Equivariant *KK*-theory for inverse semigroups

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October 28 2010

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KK^S

Definition

S **inverse semigroup**, when S semigroup and every $s \in S$ has *unique* inverse $s^* \in S$:

$$ss^*s = s, \qquad s^*ss^* = s^*$$

E = idempotent set of S

Source $= s^*s$ Range $= tt^*$ in E Commute all!

Definition

$\mathcal{E}$ Hilbert module. **Partial Isometry** U, when U linear map on $\mathcal{E}$, norm-isometrically mapping a complemented subspace $\mathcal{E}_0$ to another $\mathcal{E}_1$, and vanishing on $\mathcal{E}_0^\perp$.

Inverse partial isometry $= U^*$

U, U^* must respect $\mathbb{Z}/2$-grading

$A = C^*$-algebra, Hilbertmodule over itself

Definition

A **Hilbert C^*-algebra**, when there is action $\alpha : S \to \mathrm{PartIso}(A) \cap \mathrm{End}(A)$, i.e. α is homomorphism of inverse semigroups, and (where $\alpha_s(a) = s(a)$)

$$\langle s(a), b \rangle \quad = \quad s\langle a, s^*(b) \rangle \qquad \forall a, b \in A, s \in S$$

Self-adjoint projections $\alpha_{ss^*}, \alpha_{s^*s}$ in center of multiplier algebra of A (i.e. center of $\mathcal{L}(A)$) $\forall s \in S$

Partial isometry $\alpha_s : A \to A$ mapping $\alpha_{s^*s}(A)$ onto $\alpha_{ss^*}(A)$

Definition

$\pi : A \to B$ S-**equivariant** homomorphism if $\pi \circ s = s \circ \pi$.

Definition

$\mathcal{E}$ S-**Hilbert module** over A, when action $U : S \to \mathrm{PartIso}(\mathcal{E})$, i.e. U is homomorphism of inverse semigroups, and

$$\begin{aligned} \langle U_s(\xi), \eta \rangle &= s\langle \xi, U_s^*(\eta)\rangle \\ U_s(\xi a) &= U_s(\xi)s(a) \qquad\qquad \forall \xi, \eta \in \mathcal{E}, a \in A, s \in S \end{aligned}$$

S-action (= homomorphism of inverse semigroups) on $\mathcal{L}(\mathcal{E})$:

$$S \longrightarrow \mathcal{L}(\mathcal{E}) \qquad s(T) = U_s T U_s^* \qquad \forall T \in \mathcal{L}(\mathcal{E}), s \in S$$

But $s(ST) \neq s(S)s(T)$! Not in $\mathrm{End}(\mathcal{L}(\mathcal{E}))$. Not Hilbert C^*-algebra !

Definition

A, B Hilbert C^*-algebras. $\mathcal{E}$ S-Hilbert B-module. $\pi : A \to \mathcal{L}(\mathcal{E})$ $*$-homomorphism. π is **S-equivariant representation** if

$$\begin{aligned} [\pi(a), U_s U_s^*] &= 0 \\ U_s \pi(a) U_s^* &= \pi(s(a)) U_s U_s^* \qquad \forall a \in A, s \in S \end{aligned}$$

Definition

$(\pi, \mathcal{E}, T)$ is (A, B)-**cycle**, when A, B Hilbert C^*-algebras, $\mathcal{E}$ is countably generated $\mathbb{Z}/2$-graded S-Hilbert B-module, π is S-equivariant representation on $\mathcal{E}$, $T \in \mathcal{L}(\mathcal{E})$ is odd, $[T, A] \subseteq \mathcal{K}(\mathcal{E})$, and

$$T - T^*, \quad T^2 - 1, \quad [U_s U_s^*, T], \quad U_s T U_s^* - T U_s U_s^*$$

are elements in $\{X \in \mathcal{L}(\mathcal{E}) | \, aX, \, Xa \in \mathcal{K}(\mathcal{E}) \; \forall a \in A\}$ for all $s \in S$.

