Simplicial volume of non-compact manifolds

Clara Löh

April 2008

Abstract. Degree theorems are statements bounding the mapping degree in terms of the volumes of the domain and target manifolds. A possible strategy to obtain such degree theorems is to compare the Riemannian volume with a suitable topological invariant, e.g., the simplicial volume. In this talk, the simplicial volume of non-compact manifolds is studied and corresponding degree theorems for non-compact manifolds are derived.

This talk is based on joint work with Roman Sauer [9, 8].

1 Introduction – degree theorems

A *degree theorem* is a statement of the following type: For all proper, continuous maps $f\colon N \longrightarrow M$ between Riemannian manifolds of a particular type with finite volume, the degree of f can be bounded in terms of the volume:

$$|\deg f| \leq \operatorname{const}_{\dim M} \cdot \frac{\operatorname{vol} N}{\operatorname{vol} M}.$$

Gromov successfully applied the following strategy to obtain such degree theorems for negatively curved manifolds [4]: Find a topological replacement v of the Riemannian volume such that

- For all proper, continuous maps $f\colon N \longrightarrow M$ one has $|\deg f| \cdot v(M) \leq v(N)$.
- For all suitable domain manifolds N one has $v(N) \leq \operatorname{const}_{\dim N} \cdot \operatorname{vol} N$.
- For all suitable target manifolds M one has $v(M) \geq \operatorname{const}_{\dim M} \cdot \operatorname{vol} M$.

An example of an adequate topological replacement of the Riemannian volume in this sense is the (Lipschitz) simplicial volume.

Clara Löh · clara.loeh@uni-muenster.de · Mathematisches Institut · WWU Münster · Einsteinstr. 62, 48149 Münster, Germany

This talk is based on joint work with Roman Sauer.

Based on the methods developed by Besson, Courtois, and Gallot, degree theorems for (most) locally symmetric spaces of non-compact type with finite volume were obtained by Connell and Farb [2].

Generalising Gromov's non-vanishing results to the simplicial volume of *closed* locally symmetric spaces of non-compact type, Lafont and Schmidt derived corresponding degree theorems in the closed case [6].

In the following, we show to what extent the results by Lafont and Schmidt can be generalised to the non-compact case. We start in Section 2 with a review of the definition and basic properties of the simplicial volume. In Section 3, we explain two versions of simplicial volume for non-compact manifolds along with their respective advantages and disadvantages. Finally, in Section 4, we use the results about the simplicial volume of non-compact manifolds to deduce a degree theorem.

2 Simplicial volume – the compact case

The simplicial volume is a homotopy invariant invented by Gromov [4] measuring how efficiently the fundamental class of a manifold can be represented by $\mathbf{R}$-fundamental cycles; i.e., the simplicial volume gives an indication of how difficult it is to triangulate the manifold in question. After stating the concise definition of the simplicial volume for compact manifolds in Section 2.1, a survey of important properties of the simplicial volume is given in Section 2.2.

2.1 Definition

Definition (2.1) (Simplicial volume – compact case)**.** For an oriented, closed, connected n-manifold M, the *simplicial volume* of M is defined by

$$\|M\| := \inf \bigl\{ \|c\|_1 \bigm| c \in C_n(M) \text{ is an } \mathbf{R}\text{-fundamental cycle of } M \bigr\};$$

here, $\|c\|_1 := \sum_\sigma |a_\sigma|$ for $c = \sum_\sigma a_\sigma \cdot \sigma \in C_*(M)$ (in reduced form). ◇

2.2 Properties and examples

Degree estimate. Let $f\colon N \longrightarrow M$ be a continuous map of oriented, closed, connected manifolds (of the same dimension). Then

$$|\deg f| \cdot \|M\| \leq \|N\|.$$

Homotopy invariance. If the two oriented, closed, connected manifolds M and N are homotopy equivalent, then $\|M\| = \|N\|$.

