to the standard theory of a diffusion coefficients in the presence of surrounding species (CP. [16, 42]).

5.3 Large Time Asymptotics

In this subsection we prove that the solution $r(t)$, $b(t)$, $\rho(t)$ converges strongly in L^1 towards the equilibrium solutions r_∞ , b_∞ and ρ_∞ for arbitrary (large) initial data with bounded entropy. The strategy is to show that the relative entropy

$$RE[(r, b)|(r_\infty, b_\infty)] := \int_\Omega r \log \frac{r}{r_\infty} dx + \int_\Omega b \log \frac{b}{b_\infty} dx + \int_\Omega (1 - \rho) \log \frac{(1 - \rho)}{(1 - \rho_\infty)} dx \quad (5.8)$$

tends to zero and then to apply a Czar–Kullback type inequality, namely

$$\|r - r_\infty\|_{L^1} \leq C_{CK} RE[(r, b)|(r_\infty, b_\infty)]^{1/2} \quad (5.9)$$

to obtain convergence in L^1 . For the proof of (5.9) we refer to [43]. We recall that, due to the conservation of the total mass we have

$$\int_\Omega r \log \frac{r}{r_\infty} dx = \int_\Omega r \log r dx - \int_\Omega r \log r_\infty dx = \int_\Omega r \log r dx - \int_\Omega r_\infty \log r_\infty dx$$

and

$$\int_{\Omega} r \log \frac{r}{r_{\infty}} dx = \int_{\Omega} r_{\infty} \left(\frac{r}{r_{\infty}} \log \frac{r}{r_{\infty}} - \frac{r}{r_{\infty}} + 1 \right) dx \geq 0$$

which implies $RE[(r, b)|(r_{\infty}, b_{\infty})] \geq 0$ after analogous computations on b and ρ in the relative entropy. In particular $RE[(r, b)|(r_{\infty}, b_{\infty})] = 0$ if and only if $(r, b) = (r_{\infty}, b_{\infty})$.

We are now ready to state the main theorem of this section. Let us remark that the result holds in any space dimension.

Theorem 5.1 (L^1 -stability of equilibria for large data). *Let r_0 and b_0 be such that*

$$RE[(r_0, b_0)|(r_{\infty}, b_{\infty})] < +\infty.$$

Then, the solution $(r(t), b(t))$ satisfies

$$\lim_{t \rightarrow +\infty} [\|r(t) - r_{\infty}\|_{L^1(\Omega)} + \|b(t) - b_{\infty}\|_{L^1(\Omega)}] = 0.$$

Proof. Due to the entropy identity

$$\begin{aligned} \frac{d}{dt} RE[(r(t), b(t))|(r_{\infty}, b_{\infty})] &= -4 \int_{\Omega} (1 - \rho) |\nabla \sqrt{r}|^2 dx - 4 \int_{\Omega} (1 - \rho) |\nabla \sqrt{b}|^2 dx \\ &\quad - 4 \int_{\Omega} \rho |\nabla \sqrt{1 - \rho}|^2 dx - 2 \int_{\Omega} |\nabla \rho|^2 dx, \end{aligned} \quad (5.10)$$

and to the consequent inequality

$$\begin{aligned} \frac{d}{dt} RE[(r(t), b(t))|(r_{\infty}, b_{\infty})] &\leq \\ &-4 \int_{\Omega} (1 - \rho) |\nabla \sqrt{r}|^2 dx - 4 \int_{\Omega} (1 - \rho) |\nabla \sqrt{b}|^2 dx - 4 \int_{\Omega} |\nabla \sqrt{1 - \rho}|^2 dx, \end{aligned} \quad (5.11)$$

we obtain, for a subsequence $t_k \rightarrow +\infty$ as $k \rightarrow +\infty$,

$$\|\nabla \rho_k\|_{L^2}^2 + \|\nabla \sqrt{1 - \rho_k}\|_{L^2}^2 + \int_{\Omega} (1 - \rho_k) |\nabla \sqrt{r_k}|^2 dx + \int_{\Omega} (1 - \rho_k) |\nabla \sqrt{b_k}|^2 dx \rightarrow 0 \quad (5.12)$$

as $k \rightarrow +\infty$, where we use the subscript k to denote evaluation at time t_k . Moreover, similarly to the strategy used above to prove global existence, and due to $r_k \leq 1$ we have