Definition

$KK^S(A, B) = \{(A, B)\text{-cycles}\}/\text{homotopy}$

Theorem

There is associative Kasparov product

$$KK^S(A, B) \otimes KK^S(B, C) \to KK^S(A, C)$$

Definition

A Hilbert C^*-algebra. $A \rtimes S$ is enveloping C^*-algebra of involutive Banach-algebra

$$\ell^1(S, A) = \{\sum_{s \in S} a_s s \,(\text{formal sum}) |\, a_s \in ss^*(A), \sum \|a_s\| < \infty\}$$

$$\Big(\sum_{s \in S} a_s s\Big)^* = \sum_{s \in S} s^*(a_s^*) s^*, \qquad \sum_{s \in S} a_s s \cdot \sum_{t \in S} b_t t = \sum_{s,t \in S} a_s s(b_t) st$$

Theorem

There is descent homomorphism

$$j^S : KK^S(A, B) \to KK(A \rtimes S, B \rtimes S)$$
$$j^S(\pi, \mathcal{E}, T) = ((\pi \otimes 1) \rtimes (U \otimes \beta), \mathcal{E} \otimes_B (B \rtimes S), T \otimes 1)$$

Respects Kasparov product.

Definition

Let $\mathcal{G}$ groupoid.

A **slice** is open subset O of $\mathcal{G}$ on which range and source maps injective.

$\mathcal{G}$ ***r*-discrete** when every point of $\mathcal{G}$ in slice.

$$\begin{aligned} \text{Slice} \cdot \text{Slice} &= \text{Slice} \qquad \{g\} \cdot \{h\} = \{gh \in \mathcal{G}\} \\ \text{Slice}^{-1} &= \text{Slice} \qquad \{g\}^{-1} = \{g^{-1}\} \end{aligned}$$

Definition

Full inverse semigroup (of slices) of $\mathcal{G}$: Set S of open slices of $\mathcal{G}$ covering $\mathcal{G}$ and forming inverse semigroup.

Example: $\mathcal{G} = \{1, \ldots, n\} \times \{1, \ldots, n\}$
slice $= s = \{(1,2),(2,3),(3,4)\}$
$ss^* = \{(1,1),(2,2),(3,3)\}$
$X = \mathcal{G}^{(0)} = \{(1,1), \ldots, (n,n)\}$
Idempotent elements E of S = Subsets of X
$\Longrightarrow 1_e \in C_0(X)$ for $e \in E$

$C_0(X)$ dimension n

$C^*(E) = \mathbb{C} \rtimes E$ = set of formal sums $\sum_{e \in E} a_e e$ (dimension is 2^n)
$C_0(X) \neq C^*(E) \leftarrow$ Universal !

Theorem (Paterson)

Every inverse semigroup S is full inverse semigroup of slices of (usually non-Hausdorff) universal groupoid $\mathcal{G}_S$.

$X := \mathrm{Spec}(C^*(E))$ *totally disconnected, loc. comp. Hd.*

$C_0(X) = C^*(E)$

Each $e \in E$: carrier of e clopen in X

$\mathcal{G}_S := X \times S / \sim$ *locally compact, r-discrete*

with $(x, s) \sim (y, t)$ when $x = y$, $\exists e \in E$, $e(x) = e(y) = 1$, $e \leq s^* s, t^* t$

Let $\mathcal{G}$ Hausdorff r-discrete groupoid

Let S full inverse semigroup of slices of $\mathcal{G}$

Let E idempotent set of S

Assume all elements of E are **clopen**

Write $X = \mathcal{G}^{(0)}$ $\qquad (1_e \in C_0(X) \qquad \forall e \in E)$

Definition

$\widehat{KK^S}(A,B)$ defined like $KK^S(A,B)$, but following modifications:

- A and B are $C_0(X)$-algebras
- (Compatibility) $1_e \cdot a = e(a) \qquad \forall e \in E, a \in A$
- (Compatibility) similarly for B and $\mathcal{E}$

Remark

- *(Compatibility)* $e(a) \cdot \xi = a \cdot U_e(\xi)$
- *(Compatibility)* $U_e(\xi) \cdot b = \xi \cdot e(b) \qquad \forall \xi \in \mathcal{E}, a \in A, b \in B, e \in E$

Definition

An action of S on A is an inverse semigroup homomorphism $\alpha : S \to \mathrm{PAut}(A)$, i.e. $\alpha_s : I_s \to J_s$ isomorphism between ideals of A, and

$$\alpha_1 = 1$$

Definition

Sieben S-equivariant representation π on S-Hilbert space:
$U : S \to \mathrm{PartIso}(H)$ with
$\pi(I_s)H$ is inital space of α_s, $\pi(J_s)H$ is range space of α_s, and

$$\pi(s(a)) = U_s \pi(a) U_s^* \qquad \forall a \in A, s \in S$$

Definition (Sieben)

$A \widehat{\rtimes} S$ = C^*-algebra of universal Sieben S-equivariant representations.