Self-maps. Let $f\colon M \longrightarrow M$ be a continuous self-map of the oriented, closed, connected manifold M with $|\deg f| \geq 2$. Then $\|M\| = 0$.

Example (2.2) (Spheres, tori)**.** In particular, the simplicial volume of spheres and tori of non-zero dimension is zero. ◇

Bounded cohomology and ℓ^1-homology. The simplicial volume admits a description in terms of the algebraic tools bounded cohomology and ℓ^1-homology [4, 7].

Example (2.3) (Amenable fundamental group)**.** In particular, the simplicial volume of oriented, closed, connected manifolds (of non-zero dimension) with amenable fundamental group vanishes [4]. ◇

Products. If M and N are oriented, closed, connected manifolds, then

$$\|M\| \cdot \|N\| \leq \|M \times N\| \leq \text{const}_{\dim M + \dim N} \cdot \|M\| \cdot \|N\| .$$

The inequality on the right follows from the explicit description of the homological cross-product via triangulations of products of simplices. The inequality on the left is a consequence of the fact that the cross-product on singular cohomology is continuous with respect to the supremum norm [4, 1].

Hyperbolic manifolds. If M is an oriented, closed, connected hyperbolic n-manifold, then

$$\|M\| = \frac{\operatorname{vol} M}{v_n},$$

where v_n ist the volume of a regular, ideal hyperbolic n-simplex [4, 11].

This shows in particular that the Riemannian volume of oriented, closed, connected hyperbolic manifolds is a homotopy invariant – an important step in Gromov's proof of Mostow rigidity.

For later reference and to give an impression of the nature of the simplicial volume, we briefly review the proof of $\|M\| \geq \operatorname{vol} M / v_n$ for closed hyperbolic n-manifolds:

Proof. If M is an oriented, closed, connected hyperbolic n-manifold, then we can "straighten" any singular simplex by lifting it to the universal covering, replacing the lifted simplex by the geodesic simplex with the same vertices, and projecting the resulting simplex back to M. This chain map, denoted by str, induces the identity on top homology and does not increase the norm. Hence, we obtain

$$\begin{aligned} \operatorname{vol} M = \int_{\operatorname{str} c} \operatorname{vol}_M &= \sum_{j=0}^{r} a_j \cdot \int (\operatorname{str} \sigma_j)^* \operatorname{vol}_M \\ &\leq \sum_{j=0}^{r} |a_j| \cdot v_n \leq \|c\|_1 \cdot v_n \end{aligned}$$

for all (smooth) fundamental cycles $c = \sum_{j=0}^{r} a_j \cdot \sigma_j$ of M; taking the infimum over all fundamental cycles gives the desired estimate $\|M\| \geq \operatorname{vol} M / v_n$. □

Negative curvature. Using Thurston's straightening technique, Inoue and Yano generalised the result on hyperbolic manifolds: the simplicial volume of all oriented, closed, connected Riemannian manifolds with negative sectional curvature is non-zero [5].

Locally symmetric spaces. Lafont and Schmidt extended the non-vanishing of the simplicial volume even further, namely to all oriented, closed, connected locally symmetric manifolds of non-compact type [6].

Proportionality principle. For all Riemannian manifolds M and N with isometric universal coverings, we have

$$\frac{\|M\|}{\operatorname{vol} M} = \frac{\|N\|}{\operatorname{vol} N}$$

by a result of Gromov and Thurston [4, 11, 10]; this can also be viewed as a generalisation of the hyperbolic case.

Volume estimate. A classical result of Gromov shows that the simplicial volume provides a topological lower bound for the minimal volume; more precisely,

$$\|M\| \leq \operatorname{const}_{\dim M} \cdot \operatorname{minvol} M$$

for all oriented, closed, connected, smooth manifolds M [4]. Here the *minimal volume* of M is defined by

$$\operatorname{minvol} M := \inf \bigl\{ \operatorname{vol}(M, g) \bigm| g \text{ is a complete Riemannian metric on } M \text{ with } |\sec(g)| \leq 1 \bigr\}.$$

Combining the volume estimate with the positivity of the simplicial volume of closed locally symmetric spaces of non-compact type yields – as indicated in the introduction – a degree theorem for continuous maps into closed locally symmetric spaces.

