Equivariant KK-theory for semimultiplicative sets

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Semimultiplicative set

Semimultiplicative set G : set G with subset $G^{(2)} \subseteq G \times G$ and associative multiplication $G^{(2)} \to G, (a,b) \mapsto ab$:

$$a(bc) \quad \text{is defined} \quad \Longleftrightarrow \quad (ab)c \quad \text{is defined}$$

and

$$a(bc) = (ab)c \qquad \text{if defined}$$

Examples for G : groups, groupoids, semigroups, inverse semigroups, small categories

Left regular representation

Injective left multiplication : Assume $\forall g \in G$, the left multiplication operator $L_g : h \mapsto gh$ is injective (there where defined)

Reduced C^*-algebra $C_r^*(G)$: C^*-algebra generated by the operators $\lambda_g \in B(\ell^2(G))$

$$\lambda_g(\delta_h) = 1_{\{gh \text{ is defined}\}} \delta_{gh} \qquad\qquad (g, h \in G)$$

Example 1

Finitely aligned higher rank graph Λ

Raeburn–Sims–Yeend : **Toeplitz–Cuntz–Krieger algebra** $\mathcal{T}C^*(\Lambda)$

Make infinite path space Λ^* a semimultiplicative set:

$\lambda \circ \mu = \lambda\mu \qquad \text{if } \lambda \in \Lambda, \mu \in \Lambda^*$

otherwise composition undefined $(\lambda \in \Lambda^* \backslash \Lambda)$

$$C_r^*(\Lambda^*) \cong \mathcal{T}C^*(\Lambda)$$

$$K_0(\mathcal{T}C^*(\Lambda)) = \bigoplus_{\nu \in \Lambda^{(0)}} \mathbb{Z}, \qquad\qquad K_1(\mathcal{T}C^*(\Lambda)) = 0$$

Example 2

Want graph C^*-algebra, but relax relations induced by vertex set $\Lambda^{(0)}$

$\longrightarrow$ **semigraph C^*-algebra**

for instance C^*-algebras of labelled graphs by BATES–PASK

Or:

another Toeplitz algebra for Λ :

$C^*(T_\Lambda)$ defined like $\mathcal{T}C^*(\Lambda)$, but without $s_{s(\lambda)} = s_\lambda^* s_\lambda$ relations ($\lambda \in \Lambda$)

There is a semimultiplicative set G:

$$C_r^*(G) \cong C^*(T_\Lambda)$$

$$K_0(\mathcal{T}C^*(\Lambda)) \subset K_0(C^*(T_\Lambda)), \qquad K_1(C^*(T_\Lambda)) = 0$$

(proper inclusion iff $|\Lambda^{(0)}| < \infty$)

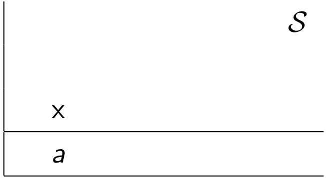
Example 3

One-sided shift $\mathcal{S}$ of finite type in dim. 2 : alphabet Σ
$\mathcal{S} \subseteq \Sigma^{\mathbb{N}\times\mathbb{N}}$, finite failures in $\mathcal{S}$ allowed
Alphabet $\mathcal{A}$: $S_a, T_b \qquad a, b \in \Sigma^{\mathbb{N}}$
Hilbert space $\ell^2(\mathcal{S})$
Rank 2 **Exel–Laca algebra** $C^*(\mathcal{S})$ **:** C^*-subalgebra of $B(\ell^2(\mathcal{S}))$
generated by $\pi(S_a), \pi(T_b) \qquad (a, b \in \Sigma^{\mathbb{N}})$
K**-theory :**

$$\begin{aligned} K_0(C^*(\mathcal{S})) &= \text{subring in } C^*(\mathcal{S}) \text{ generated by} \\ &\quad \text{source projections of } \pi(S_a), \pi(T_b) \quad (a, b \in \Sigma^{\mathbb{N}}) \end{aligned}$$

$$K_1(C^*(\mathcal{S})) = 0$$

Operators acting on $\mathcal{S}$



$$\pi(S_a)\delta_x \quad = \quad 1_{\{ax \in \mathcal{S}\}}\delta_{ax}$$

$$\pi(T_b)\delta_x \quad = \quad 1_{\{bx \in \mathcal{S}\}}\delta_{bx}$$

$$S_a = (a_1, a_2, a_3, a_4, a_5, \ldots) \qquad a_i \in \Sigma$$

$$T_b = (b_1, b_2, b_3, b_4, b_5, \ldots) \qquad b_i \in \Sigma$$

Motivation for G-equivariant KK-theory

PATERSON: Translation r-**discrete groupoid** $\mathcal{G} \leftrightarrow$ **inverse semigroup** S