Remark

"Unstrict": $A \rtimes S$!

Notation: S-C^*-algebra $=$ S-Hilbert C^*-algebra $+$ $C_0(X)$-algebra

Theorem (Quigg–Sieben)

Isomorphism of categories

$$[\mathcal{G}\text{-}C^*\text{-algebras}] \longleftrightarrow [S\text{-}C^*\text{-algebras}]$$

$$a_{s(g)} \xrightarrow{\alpha_g} a_{r(g)} \quad = \quad s(a)_{r(g)} \qquad \forall a \in A, s \in S, g \in s$$

Starting with a slice s you have a table $s = \{g\}$ $(g \in \mathcal{G})$. Realize action s by arrows α_g.

Theorem (Quigg–Sieben)

$$A \rtimes \mathcal{G} \quad \cong \quad A \widehat{\rtimes} S$$

$$(a_{r(g)})_{g \in s} \quad = \quad a \widehat{\rtimes} s \qquad \forall a \in A, s \in S$$

Theorem

There is an isomorphism

$$KK^{\mathcal{G}}(A,B) \xrightarrow{\rho} \widehat{KK^{S}}(A,B)$$

Respects functoriality, Kasparov product, and descent homomorphism:

$$\begin{array}{ccc} KK^{\mathcal{G}}(A,B) & \xrightarrow{\rho} & \widehat{KK^{S}}(A,B) \\ j^{\mathcal{G}} \downarrow & & \downarrow \widehat{j^{S}} \\ KK(A \rtimes \mathcal{G}, B \rtimes \mathcal{G}) & \longrightarrow & KK(A \widehat{\rtimes} S, B \widehat{\rtimes} S) \end{array}$$

Bottom arrow by Quigg–Sieben isomorphism of crossed products.

Definition

$\widehat{j^{S}}$ defined in above way !

Theorem (Khoshkam–Skandalis)

Let S any inverse semigroup.

$$
\begin{aligned}
(A \rtimes E) \widehat{\rtimes} S &\cong A \rtimes S \\
(a \rtimes e) \widehat{\rtimes} s &= a \rtimes es \qquad\qquad \forall a \in A, e \in E, s \in S
\end{aligned}
$$

Theorem (Tu)

There is a Baum–Connes map for groupoids:

$$\lim_{Y \subseteq \underline{E}\mathcal{G}} KK^{\mathcal{G}}(C_0(Y), A) \longrightarrow K(A \rtimes \mathcal{G})$$

Theorem

There is a Baum–Connes map for Sieben's crossed product:

$$\lim_{Y \subseteq \underline{E}\mathcal{G}} \widehat{KK^S}(C_0(Y), A) \quad \longrightarrow \quad K(A \widehat{\rtimes} S)$$

And Khoshkam–Skandalis' crossed product:

$$\lim_{Y \subseteq \underline{E}\mathcal{G}_S} \widehat{KK^S}(C_0(Y), A \rtimes E) \quad \longrightarrow \quad K(A \rtimes S)$$

If you start with any S and take $\mathcal{G} = \mathcal{G}_S$, it must checked to be Hausdorff !

Expansion

Let S any inverse semigroup

Define $\mathcal{G} := \mathcal{G}_S$ universal groupoid for S. Assume Hausdorff.

WHOLE SECTION !

Define $X := \mathcal{G}^{(0)}$

Then $C_0(X) \cong C^*(E)$ universal C^*-algebra generated by E (abelian)

Let A Hilbert C^*-algebra

Then $A \rtimes E$ is S-Hilbert C^*-algebra:

$$s(a \rtimes e) = s(a) \rtimes ses^* \qquad \forall e \in E, a \in e(A), s \in S$$

Then $A \rtimes E$ is $C_0(X)$-algebra + S-algebra (compatible)

Do we have map

$$KK^S(A, B) \longrightarrow \widehat{KK^S}(A \rtimes E, B \rtimes E) \qquad ?$$

Example: $S = E = \{p, P\}$ where $p < P$

$$C^*(E) = \{\lambda p + \mu P\} \cong C_0(\{p, P - p\}) = C_0(X)$$

$\mathbb{C}$ is Hilbert C^*-algebra with trivial action

By universality of $C^*(E)$: $\mathbb{C}$ is also $C_0(X)$-algebra

$C_0(X)$-action on $\mathbb{C}$: $P(1) = 1$ and $p(1) = 1 \Longrightarrow (P - p)(1) = 0$

$$\mathbb{C} \rtimes E = C_0(\{p, P - p\})$$

$C_0(X)$-action is multiplication

Expanded somehow!

