Houston Journal of Mathematics
Volume 76, No. 1, 2050

REPRESENTATIONS OF CROSSED PRODUCTS BY CANCELLING ACTIONS AND APPLICATIONS

BERNHARD BURGSTALLER

Communicated by Kenneth R. Davidson

Abstract. The algebraic crossed product of a cancelling action of an amenable group on a locally matricial algebra allows only one faithful C^*-representation. We give examples, and actually prove a more general uniqueness theorem for algebraic G-bundles.

1. Introduction

If G is an amenable group then the full and reduced C^*-algebras, $\mathbb{C} \rtimes G$ and $\mathbb{C} \rtimes_{\mathrm{r}} G$, are isomorphic, but usually there exist many different C^*-norms on the algebraic crossed product $\mathbb{C} \rtimes_{\mathrm{alg}} G$. In contrast, we shall prove the following uniqueness theorem: if A is a locally matricial $*$-algebra (i.e. an algebraic direct limit of finite dimensional C^*-algebras), G an amenable group, and α a cancelling G-action on A (see Section 4) then the canonical map $A \rtimes_{\mathrm{alg}} G \to C^*(A) \rtimes_\alpha G$ is the only existing C^*-representation, up to isomorphism, which is faithful on A and has dense image, see Proposition 5. In particular, there exists only one C^*-norm on $A \rtimes_{\mathrm{alg}} G$.

In Section 4.2 we show that the canonical shift action on an infinite tensor product, indexed by a group, of copies of $n \times n$-matrices is a cancelling action. In Section 4.3 we give a tractable and sufficient condition when a group action on a locally compact zero-dimensional space T gives rise to a cancelling action on (a dense subalgebra of) $C_0(T)$. In Section 4.4 we show that an aperiodic subshift T of a full shift Ω^G (Ω discrete), where G is a group, induces a cancelling action on

2000 *Mathematics Subject Classification.* 46L55; 46L05.

Key words and phrases. crossed product, amenable group, representation, uniqueness, shift space, zero-dimensional, amenable Fell bundle, purely infinite, ideals.

The author was supported by the Austrian Schrödinger stipend J2471-N12.

$C_0(T)$. Although it does not directly apply to our setting as the circle S^1 is not a discrete set, let us remark that subshifts of $(S^1)^G$ for amenable groups G appear, for instance, in Deninger [5] and Deninger and Schmidt [6]; Schmidt commuted to us that one may take the same construction as in [6] (see the first paragraph of the introduction in [6]) but with S^1 replaced by $\mathbb{Z}/p\mathbb{Z}$, $p \in \mathbb{N}$.

Actually, we shall consider a somewhat more general setting than cancelling actions. We start with a $*$-algebra X generated by an alphabet $\mathcal{A}$, a homomorphism ϕ from the set of nonzero words to a (usually amenable) group G, and show a uniqueness theorem for the universal C^*-representation $X \to C^*(X)$ under some assumptions in Theorem 2.1. The uniqueness is relying on the fact that a sufficiently injective representation of X in a C^*-algebra gives always one and the same Fell bundle, see Proposition 2; the uniqueness for amenable groups G follows then from Exel's work [7] on the amenability of Fell bundles. We further state pure infiniteness and ideal structure results in Theorems 2.2 and 2.3, generalizing effortlessly corresponding known results when the bundle group is abelian [4]. For an abelian bundle group the stabilization of $C^*(X)$ by compact operators can always be realized as a crossed product by a cancelling action, see Proposition 6. In Section 3 we show that a (essentially) faithful representation of X even exists if all words in X are partial isometries, see Theorem 3.1 and Corollary 1. The core of the proof is relying on a left regular representation of X. This existence theorem is not obvious in general, but clear for crossed products, where one is readily provided with the left regular representation.

The Sections 2-3 are technical, and the reader interested in cancelling actions and examples may directly go to Section 4, after maybe reading the first page of Section 2 for gathering some notation.

2. Amenable cancelling systems

In this section we will generalize the discussion of [2, Sections 2-3].

In this note, a $*$-algebra X is an algebra over the complex numbers endowed with an involution. If $\mathcal{A}$ is a set then we denote by $\mathbb{F}_{\mathcal{A}}$ the free nonunital $*$-algebra generated by $\mathcal{A}$. If a $*$-algebra X is generated by a subset $\mathcal{A}$ of X as a $*$-algebra, then we say that $(X, \mathcal{A})$ is a *system of generators and relations*, and write $\mathbb{I}_X$ for the kernel of the canonical epimorphism $\mathbb{F}_{\mathcal{A}} \to X$. For a subset M of $\mathbb{F}_{\mathcal{A}}$, the *universal $*$-algebra generated by $\mathcal{A}$ subject to the relations M* is the quotient of $\mathbb{F}_{\mathcal{A}}$ by the two-sided selfadjoint ideal generated by M. A projection p in a $*$-algebra X is a self-adoint and idempotent element, that is, $p = p^2 = p^*$, and a partial isometry s in X satisfies $s = ss^*s$ per definition. We introduce the usual order $p \leq q \Leftrightarrow pq = p$ for projections $p, q \in X$. We write $p \precsim q$ for Murray–von Neumann order in X, that is, when there is a partial isometry s in X such that $p \leq ss^*$ and $s^*s = q$. If M is a subset of X, then $\mathbb{P}(M)$ denotes the set of *nonzero* projections in M. We say that X is a G-graded $*$-algebra if G is a group, X is a direct sum $\bigoplus_{g \in G} X_g$ as a vector space, and multiplication and taking adjoints in X are given in such a way that $X_g X_h \subseteq X_{gh}$ and $X_g^* = X_{g^{-1}}$ $(g, h \in G)$. A $*$-homomorphism $\pi : X \to A$ is called X_e*-faithful* if π is injective on X_e, and π is called a C^**-representation* if A is a C^*-algebra. If X is also endowed with a norm and the natural projection $F : X \to X_e$ is bounded, then X is called a *topologically* G*-graded* $*$*-algebra*. Recall that a *locally matricial algebra* X is a $*$-algebra X which is the union of (a net of) finite dimensional C^*-algebras. A $*$-algebra X satisfies the C^**-property* [4] if $xx^* = 0$ implies $x = 0$, for all $x \in X$. For a system of generators and relations $(X, \mathcal{A})$, the set of *words* is