3 Simplicial volume – the non-compact case

There are two main flavours of simplicial volumes for non-compact manifolds: a topological one and a geometric one. While the topological one is a straightforward adaption of the definition to locally finite homology (Section 3.1), for the geometric version a Lipschitz constraint is added (Section 3.3).

3.1 Topological definition

Definition (3.1) (Simplicial volume – non-compact case)**.** For an oriented, connected n-manifold M the *simplicial volume* is defined by

$$\|M\| := \inf \bigl\{ \|c\|_1 \bigm| c \in C_n^{\mathrm{lf}}(M) \text{ is a locally finite } \mathbf{R}\text{-fundamental cycle of } M \bigr\}. \quad \diamond$$

Notice that the ℓ^1-norm $\|\cdot\|_1$ on the locally finite chain complex $C_*^{\mathrm{lf}}(\,\cdot\,)$ is not necessarily finite; hence also $\|M\|$ can be infinite for non-compact manifolds M.

3.2 Properties and examples – topological version

Degree estimate. Let $f\colon N \longrightarrow M$ be a proper, continuous map of oriented, connected manifolds (of the same dimension). Then

$$|\deg f| \cdot \|M\| \leq \|N\|.$$

Proper homotopy invariance. In particular, properly homotopy equivalent oriented, connected manifolds have equal simplicial volume.

Self-maps. Let $f\colon M \longrightarrow M$ be a proper, continuous self-map of the oriented, connected manifold M with $|\deg f| \geq 2$. Then $\|M\| \in \{0, \infty\}$.

Bounded cohomology and ℓ^1-homology. Also the simplicial volume of non-compact manifolds admits a description in terms of the algebraic tools bounded cohomology and ℓ^1-homology [7]; however, this description is not as convenient as in the compact case because there is no comparison map linking locally finite homology and ℓ^1-homology.

Example (3.2) (Amenable fundamental group)**.** The simplicial volume of oriented, connected manifolds with amenable fundamental group is zero or infinite. ◇

Finiteness criterion. Let $(W, \partial W)$ be an oriented, compact, connected manifold with boundary. Then the simplicial volume of the interior W° is finite if and only if the boundary ∂W is ℓ^1-invisible, i.e., if and only if the fundamental class of ∂W maps to zero under the comparison map between singular homology and ℓ^1-homology [7]. Notice that this finiteness criterion is purely topological.

Example (3.3) (Euclidean spaces)**.** A sphere is ℓ^1-invisible if and only if it has non-zero dimension. Hence $\|\mathbf{R}^n\|$ is finite if and only if $n > 1$. Because Euclidean spaces are simply connected, we obtain

$$\|\mathbf{R}^n\| = 0$$

for all $n \in \mathbf{N}_{>1}$. ◇

Example (3.4) (Cylinders)**.** For an oriented, closed, connected manifold M the simplicial volume of $M \times \mathbf{R}$ is finite (and then even equal to zero!) if and only if M is ℓ^1-invisible [7]. ◇

Products. If M and N are oriented, connected manifolds, then examining the homological cross products yields

$$\|M \times N\| \leq \mathrm{const}_{\dim M + \dim N} \cdot \|M\| \cdot \|N\|.$$

However, $\|\mathbf{R} \times \mathbf{R}\| = 0$ despite of $\|\mathbf{R}\| = \infty$. So, in general, there is no lower bound for $\|M \times N\|$ in terms of the simplicial volumes of the factors (such a lower bound does exist if one of the factors is compact [4, 7]).