Theorem

There is an expansion homomorphism (like descent)

$$\widehat{KK^S}(A,B) \xrightarrow{\epsilon} \widehat{KK^S}(A \rtimes E, B \rtimes E)$$

Respects functoriality, Kasparov product, and descent homomorphism:

$$\begin{array}{ccc}
\widehat{KK^S}(A,B) & \xrightarrow{\epsilon} & \widehat{KK^S}(A \rtimes E, B \rtimes E) \\
\downarrow & & \\
KK^S(A,B) & & \downarrow \widehat{j^S} \\
j^S \downarrow & & \\
KK(A \rtimes S, B \rtimes S) & \longrightarrow & KK((A \rtimes E)\widehat{\rtimes} S, (B \rtimes E)\widehat{\rtimes} S)
\end{array}$$

Bottom arrow by Khoshkam–Skandalis' isomorphism of crossed products.

Theorem (Khoskkam–Skandalis)

$$A \rtimes_{\mathrm{r}} S \;\cong\; (A \rtimes E) \rtimes_{\mathrm{r}} \mathcal{G}$$

Definition

Define descent homomorphism for reduced crossed product

$$\begin{array}{ccc}
\widehat{KK^S}(A,B) & \stackrel{\epsilon}{\longrightarrow} & \widehat{KK^S}(A \rtimes E, B \rtimes E) \\
 & & \downarrow \rho^{-1} \\
\downarrow \widehat{j_{\mathrm{r}}^S} & & KK^{\mathcal{G}}(A \rtimes E, B \rtimes E) \\
 & & \downarrow j_{\mathrm{r}}^{\mathcal{G}} \\
KK(A \rtimes_{\mathrm{r}} S, B \rtimes_{\mathrm{r}} S) & \longleftarrow & KK((A \rtimes E) \rtimes_{\mathrm{r}} \mathcal{G}, (B \rtimes E) \rtimes_{\mathrm{r}} \mathcal{G})
\end{array}$$

ρ^{-1} is isomorphism $\widehat{KK^S} \cong KK^{\mathcal{G}}$.

Theorem

Let $\mathcal{G}_S$ not necessarily Hausdorff.

There is an expansion homomorphism

$$KK^S(A, B) \xrightarrow{\epsilon} KK^S(A \rtimes E, B \rtimes E)$$

Respects functoriality and Kasparov product.

Expansion ϵ of a cycle $(\mathcal{E}, T)$ in $KK^S(A, B)$ gives a cycle $(\mathcal{E}', T') = (\mathcal{E} \otimes_B (B \rtimes E), T')$ in $KK^S(A \rtimes E, B \rtimes E)$ with

- compatible multiplication between $A \rtimes E$ and $\mathcal{E}'$,
- incompatible one between $\mathcal{E}'$ and $B \rtimes E$

Replace $\mathcal{E}'$ by balanced tensor product

$$\mathcal{E} \otimes_B^{C_0(X)} (B \rtimes E)$$

Theorem

Let E be finite and S unital !

There is a 'compatible' expansion isomorphism

$$KK^S(A, B) \xrightarrow{\delta} \widehat{KK^S}(A \rtimes E, B \rtimes E)$$

Respects functoriality.

Theorem

Let S be finite and unital.

There is a Green–Julg isomorphism

$$KK^S(\mathbb{C},A) \xrightarrow{\delta} \widehat{KK^S}(C_0(X), A \rtimes E) \xrightarrow{\widehat{\mu}} K((A \rtimes E)\widehat{\rtimes} S)$$

$$\searrow \mu^S \qquad\qquad \downarrow \; K(A \rtimes S)$$

δ = compatible expansion isomorphism

$\widehat{\mu}$ = Baum-Connes isomorphism

$\downarrow$ by Khoshkam–Skandalis isomorphism of crossed products

$\underline{E}\mathcal{G} = \mathcal{G}^{(0)} =: X$

$\mathbb{C} \rtimes E = C_0(X)$