$$W = \{a_1 \ldots a_n \in X |\, n \geq 1,\, a_i \in \mathcal{A} \cup \mathcal{A}^*\}.$$

Definition. The system $\Gamma = (X, \mathcal{A}, \phi, G)$ is called a *balance system* if $(X, \mathcal{A})$ is a system of generators and relations, G is a discrete group, and ϕ (the *balance function*) is a map $\phi : W \backslash \{0\} \to G$ satisfying $\phi(xy) = \phi(x)\phi(y)$ and $\phi(z^*) = \phi(z)^{-1}$ for all $x, y, z \neq 0$, $xy \neq 0$.

In this section $\Gamma = (X, \mathcal{A}, \phi, G)$ denotes a balance system. We write W_g for the set of nonzero words in W with balance g $(g \in G)$, i.e. $w \in W_g$ if and only if $w \in W$, $w \neq 0$ and $\phi(w) = g$. The fiber X_g is defined as the linear span of W_g in X, i.e. $X_g = \mathrm{span}(W_g) \subseteq X$. A nonzero word w is called *zero-balanced* (or neutral-balanced) if $\phi(w) = e$ (the neutral element of G), and otherwise called *nonzero-balanced*. If we want to emphasize to which balance system the distinguished sets refer then we shall write $W^{(\Gamma)}, X^{(\Gamma)}, W_g^{(\Gamma)}$ and $X_g^{(\Gamma)}$ rather than W, X, W_g and X_g. Two balance systems $\Gamma = (X, \mathcal{A}, \phi, G)$ and $\Lambda = (Y, \mathcal{B}, \psi, G)$ are said to be *isomorphic* if there is a $*$-isomorphism $\varphi : X \to Y$ such that $\varphi(X_g) = Y_g$ for all $g \in G$. Notice that Γ is isomorphic to $\Gamma_W = (X, W, \phi, G)$.

Definition. A balance system Γ is called a *cancelling system* if X_e is a locally matricial algebra and the following condition (C') is satisfied.

- (C') For every nonzero-balanced word x in $W \backslash W_e$ and every nonzero projection p in X_e there is a nonzero projection q in X_e such that $q \leq p$ and $qxq = 0$.

If X also satisfies the C^*-property then Γ is sometimes called a C^**-cancelling system*.

Each of the following conditions is equivalent to (C') by an elementary argument which uses Murray-von Neumann equivalence of projections and induction, see [2, Lemma 2.5] or [3, Lemma 2.3].

(C") For all $x \in \sum_{g \in G \backslash \{e\}} X_g$ and all $p \in \mathbb{P}(X_e)$ there exists $q \in \mathbb{P}(X_e)$ such that $q \le p$ and $qxq = 0$.

(C*) There exist subsets $P \subseteq \mathbb{P}(X_e)$ and $B \subseteq X$ such that for each $q \in \mathbb{P}(X_e)$ there is a $p \in P$ satisfying $p \precsim q$ in X_e, $W \backslash W_e \subseteq \mathrm{span}(B)$, and for all $x \in B, p \in P$ there exists $q \in P$ such that $q \le p$ and $qxq = 0$.

Using condition (C") rather than (C'), one may verify that if Λ and Γ are isomorphic balance systems, then Λ is a cancelling system if and only if Γ is one. Given two balance systems $\Gamma_i = (X_i, \mathcal{A}_i, \phi_i, G_i)$ we define the tensor product system $\Gamma_1 \otimes \Gamma_2 = (X_1 \otimes X_2, \mathcal{A}_1 \times \mathcal{A}_2, \phi_1 \otimes \phi_2, G_1 \times G_2)$, where $(\phi_1 \otimes \phi_2)(a_1 \otimes a_2) = (\phi_1(a_1), \phi_2(a_2))$.

Balance systems for locally matricial algebras were also considered in [2, Section 2], though they were not called this way, but subsumed under two conditions called (A_0) and (B). The algebra X is always written as a quotient $\mathbb{F}_{\mathcal{A}}/\mathbb{I}_X = X$ in [2]. There appears a further property (C) in [2] which is stronger than (C'). It states that for all nonzero-balanced words $x \in W$ and all $p, p_1, p_2 \in \mathbb{P}(X_e)$ there are smaller projections $q \le p, q_1 \le p_1, q_2 \le p_2$ in $\mathbb{P}(X_e)$ such that $qxq = 0$ and $q_1 x q_2 = 0$.