QUIGG–SIEBEN: Translation **crossed product** $A \rtimes \mathcal{G} \leftrightarrow$ **crossed product** $A \rtimes^{Sieben} S$

It is tempting to seek translation

$$KK^{\mathcal{G}}(A, B) \cong KK^{S}(A, B) \qquad (!?)$$

Let E denote idempotent set of S

KHOSHKAM–SKANDALIS: Translation **crossed product** $A \rtimes \mathcal{G} \leftrightarrow$ **crossed product** $(A \rtimes E) \rtimes^{Sieben} S$

$A \rtimes E$ is $C_0(X)$-algebra (X spectrum of $C^*(E)$)

$$KK^{S}(A, B) \longrightarrow KK^{S}(A \rtimes E, B \rtimes E) \qquad (!?)$$

Take $S =$ inverse semigroup generated by $\lambda(G)$ (!?)

G-Hilbert C^*-algebra

$\mathcal{E}$ = Hilbert module over C^*-algebra B

Partial isometry U on $\mathcal{E}$: complemented subspaces $\mathcal{E}_i \subseteq \mathcal{E}$, linear map

$$U : \mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2 \longrightarrow \mathcal{E}_3 \oplus \mathcal{E}_4 = \mathcal{E}$$

$U|_{\mathcal{E}_1} : \mathcal{E}_1 \to \mathcal{E}_3$ norm-isometric, $U|_{\mathcal{E}_2} = 0$

Have inverse partial isometry U^*

G-Hilbert C^*-algebra : Hilbert B-module B with $\langle x, y\rangle = x^* y$

Action $\alpha : G \to \mathrm{PartIso}(B) \cap \mathrm{End}(B)$

$$\alpha_{gh} = \alpha_g \alpha_h \qquad \text{if } gh \text{ is defined } (g, h \in G)$$

α_g, α_g^* grading preserving

Notation: $\alpha_g(b) = g(b)$

Further required relations:

$$\langle g(x), y\rangle = g\langle x, g^*(y)\rangle, \qquad \langle g^*(x), y\rangle = g^*\langle x, g(y)\rangle$$

G-Hilbert module

G-**Hilbert module :** $\mathcal{E}$ Hilbert module over G-Hilbert C^*-algebra B

Action $U : G \to \mathrm{PartIso}(\mathcal{E})$

U_g, U_g^* grading preserving

Further required relations: $(x \in \mathcal{E}, b \in B)$

$$U_g(xb) = U_g(x)g(b), \qquad U_g^*(xb) = U_g^*(x)g^*(b)$$

$$\langle U_g(x), y\rangle = g\langle x, U_g^*(y)\rangle, \qquad \langle U_g^*(x), y\rangle = g^*\langle x, U_g(y)\rangle$$

G-**equivariant** $*$-**homomorphism :** A and B G-Hilbert C^*-algebras, $\pi : A \to B$ $*$-homomorphism

$$\pi(g(a)) = g(\pi(a)), \qquad \pi(g^*(a)) = g^*(\pi(a))$$

$\forall g \in G, a \in A$

G-equivariance

$\mathcal{E}$ G-Hilbert module

linear G-actions on $\mathcal{L}(\mathcal{E})$: $g(T) = U_g T U_g^*, \qquad g^*(T) = U_g^* T U_g$ $\forall T \in \mathcal{L}(\mathcal{E})$

$$g(ST) \neq g(S)g(T) \qquad \text{in general}$$

G-equivariant representation : A G-Hilbert C^*-algebra

$\pi : A \to \mathcal{L}(\mathcal{E})$ $*$-homomorphism

$$U_g \pi(a) U_g^* = \pi(g(a)) U_g U_g^*$$

$$U_g^* \pi(a) U_g = \pi(g^*(a)) U_g^* U_g$$

$$[U_g U_g^*, \pi(a)] = 0, \qquad [U_g^* U_g, \pi(a)] = 0$$

Inspired by $C_r^*(G)$: $U =$ shift on $\ell^2(G)$, $\pi : \mathbb{C} \to B(\ell^2(G))$ trivial

G-cycles

A, B G-Hilbert C^*-algebras

G-**Hilbert** (A, B)-**bimodule** $\mathcal{E}$ **:** G-Hilbert B-module $\mathcal{E}$ together with G-equivariant representation $\pi : A \longrightarrow \mathcal{L}(\mathcal{E})$

G-**Cycle** $(\mathcal{E}, T)$ **over** (A, B) **:** $\mathcal{E} =$ countably generated G-Hilbert (A, B)-bimodule, $T \in \mathcal{L}(\mathcal{E})$ odd