$$\begin{aligned} \int_{\Omega} |\nabla (r_k \sqrt{1 - \rho_k})|^2 dx &\leq 2 \int_{\Omega} (1 - \rho_k) |\nabla r_k|^2 dx + 2 \int_{\Omega} r_k^2 |\nabla \sqrt{1 - \rho_k}|^2 dx \\ &\leq 2 \int_{\Omega} \frac{(1 - \rho_k)}{r_k} |\nabla r_k|^2 dx + 2 \int_{\Omega} r_k^2 |\nabla \sqrt{1 - \rho_k}|^2 dx \\ &= 8 \int_{\Omega} (1 - \rho_k) |\nabla \sqrt{r_k}|^2 dx + 2 \int_{\Omega} |\nabla \sqrt{1 - \rho_k}|^2 dx \rightarrow 0 \end{aligned}$$

as $k \rightarrow +\infty$ in view of (5.12). Consequently, we have

$$r_k \sqrt{1 - \rho_k} \rightarrow \sigma \quad \text{in } L^2$$

and by weak lower semi-continuity

$$\int_{\Omega} |\nabla \sigma|^2 dx \leq \liminf_{k \rightarrow +\infty} \int_{\Omega} |\nabla r_k \sqrt{1 - \rho_k}|^2 dx = 0$$

which implies $\sigma \equiv \text{constant}$. Moreover, possibly choosing a subsequence of t_k , we have r_k and b_k converging to r^* and b^* respectively in the weak* L^∞ topology. This is due to $\rho_k \leq 1$. Therefore, since $\sqrt{1-\rho_k}$ is strongly compact in L^2 , then $\sqrt{1-\rho_k}r_k$ converges (along a subsequence) to $\sqrt{1-\rho^*}r^*$ in the weak- L^2 sense. By uniqueness of the weak limit, then, $\sigma = \sqrt{1-\rho^*}r^*$. Therefore, r^* and b^* must be constant. As $r_\infty = \frac{1}{|\Omega|} \int_\Omega r_k dx$ and r_k converges weakly in L^1 to r^* , we clearly have $r^* = r_\infty$ and $b^* = b_\infty$.

Now, we use the above considerations as follows: let λ be the weak* limit of $r_k \log r_k$. Then,

$$\begin{aligned} r_k \sqrt{1-\rho_k} \log r_k \sqrt{1-\rho_k} &= r_k \sqrt{1-\rho_k} \log r_k + r_k \sqrt{1-\rho_k} \log \sqrt{1-\rho_k} \\ &\rightharpoonup \lambda \sqrt{1-\rho_\infty} + r_\infty \sqrt{1-\rho_\infty} \log \sqrt{1-\rho_\infty} \end{aligned}$$

in weak L^2 sense as $k \rightarrow +\infty$. The left hand side above converges strongly almost everywhere (possibly along a subsequence) to $r_\infty \sqrt{1-\rho_\infty} \log r_\infty \sqrt{1-\rho_\infty}$ and this implies

$$r_k \log r_k \rightharpoonup r_\infty \log r_\infty$$

weakly in L^2 . Therefore, by definition of weak L^2 convergence and by analogous computations for b and ρ (we have actually a much stronger compactness for ρ due to $\nabla \rho_k \rightarrow 0$ in L^2) we obtain

$$RE[(r_k, b_k)|(r_\infty, b_\infty)] \rightarrow 0$$

as $k \rightarrow +\infty$. Since $RE[(r(t), b(t))|(r_\infty, b_\infty)]$ is monotone in time, the desired assertion is proved. (5.9). □

6 Conclusions and Open Issues

In this paper we have analyzed a diffusion model with size exclusion effects, which lead to nonlinear cross-diffusion terms as well as nonlinear coupling in the mobility. In the proofs for existence and large-time convergence one observes that there is a crucial effect of the total volume density ρ . We only obtain regularity for the individual density in regions where ρ is bounded away from its maximal value one, which is somehow reasonable from the microscopic interpretation of the model - if the particles are densely packed there can be arbitrary mixtures of red and blue ones. It is an interesting question also from a practical point of view if and how clusters can form in such models - we would expect such effects

Several questions on the specific model and on more general systems of nonlinear cross-diffusion equations of the form (3.12) remain open. The first natural question is to ask for uniqueness, which one obviously expects in this kind of problem for convex entropies as considered in this paper. However, it remains unclear how to carry out such a proof far from equilibrium. Another question is the possibility to obtain exponential convergence to equilibrium for coupled and possibly degenerate problems.

The cross-diffusion model considered in this paper can be applied to simulate ion flux through narrow pores such as membrane ion channels, if the potentials are computed self-consistently via Poisson equations. Certainly, the analysis of such models with electrostatic or other long-range interactions will be a challenging tasks.

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