Definition. Define $\Gamma_{\mathcal{P}} = (X_{\mathcal{P}}, \mathcal{P}, \phi_{\mathcal{P}}, \{e\})$ to be the cancelling system given by the formal alphabet $\mathcal{P} = \{p_{1,c_2,\ldots,c_n} | \, n \ge 1, \, c_i \in \{1,2\}\}$ and the universal $*$-algebra $X_{\mathcal{P}}$ generated by $\mathcal{P}$ subject to the relations $p = p^2 = p^*$, $pq = qp$, $p_1 p = p$ and $p_{1,c_2,\ldots,c_n} = p_{1,c_2,\ldots,c_n,1} + p_{1,c_2,\ldots,c_n,2}$ for all $p, q \in \mathcal{P}$ and all $c_i \in \{1,2\}$. In other words, $\mathcal{P}$ forms a kind of tree consisting of commuting projections which are mutually either orthogonal or comparable.

The following lemma was proven in [3, Theorem 2.1] when G is an abelian group, but the proof works in the same way for non-commutative G.

Lemma 1. *$\Gamma \otimes \Gamma_{\mathcal{P}}$ is a cancelling system and satisfies condition (C).*

In the next proposition we prove that X is a G-graded $*$-algebra when it satisfies the C^*-property, and show how we can test the C^*-property.

Proposition 1. *(a) If $x = \sum_{g \in G} x_g$ $(x_g \in X_g)$ is an elements of X and $x^* x = 0$ then $x_g^* x_g = 0$ for all $g \in G$.*

(b) If $x^* x$ implies $x = 0$ for all fiber elements $x \in \bigcup_{g \in G} X_g$ then X satisfies the C^*-property.

(c) If X satisfies the C^*-property then X is the G-graded $*$-algebra $X \cong \bigoplus_{g\in G} X_g$.

Proof. The system $\Lambda = \Gamma \otimes \Gamma_{\mathcal{P}}$ satisfies conditions (A_0), (B) and (C) of [2] by Lemma 1. So assume $x \in X^{(\Lambda)}$ and $y = x^*x = 0$. Write $y = \sum_{g\in G} y_g$ for $y_g \in X_g^{(\Lambda)}$. By [2, Proposition 2] we may choose $Q \in X_e^{(\Lambda)}$ such that $QyQ = Qy_eQ$ and Qy_eQ is nonzero if y_e is nonzero. Thus, since $QyQ = 0$, $y_e = 0$. Hence $y_e = \sum_{g\in G} x_g^* x_g = 0$, and so $x_g^* x_g = 0$ for all $g \in G$, and thus $x_g = 0$ for all $g \in G$ by the assumed condition. This proves the claims not only for the system Λ, but also for Γ since $X^{(\Gamma)} \cong X^{(\Gamma)} \otimes \mathbb{C}p_1 \subseteq X^{(\Lambda)}$. □

In the next proposition (part (c)) we shall come to the point why cancelling systems are defined the way they are. The point is that the closure of the image of *every* X_e-faithful C^*-representation π of X is the closure of one and the same unique Fell bundle which is independent of π.

Proposition 2. *Let $\pi : X \to A$ be a X_e-faithful C^*-representation. Then*

(a) π is isometric on X_e,

(b) π is injective if and only if X satisfies the C^*-property,

(c) $\overline{\pi(X)}$ is a topologically G-graded C^*-algebra, since it is the closure of the G-graded $*$-algebra and Fell bundle $\bigoplus_{g\in G} \overline{\pi(X_g)}$, which is independent of π, and

(d) the canonical projection $F_\pi : \overline{\pi(X)} \to \overline{\pi(X_e)}$ is a contractive conditional expectation.

Proof. (a) Since X_e is a locally matricial algebra, π is isometric on X_e. (d) The system $\Lambda = \Gamma \otimes \Gamma_{\mathcal{P}}$ satisfies the conditions (A_0), (B) and (C) of [2] by Lemma 1. Write $\iota : X_{\mathcal{P}} \to C^*(X_{\mathcal{P}})$ for the identical embedding. By [2, Lemma 2.6], applied to the homomorphism $\pi \otimes \iota : X \otimes X_{\mathcal{P}} \to A \otimes C^*(X_{\mathcal{P}})$, we obtain a canonical contractive conditional expectation F. Restricting F to $X \otimes \mathbb{C}p_1$ yields the desired conditional expectation F_π. (c) Let $x = \sum_{g\in G} x_g$, where $x_g \in X_g$. Assume that $\pi(x) = 0$. Then $0 = F_\pi \pi(x^*x) = \sum_g \pi(x_g^* x_g)$. Thus $\pi(x_g^* x_g) = 0$ for all $g \in G$, and so $\pi(x_g) = 0$ for all $g \in G$. This proves that we have a G-graded algebra $\pi(X)$. Since π is faithful on X_e, one has $\|\pi(y_g)\|^2_{\pi(X_g)} = \|\pi(y_g^* y_g)\|_{\pi(X_e)} = \|y_g^* y_g\|_{X_e}$ for all $g \in G, y_g \in X_g$, and so all fibers $\pi(X_g)$ give the same closure $\overline{\pi(X_g)}$ independent from π. (b) If X satisfies the C^*-property, then (by a computation as before) $\pi(x) = 0$ implies $\pi(x_g^* x_g) = 0$ and so $x_g^* x_g = 0$ (since π is faithful on X_e), and so $x_g = 0$ for all $g \in G$, and so $x = 0$. □

Definition. If $\Gamma = (X, \mathcal{A}, \phi, G)$ is a cancelling system then we denote by $C^*(X)$ (or $C^*(\Gamma)$) the enveloping C^*-algebra of X, and write $\pi_{C^*} : X \to C^*(X)$ for the universal C^*-representation of X.