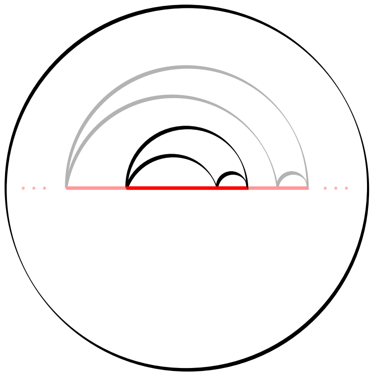

Figure (3.5): The straightening of locally finite chains in the hyperbolic plane is not necessarily locally finite

Hyperbolic manifolds. It is one of the confusing aspects of the simplicial volume of non-compact manifolds that $\|\mathbf{H}^n\| = 0$ for all $n \in \mathbf{N}_{>1}$ despite of $\mathbf{H}^n$ being of (constant) negative curvature and having infinite volume.

In order to understand this phenomenon, we exhibit where the non-vanishing proof for the compact case breaks down in the non-compact case: If M is not compact, then the straightening of a locally finite chain is not necessarily locally finite again (cf. Figure (3.5)). Therefore, the straightening argument is in general in the non-compact setting not applicable.

On the other hand, if M is an oriented, connected hyperbolic manifold with finite volume, then [11, 3]

$$\|M\| = \frac{\operatorname{vol} M}{v_n}.$$

Vanishing results. Using Gromov's vanishing result on amenable covers, we derive the following vanishing result [9]: If M is an oriented, connected n-manifold such that the classifying space $B\pi_1(M)$ of the fundamental group admits a finite model of dimension at most $n-2$ and M possesses a compactification with nice boundary, then $\|M\| = 0$.

This vanishing result can be used ot show that the simplicial volume of non-compact aspherical manifolds is zero in many cases.

Locally symmetric spaces. More drastically, for any locally symmetric space of non-compact type with $\mathbf{Q}$-rank at least 3 with finite volume, the simplicial volume is zero [9]; this is a consequence of the vanishing result above – the finiteness condition on the corresponding classifying space and the existence of a nice compactification can easily be verified by looking at the Borel-Serre compactification.

Proportionality principle. The vanishing result for locally symmetric spaces of higher $\mathbf{Q}$-rank shows that in general there cannot be a proportionality principle for non-compact manifolds.

The first counterexamples were given by Gromov [4] in the form of products of three open hyperbolic manifolds of finite volume.

Volume estimate. Gromov's volume estimate also holds in the non-compact case, i.e., for all oriented, connected, smooth manifolds M, the simplicial volume provides a lower bound for the minimal volume:

$$\|M\| \leq \text{const}_{\dim M} \cdot \text{minvol}\, M.$$

3.3 Geometric definition

Definition (3.6) (Lipschitz simplicial volume)**.** Let M be an oriented, connected, Riemannian n-manifold. Then the *Lipschitz simplicial volume* of M is defined by

$$\|M\|_{\text{Lip}} := \inf \bigl\{ \|c\|_1 \bigm| c \in C_n^{\text{lf}}(M) \text{ is a locally finite } \mathbf{R}\text{-fundamental cycle} \\ \text{with } \text{Lip}\, c < \infty \bigr\};$$

here, $\text{Lip}\, c := \sup \{\text{Lip}\, \sigma \mid \sigma \in \text{supp}\, c\}$ for all $c \in C_*^{\text{lf}}(M)$. ◇

Notice that $\|M\|_{\text{Lip}}$ depends on the Riemannian metric on M.

3.4 Properties and examples – geometric version

Adding the Lipschitz constraint enforces a more geometric behaviour of $\|\cdot\|_{\text{Lip}}$ in comparison with $\|\cdot\|$ – for example locally finite Lipschitz chains can easily be integrated over the volume form, and, in the presence of non-positive sectional curvature, there is a straightening on locally finite Lipschitz chains; however, this link with Riemannian geometry comes at a price: we loose the ability to access this invariant algebraically by bounded cohomology or ℓ^1-homology.