$\forall a \in A$ these operators of $\mathcal{L}(\mathcal{E})$ lie in $\mathcal{K}(\mathcal{E})$:

$$\begin{array}{rl} & [a, T] \\ a(T - T^*), & (T - T^*)a \\ a(T^2 - 1), & (T^2 - 1)a \\ a(U_g T U_g^* - T U_g U_g^*), & (U_g T U_g^* - T U_g U_g^*)a \\ a[T, U_g U_g^*], & [T, U_g U_g^*]a \\ a[T, U_g^* U_g], & [T, U_g^* U_g]a \end{array}$$

Set of cycles : $\mathbb{E}^G(A, B)$

G-equivariant KK-theory

Addition of cycles : $(\mathcal{E}_1, T_1) \oplus (\mathcal{E}_2, T_2) = (\mathcal{E}_1 \oplus \mathcal{E}_2, T_1 \oplus T_2)$

G-**equivariant** KK-**theory :** Abelian group

$$KK^G(A, B) = \mathbb{E}^G(A, B)/\text{homotopy}$$

Functoriality : KK^G has usual functoriality in A and B

Kasparov product : A, B, C G-Hilbert C^*-algebras, A separable

There is a bilinear map

$$\otimes_B : KK^G(A, B) \otimes KK^G(B, C) \longrightarrow KK^G(A, C)$$

Associativity : Kasparov product is associative

$$(x \otimes_B y) \otimes_C z = x \otimes_B (y \otimes_C z)$$

Full crossed product

Convolution algebra :

A G-Hilbert C^*-algebra

$G^* =$ set of expressions $g_1^{\epsilon_1} \dots g_n^{\epsilon_n}$ where $g_i \in G$, $\epsilon_i \in \{1, *\}$

$A \rtimes_{\mathrm{alg}} G =$ formal sums $\sum_{g \in G^*} a_g g$ where $a_g \in gg^*(A)$

$$\Big(\sum_{g \in G^*} a_g g \Big)^* = \sum_{g \in G^*} g^*(a_g^*) g^*$$

$$\Big(\sum_{g \in G^*} a_g g \Big) \Big(\sum_{h \in G^*} b_h h \Big) = \sum_{g, h \in G^*} a_g g(b_h) gh$$

Crossed product : $A \rtimes G =$ universal C^*-algebra for $A \rtimes_{\mathrm{alg}} G$ under covariant representation on Hilbert space

G **inverse semigroup :** $A \rtimes G =$ universal C^*-algebra for $\ell^1(G, A)$ (in accordance with KHOSHKAM–SKANDALIS)

Reduced crossed product

G has injective left multiplication

A G-Hilbert C^*-algebra represented on H

A has **transferred injective multiplication :**

$$gh \quad \text{is defined} \qquad \Longrightarrow \qquad \alpha_g^* \alpha_g \alpha_h = \alpha_h$$

Reduced covariant representation :

$$U_g : \ell^2(G, H) \to \ell^2(G, H) : U_g(\xi \delta_h) = 1_{\{gh \text{ is defined}\}} \xi \delta_{gh}$$

$$\pi : A \to B(\ell^2(G)) : \pi(a)(\xi \delta_g) = (\pi(g^*(a))\xi)\delta_g$$

Reduced crossed product : $A \rtimes_r G = C^*$-subalgebra of $B(\ell^2(G, H))$
generated by $\pi(a)U_g \quad (a \in A, g \in G)$

Trivial action on $\mathbb{C}$: $\mathbb{C} \rtimes_r G = C_r^*(G)$

Strong crossed product

Strong G-action U on Hilbert H : $U_g U_h = 0$ if gh not defined

Strong crossed product : $A \rtimes_s G =$ universal C^*-algebra for $A \rtimes_{\mathrm{alg}} G$ under G-equivariant representation on Hilbert space with strong G-action

$$a_g g * b_h h = 0 \qquad \text{if} \quad gh \quad \text{not defined}$$

Discrete groupoid G : $\mathbb{C} \rtimes_s G =$ usual groupoid C^*-algebra of G

Descent homomorphism

H, G semimultiplicative sets, G with 1

A, B $H \times G$-Hilbert C^*-algebras

There is a 'descent' homomorphism : (for full, strong, reduced)

$$j^G : KK^{H\times G}(A,B) \to KK^H(A \rtimes G, B \rtimes G)$$

$$j^G[(\mathcal{E}, T)] = (\mathcal{E} \otimes_B (B \rtimes G), T \otimes 1)$$

Respects Kasparov product :

$$j^G(x \otimes_B y) = j^G(x) \otimes_{B\rtimes G} j^G(y)$$

Remark : In all occurring $G \times H$-Hilbert modules: H-actions (h, h^*) commute with G-actions (g, g^*)

Reduced : G injective left multiplication + non-degenerate , transferred injective multiplication , trivial G-action on B