Note that the enveloping C^*-algebra $C^*(X)$ exists as X_e is a locally matricial algebra, and so

$$\sup_\pi \Big\| \pi\Big(\sum_g x_g\Big)\Big\| \le \sup_\pi \sum_g \|\pi(x_g)\| \le \sum_g \|x_g^* x_g\|_{X_e}^{1/2} < \infty.$$

Definition. A cancelling system Γ is called *amenable* if the universal representation π_{C^*} is faithful on X_e and its conditional expectation $F_{\pi_{C^*}}$ of Proposition 2.(d) is faithful.

We shall see in the next section that X_e-faithfulness of π_{C^*} is automatic in many situations. If π_{C^*} is faithful on X_e then $C^*(X)$ coincides with the full cross sectional C^*-algebra $C^*(\pi_{C^*}(X))$ (see [7]), where $\pi_{C^*}(X)$ (by sloppy notation) denotes the G-graded algebra (or Fell bundle) of Proposition 2.(c). By [8, 1.9], $F_{\pi_{C^*}}$ is faithful if and only if the Fell bundle $\pi_{C^*}(X)$ is amenable in the sense of Exel [7, 4.1].

The following proposition follows immediately from the fact that a Fell bundle over an amenable group is amenable [7, 4.7].

Proposition 3. *If G is an amenable group and the universal representation π_{C^*} is faithful on X_e then the cancelling system is amenable.*

Theorem 2.1 (Uniqueness)**.** *A cancelling system $\Gamma = (X, \mathcal{A}, \phi, G)$ is amenable if and only if the canonical homomorphism π_{C^*} from X to the enveloping C^*-algebra $C^*(X)$ is faithful on X_e, and for any X_e-faithful C^*-representation $\pi : X \to A$ the induced map $\rho : C^*(X) \to A$ is injective.*

Proof. If Γ is amenable, the conditions ('uniqueness') follow directly from Proposition 2.(c) and [7, 4.2]. To prove the reverse implication, notice that $C^*(X)$ is the closure $C^*(\pi_{C^*}(X))$ of the Fell bundle $\pi_{C^*}(X)$. Also, there is a homomorphism $\rho : C^*(X) \cong C^*(\pi_{C^*}(X)) \to C^*_\lambda(\pi_{C^*}(X))$, the left regular representation introduced in [7]. Since the left regular representation is injective on $\pi_{C^*}(X_e) \cong X_e$, ρ is injective by assumption. Hence $\pi_{C^*}(X)$ is an amenable Fell bundle by [7, 4.1]. □

In the next two theorems $\Gamma = (X, \mathcal{A}, \phi, G)$ denotes an amenable cancelling system.

Theorem 2.2. *$C^*(X)$ is purely infinite if and only if each nonzero projection in X_e is infinite in $C^*(X)$.*

Proof. This can be proved in exactly the same way as [4, Theorem 3.2]. □

Theorem 2.3. *Assume that X is injectively embedded in $C^*(X)$ (confer Proposition 4). Write Σ for the set of closed two-sided ideals in $C^*(X)$. For any $*$-subalgebra Z of X_e define*

$$\Sigma_Z = \{J \cap Z |\, J \textit{ is a two-sided self-adjoint ideal in } X\}.$$

(a) If $Z = X_e$, then Σ_Z is a lattice of ideals of X_e and consists exactly of those two-sided self-adjoint ideals J in X_e which satisfy $aJb \subseteq J$ whenever $a, b \in W$ and $\phi(a)\phi(b) = e$.

(b) For every $*$-subalgebra Z of X_e there is an injective map $\Phi_Z : \Sigma_Z \to \Sigma$ such that $\Phi_Z(J)$ is the closed two-sided ideal in $C^*(X)$ generated by J ($J \in \Sigma_Z$), and conversely, $\Phi_Z^{-1}(I) = I \cap Z$ for a two-sided closed ideal I in $C^*(X)$ in the image of Φ_Z.

Proof. This was proved in [4] (Lemma 4.4 and Theorem 4.6) when G is abelian, but the proof works also for general G. □

3. Faithful representations

The following useful proposition shows how to convert a cancelling system into a C^*-cancelling system. In particular, the new X embeds injectively in $C^*(X)$ by Proposition 2.(b).

Proposition 4. *Let $\Gamma = (X, \mathcal{A}, \phi, G)$ be a cancelling system. Let J be the two-sided self-adjoint ideal in X generated by $\{x \in \bigcup_{g\in G} X_g |\, x^*x = 0\}$, and $\sigma : X \to X/J$ the quotient map. Then $\Gamma_{C^*} = (\sigma(X), \sigma(\mathcal{A}), \phi\sigma^{-1}, G)$ is a cancelling system, $\sigma(X)$ satisfies the C^*-property, $\sigma|_{X_e} : X_e \to \sigma(X_e)$ is an isomorphism, and $\sigma(X) = \bigoplus_{g\in G} \sigma(X_g)$ is G-graded. Of course, $C^*(X) = C^*(\sigma(X))$.*

If $\pi : X \to A$ is an X_e-faithful C^*-representation then $\ker(\pi) = J$.

Proof. In this proof, W, X, W_g and X_g are related to the system Γ.

Step 1. Let $v, w \in W$ such that $\sigma(v) = \sigma(w)$. Then

$$v - w = \sum_{g\in G} \sum_i \lambda_{g,i}\alpha_{g,i}x_{g,i}\beta_{g,i}$$

for some $\lambda_{g,i} \in \mathbb{C}$, $\alpha_{g,i}, \beta_{g,i} \in W \sqcup \{1_{\mathbb{C}}\}$ and $x_{g,i} \in X_g$ such that $x_{g,i}^* x_{g,i} = 0$. Assume that $\phi(v) \neq \phi(w)$. Then, ordering words by their balance, notice that

$$0 = w - v + \sum_{g,i} \lambda_{g,i} \alpha_{g,i} x_{g,i} \beta_{g,i} = (w - j_1) + (v - j_2) + j_3 \tag{1}$$

for certain $j_1, j_2, j_3 \in J$ such that $j_1 \in X_{\phi(w)}, j_2 \in X_{\phi(v)}, j_3 \in X_\mu$. An application of Proposition 1 shows that $(w-j_1)^*(w-j_1) = 0$. Thus $w-j_1 \in J$ and so $\sigma(w) = 0$. This proves that $\phi\sigma^{-1}$ is well defined, and thus Γ_{C^*} is a balance system.

Step 2. Let $w \in J \cap X_e$. We claim that $w \in K$, where

$$\begin{aligned} K \quad = \quad & \operatorname{span}\{\, \alpha y \beta \in J \cap X_e \,|\, \alpha \in X_{g_1} \sqcup \{1\}, \beta \in X_{g_2} \sqcup \{1\}, y \in X_{g_3}, \\ & y^* y = 0, \phi(g_1)\phi(g_2)\phi(g_3) = e\}. \end{aligned}$$

Indeed, similarly as in (1) we may choose $j_1 \in K$ and $j_2 \in \bigcup_{g \in G \setminus \{e\}} X_g$ such that $w - j_1 + j_2 = 0$. An application of Proposition 1 shows that $(w - j_1)^*(w - j_1) = 0$. Hence $w - j_1 = 0$ since $w - j_1 \in X_e$ and X_e is a pre-C^*-algebra.

Step 3. We claim that $K = \{0\}$. Let $j = \alpha y \beta \in K$ for α, y, β as described in the definition of K. Define $\gamma = y\beta\beta^* y^*$. Then $\gamma^*\gamma = 0$. Thus $\gamma = 0$ since $\gamma \in X_e$. Then we get $jj^* = \alpha\gamma\alpha^* = 0$. Hence $j = 0$ since $j \in X_e$.

Step 4. Steps 2-3 show that $J \cap X_e = \{0\}$. First, this proves that $\sigma|_{X_e^{(\Gamma)}} : X_e^{(\Gamma)} \to X_e^{(\Gamma_{C^*})}$ is an isomorphism. It is now easy to check that Γ_{C^*} is a cancelling system. Second, by the criterion in Proposition 1 this proves that Γ_{C^*} is a C^*-cancelling system. Indeed, $x_g \in X_g$ and $\sigma(x_g)^*\sigma(x_g) = 0$ imply that $x_g^* x_g \in J \cap X_e = \{0\}$, and hence $x_g \in J$, and so $\sigma(x_g) = 0$.

Step 5. Notice that $J \subseteq \ker(\pi)$. On the other hand, if $x = \sum_g x_g \in \ker(\pi)$, $x_g \in X_g$, then $0 = F_\pi \pi(x^* x) = \sum_g \pi(x_g^* x_g)$ by Proposition 2. Hence $x_g^* x_g = 0$ for all g since π is faithful on X_e, and so $x_g \in J$ for all g, and $x \in J$. Hence $J = \ker(\pi)$. □

Lemma 2. *Let $X = \bigoplus_{g \in G} X_g$ be a G-graded $*$-algebra satisfying the C^*-property and suppose that X_e is endowed with a C^*-norm. Assume further that for all $y \in \bigcup_{g \in G} X_g$ there exits $M_y \geq 0$ such that $\|y^* a y\| \leq M_y \|a\|$ for all $a \in X_e$. Then there exists a left-regular representation $\lambda : X \to C_r^*(X)$ in the sense of Exel* [7] *such that λ is isometric on X_e.*

Proof. We follow the approach in [7]. Notice that X is a 'pre-Hilbert module' over the *pre-C^**-algebra X_e with algebraic operations inherited from X and with X_e-valued inner product $\langle x, y \rangle = \sum_{g \in G} x_g^* y_g$ for $x = \sum_g x_g$ and $y = \sum_g y_g$, where $x_g, y_g \in X_g$. By taking the norm closure of X_e, we can copy the usual proofs for pre-Hilbert modules to show that $\|x\|_X = \|\langle x, x \rangle\|^{1/2}$ is a norm on X

which coincides with the given C^*-norm on X_e, and satisfies $\|xa\| \leq \|x\|\|a\|$ and $\|\langle x,y\rangle\| \leq \|x\|\|y\|$ for all $x,y \in X$ and $a \in X_e$. The completion $\overline{X}$ of X then becomes a Hilbert module over $\overline{X}_e$. We consider the left regular representation $\lambda : X \to \mathbb{B}(\overline{X})$, where $\lambda(x)$ is determined by $\lambda(x)y = xy$ (multiplication in X) for $x \in X_g$ and $y \in X$. We will show that $\lambda(x)$ is bounded, and extend $\lambda(x)$ to $\overline{X}$ by continuity. Take $x \in X_g, y \in X$. By Cauchy-Schwartz's inequality we have

$$\|xy\|_X^2 = \|\langle xy,xy\rangle\|_{X_e} = \|\langle y,x^*xy\rangle\| \leq \|y\|\|x^*xy\|$$
$$\ldots \leq \|y\|\|y\|^{1/2}\|(x^*x)^2y\|^{1/2} \leq \ldots \leq \|y\|^{2-1/(2^n)}\|(x^*x)^{2^n}y\|^{1/(2^n)}.$$

Since

$$\|ay\|^2 = \Big\| \sum_g y_g^* a^* a y_g \Big\| \leq \sum_g M_{y_g}\|a^*a\|$$

for all $a \in X_e$ by assumption, we get $\|xy\|^2 \leq \|y\|^2\|x^*x\|$ by letting n tend to infinity. Hence $\lambda(x)$ is bounded by $\|x\|$ and can be continuously extended to $\overline{X}$. On the other hand, $\|\lambda(x)x^*\| = \|x\|^2$ for $x \in X_e$. Hence $\|\lambda(x)\| = \|x\|$. By linearity we extend λ to X. □

Definition. The *left regular representation* of a cancelling system Γ is the composition $\lambda \circ \sigma$ of the quotient map σ of Proposition 4 with the left regular representation $\lambda : X^{(\Gamma_{C^*})} \to C_r^*(X^{(\Gamma_{C^*})})$ of Lemma 2 (recall Proposition 1), provided that λ of Lemma 2 exists.

Theorem 3.1. *The following conditions are equivalent for a cancelling system* $\Gamma = (X, \mathcal{A}, \phi, G)$.

(1) $\forall y \in W$, $\exists M_y \geq 0$, $\forall a \in X_e \qquad \|y^*ay\|_{X_e} \leq M_y\|a\|_{X_e}$.

(2) *The left regular representation* $\lambda : X \to C_r^*(X)$ *exists and is faithful on* X_e.

(3) *The universal representation* π_{C^*} *of* X *is faithful on* X_e.

Each of the following conditions is sufficient for one of the above conditions.

(4) $W \subseteq \bigcup_{g\in G} \operatorname{span}\{x \in X_g \mid$ *x is a partial isometry*$\}$.

(5) W *consists of partial isometries.*

Proof. The implications (2) ⇒ (3) ⇒ (1) and (5) ⇒ (4) are trivial. (1) ⇒ (2): By Theorem 3.1.(1) we have

$$(2) \qquad \|y^*az\| = \|y^*azz^*a^*y\|^{1/2} \leq M_y^{1/2}\|azz^*a^*\|^{1/2} \leq M_y^{1/2}\|zz^*\|^{1/2}\|a\|$$

for all $g \in G$, $y,z \in W_g$ and $a \in X_e$. From this, Theorem 3.1.(2) easily follows from Proposition 1 and Lemma 2 in case that Γ is a C^*-cancelling system. If Γ is only a cancelling system, then we take the C^*-cancelling system Γ_{C^*} of

Proposition 4 instead of Γ and observe that Theorem 3.1.(1) also holds with respect to Γ_{C^*}. Then we put $\lambda^{(\Gamma)} = \lambda^{(\Gamma_{C^*})}\sigma$, where σ is the quotient map of Proposition 4.

(4) $\Rightarrow$ (1): Take $g \in G$ and $w = \sum_{s=1}^n \alpha_s y_s \in W$, where $y_s y_s^* y_s = y_s$ and $\alpha_s \in \mathbb{C}, y_s \in X_g$ for all s. Fix s and define $y = y_s$. Choose a finite dimensional C^*-algebra $\mathcal{M} \subseteq X_e$ containing yy^*. Enlarge $\{yy^*\}$ to a maximal abelian subalgebra S of $\mathcal{M}$. Then choose a self-adjoint system (e_{ij}) of matrix units $e_{ij} \in \mathcal{M}$ ($e_{ij} = 0$ is allowed) which generates $\mathcal{M}$, and such that $(e_{ii})_i$ is the collection of all minimal projections of S. Then, for all i, either $(yy^*)e_{ii} = 0$ or $(yy^*)e_{ii} = e_{ii}$. Put $\mathcal{M}' = yy^*\mathcal{M}yy^*$. Then, $\varphi : \mathcal{M}' \to X_e$ given by $\varphi(e_{ij}) = y^* e_{ij} y$ is a $*$-isomorphism. Thus we get $\|y^*ay\| = \|y^*yy^*ayy^*y\| = \|\varphi(yy^*ayy^*)\| \le \|a\|$ for all $a \in \mathcal{M}$, and consequently for all $a \in X_e$. Thus, by a similar estimate as in (2), we get $\|w^*aw\| \le M_w\|a\|$ for all $a \in X_e$. □

Using the fact that idempotent partial isometries are automatically self-adjoint, one easily sees W is an inverse semigroup (so the range and source projections commute) when W consists of partial isometries. The next corollary immediately follows from the last theorem and Proposition 3.

Corollary 1. *If $\Gamma = (X, \mathcal{A}, \phi, G)$ is a cancelling system such that G is an amenable group and the word set W consists of partial isometries then Γ is amenable.*

Corollary 2. *If Γ is a cancelling system and W consists of partial isometries then there exists an epimorphism $A \rtimes F \to C^*(\Gamma)$ of a partial crossed product $A \rtimes F$, where A is an abelian C^*-algebra and F is a free group.*

Proof. By Proposition 4 and Theorem 3.1 we have an injective homomorphism $X^{(\Gamma_{C^*})} \to C^*(\Gamma_{C^*}) = C^*(\Gamma)$. The claim then follows by [1, Proposition 2.5]. □

4. Cancelling actions

In this section G denotes a discrete group.

Definition. A group action $\alpha : G \to \mathrm{Aut}(A)$ on a $*$-algebra A is called *cancelling* if for every g in $G\backslash\{e\}$, every x in A, and every nonzero projection p in A there is a nonzero projection q in A such that $q \le p$ and $qx\alpha_g(q) = 0$.

If A is locally matricial then α extends to an action on the unique C^*-norm closure $C^*(A)$ as $\alpha_g|_B : B \to \alpha_g(B)$ is an isometric isomorphism for any finite dimensional sub-C^*-algebra B of A and $g \in G$.

Lemma 3. *An action α is cancelling if there exist a subset B of A which linearly spans A and a subset P of nonzero projections of A such that for every nonzero projection in A there is a smaller or equal one in P in Murray–von Neumann order, and*

$$\forall g \in G\backslash\{e\}, \forall x \in B, \forall p \in P, \exists q \in P \quad q \le p \ \textit{and} \ qx\alpha_g(q) = 0.$$

Proof. By a similar elementary proof as in [2, Lemma 2.4]. □

4.1. **Crossed product system.** Write $A \rtimes_{\text{alg}} G$ for the algebraic crossed product. Denote by AG the collection of all fibers of $A \rtimes_{\text{alg}} G$, i.e.

$$AG = \{ag \in A \rtimes_{\text{alg}} G |\, a \in A, g \in G\}.$$

Note that AG is a subset of $A \rtimes_{\text{alg}} G$ which is closed under multiplication and taking adjoints. Define a function $\phi : AG \to G$ by $\phi(ag) = g$. The balance system $(A \rtimes_{\text{alg}} G, AG, \phi, G)$ is called a *crossed product system*.

Proposition 5. *For a cancelling action α of an amenable group G on a locally matricial algebra A, $(A \rtimes_{\text{alg}} G, AG, \phi, G)$ is a cancelling system. That is, there exists only one C^*-norm on $A \rtimes_{\text{alg}} G$, and*

$$C^*(A \rtimes_{\text{alg}} G) = C^*(A) \rtimes G.$$

Even more, $A \rtimes_{\text{alg}} G \to C^*(A) \rtimes G$ is the only existing homomorphism in a C^*-algebra (up to isomorphism) which has dense image and is injective on A.

Proof. This follows from Theorem 2.1. □

4.2. **Shift actions on UHF algebras.**

Lemma 4. *Let G be an infinite group and $n \ge 2$. The shift action α on the algebraic infinite tensor product $X = \bigotimes_{g \in G} M_n(\mathbb{C})$ is cancelling. (So, $\alpha_g(e^h_{ij}) = e^{hg}_{ij}$ for $1 \le i, j \le n, g, h \in G$ and canonical matrix units $e^h_{ij} \in X$.)*

Proof. Denote by P the set of nonzero, finite products of diagonal entries e^g_{ii}. Write B for the set of finite products of matrix entries e^g_{ij}. Let $g \in G\backslash\{e\}, p \in P$ and $x \in B$. Since G is infinite, there exists some $s \in G$ such that $q = pe^s_{11}e^{g^{-1}s}_{22}$ is nonzero, and x and $e^s_{11}e^{g^{-1}s}_{22}$ commute. Then $qx\alpha_g(q) = pe^s_{11}e^{g^{-1}s}_{22}x\alpha_g(p)e^{gs}_{11}e^s_{22} = 0$ since e^s_{11} cancels e^s_{22}. Hence by Lemma 3, α is cancelling. □

4.3. **Action on zero-dimensional space.** Let T be a locally compact, zero-dimensional (i.e. having a basis $\mathcal{O}$ of clopen sets) Hausdorff space, and θ a continuous G-action on T. Denote by A the subalgebra of $C_0(T)$ which is linearly spanned by the characteristic functions of $\mathcal{O}$. Then A is a locally matricial algebra and $C_0(T) = C^*(A)$. Let α be the G-action on $C_0(T)$ induced by θ, i.e. $\alpha_g(f) = f \circ \theta_g^{-1}$. Say that θ is *cancelling* if α is cancelling. So, by Lemma 3 (for $P = B$ being the characteristic functions of elements of $\mathcal{O}$), we obtain a tractable condition as follows:

Lemma 5. *θ is a cancelling action if*

$$\forall g \in G \backslash \{e\}, \forall p \in \mathcal{O}, \exists q \in \mathcal{O} \quad q \subseteq p \ \text{ and } \ q \cap \theta_g(q) = \emptyset.$$

If θ is cancelling then $(A \rtimes_{\text{alg}} G, AG, \phi, G)$ is a cancelling system, and one has $C^*(A \rtimes_{\text{alg}} G) = C_0(T) \rtimes_\alpha G$. If G is amenable, then so is the said system. This example may also be considered for T just being a Hausdorff space. However, then A may be a small algebra and $C_0(T) \neq C^*(A)$.

4.4. **Shift space.** A special case of an action on a zero-dimensional space arises by a closed subshift $(T, \sigma|_T, G)$ of the full shift (Ω^G, σ, G), where Ω denotes a discrete set, and $(\sigma_g(z))_h = z(g^{-1}h)$ $(g, h \in G, z \in \Omega^G)$ the shift action. Denote the collection of cylinder sets of T (to be precise: the cylinder sets of Ω^G intersected with T) by $\mathcal{Z}$.

Lemma 6. *If the shift T is aperiodic in the sense that for every $g \in G \backslash \{e\}$, every cylinder set of T contains an element x such that $\sigma_g(x) \neq x$, then the crossed product system $(A \rtimes_{\text{alg}} G, AG, \phi, G)$ for T is cancelling.*

Proof. The cylinder sets $\mathcal{Z}$ are a basis for the topology of T. So it suffices to check the condition of Lemma 5 for $\mathcal{O} = \mathcal{Z}$. Given $p \in \mathcal{Z}$, we may choose an $x \in p$ such that $\sigma_g(x) \neq x$. Hence, there is some $h \in G$ such that $x_{g^{-1}h} \neq x_h$. Say that $a_1 \in \Omega, h_1 \in G$ such that $p = \{y \in T |\, y_{h_1} = a_1\}$ (of course, more coordinates may be forced). Then $q = \{y \in T |\, y_{h_1} = a_1, y_h = x_h, y_{g^{-1}h} = x_{g^{-1}h}\}$ is a cylinder set containing x such that $q \cap \sigma_g(q) = \emptyset$. □

4.5. **Abelian cancelling systems.** In this section we consider an *abelian* cancelling system $\Gamma = (X, \mathcal{A}, \phi, G)$, that means, G is abelian, and show that (the stabilization of) $C^*(X)$ can be expressed as a crossed product with a cancelling action. We may assume that $\phi(\mathcal{A})$ generates the whole group G. By duality theory for abelian groups there exist a compact subgroup H in $\mathbb{T}^{\mathcal{A}}$ and an isomorphism $T : G \to \widehat{H}$ such that $T(\phi(a))(\lambda) = \lambda_a$ for all $a \in \mathcal{A}, \lambda = (\lambda_a)_{a \in \mathcal{A}} \in H$.

Lemma 7. *Γ is isomorphic to $\Gamma_H = (X, \mathcal{A}, T \circ \phi, \widehat{H})$.*

Proposition 6. *For an abelian amenable cancelling system $(X, \mathcal{A}, \phi, G)$ there is a locally matricial algebra A and a cancelling action α on A such that*

$$C^*(X) \otimes \mathcal{K}(L^2(\widehat{G})) \cong C^*(A) \rtimes_\alpha G.$$

Proof. If necessary, we go over to the C^*-cancelling system of Proposition 4 (in order that X is a G-graded algebra), so that there is a well defined gauge action $\gamma : H \to \mathrm{Aut}(X)$ determined by $\gamma_\lambda(a) = \lambda_a a$ for $\lambda \in H, a \in \mathcal{A}$. We can then apply [4, Theorem 5.1], which states the claimed isomorphism (which is just Takai's duality) and says that A is locally matricial. It remains to show that α is cancelling.

By the proof of [4, Theorem 5.1], there is a family $\mathcal{X} = (X_n)_{n \in G}$ of mutually orthogonal projections in the multiplier algebra of $C^*(A) \rtimes_\alpha G$ such that $A = \mathrm{span} B$, where

$$B = \{wX_n \in A |\, w \in W, n \in G\}.$$

Further, one has $X_n w = w X_{n+\phi(w)}$ for all $w \in W, n \in G$. The action β is given by $\beta_g(wX_n) = wX_{n+g}$. Then A allows a locally matricial description, where the projections on the diagonal of the matrices are in the set

$$P = \{X_n v X_n |\, v \in \mathbb{P}(W_e), n \in G\}.$$

Hence, for any nonzero projection in A there is a smaller or equal one in P in the Murray–von Neumann order. Let $x = wX_n$ in B, and $p = X_m v X_m$ in P. If $g \in G \backslash \{e\}$, then

$$qx\beta_g(q) = X_m v X_m w X_n X_{m+g} v X_{m+g} = X_m v w v X_{m+\phi(w)} X_n X_{m+g} X_{m+g}.$$

If $qx\beta_g(q)$ happens to be nonzero then $\phi(w) = n - m \neq e$. In this case we may, by condition (C'), choose a projection $v' \in \mathbb{P}(X_e)$ such that $v' \leq v$ and $v' w v' = 0$. Setting $q = X_m v' X_m$, we get $q \leq p$ and $qx\beta_g(q) = 0$. This proves by Lemma 3 that β is cancelling. □

References

[1] B. Burgstaller. On partial crossed products and Cuntz–Krieger type algebras. *Acta Sci. Math.*, 71:643–661, 2005.

[2] B. Burgstaller. The uniqueness of Cuntz–Krieger type algebras. *J. reine angew. Math.*, 594:207–236, 2006.

[3] B. Burgstaller. Notes on Cuntz–Krieger uniqueness theorems and C^*-algebras of labelled graphs. *Quaest. Math.*, 32:229–240, 2009.

[4] B. Burgstaller and D. G. Evans. On certain properties of Cuntz–Krieger type algebras. preprint.

[5] C. Deninger. Fuglede-Kadison determinants and entropy for actions of discrete amenable groups. *J. Am. Math. Soc.*, 19(3):737–758, 2006.

[6] C. Deninger and K. Schmidt. Expansive algebraic actions of discrete residually finite amenable groups and their entropy. *Ergodic Theory Dyn. Syst.*, 27(3):769–786, 2007.

[7] R. Exel. Amenability for Fell bundles. *J. reine angew. Math.*, 492:41–73, 1997.

[8] R. Exel. Partial representations and amenable Fell bundles over free groups. *Pac. J. Math.*, 192(1):39–63, 2000.

Mathematisches Institut, Einsteinstrasse 62, 48149 Münster, Germany
E-mail address: bernhardburgstaller@yahoo.de