Comparison with the topological version. By definition, for oriented, connected, Riemannian manifolds M we have

$$\|M\| \leq \|M\|_{\text{Lip}};$$

if M is compact, then $\|M\| = \|M\|_{\text{Lip}}$ because the ordinary simplicial volume can be computed in terms of (finite) smooth fundamental cycles.

Non-degeneracy. If M is an oriented, connected Riemannian manifold with infinite volume, then integration over the volume form shows that $\|M\|_{\text{Lip}} = \infty$.

Degree estimate. Let $f\colon N \longrightarrow M$ be a proper, Lipschitz map of oriented, connected, Riemannian manifolds (of the same dimension). Then

$$|\deg f| \cdot \|M\|_{\text{Lip}} \leq \|N\|_{\text{Lip}}.$$

Products. Let M and N be oriented, connected, Riemannian manifolds. Then

$$\|M \times N\|_{\text{Lip}} \leq \text{const}_{\dim M + \dim N} \cdot \|M\|_{\text{Lip}} \cdot \|N\|_{\text{Lip}}.$$

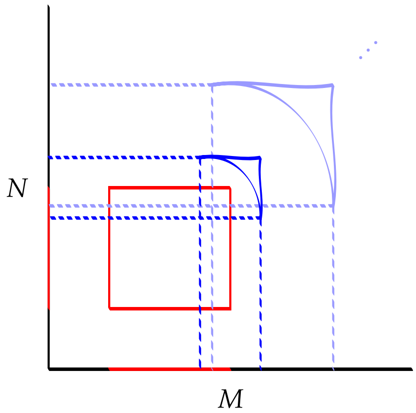

Figure (3.7): The cross-product of compactly supported chains is not necessarily compactly supported

If M and N have non-positive sectional curvature, then also the reverse inequality

$$\|M\|_{\text{Lip}} \cdot \|N\|_{\text{Lip}} \leq \|M \times N\|_{\text{Lip}}$$

holds [9].

The key to proving the non-trivial inequality is to describe the Lipschitz simplicial volume in terms of cohomology with Lipschitz compact supports. While the cross-product of two compactly supported cochains is not necessarily compactly supported (cf. Figure (3.7)), the cross-product of two cochains with Lipschitz support is again a cochain with Lipschitz support.

We need the curvature condition in order to get small fundamental cycles on $M \times N$ with the additional property that the projections of their supports to M and N are locally finite; however, it might be possible to get rid of this curvature condition by a more careful analysis.

Proportionality. Let M and N be oriented, connected, complete, Riemannian manifolds of non-positive sectional curvature with finite volume that have isometric universal coverings. Then [9]

$$\frac{\|M\|_{\text{Lip}}}{\operatorname{vol} M} = \frac{\|N\|_{\text{Lip}}}{\operatorname{vol} N}.$$

The proof of this version of the proportionality principle is similar to the one by Thurston in the compact case using measure homology and the smearing construction [11, 10]. In order to explain the rôle of the Lipschitz constraint in the above proportionality principle, we sketch the proof:

Sketch proof (see [9] for the complete proof). Let G be the Lie group of orientation preserving isometries of $\widetilde{M} = \widetilde{N}$; the fundamental group $\Gamma := \pi_1(M)$ is a lattice in G and hence G is unimodular. Let $\mu_{\Gamma\setminus G}$ be the normalised right G-invariant measure on $\Gamma \setminus G$.

We now give the definition of smearing of a Lipschitz chain: For a smooth simplex τ on $\widetilde{N}$ we write $\operatorname{smear}_\tau$ for the push-forward of $\mu_{\Gamma\setminus G}$ under the map

$$\begin{aligned} \Gamma \setminus G &\longrightarrow C^1(\Delta^*, M) \\ \Gamma \cdot g &\longmapsto \pi_M \circ (g \cdot \tau), \end{aligned}$$

where $\pi_M \colon \widetilde{M} \longrightarrow M$ is the universal covering map.

Consequently, for smooth chains $c := \sum_\sigma a_\sigma \cdot \sigma \in C_*^{\mathrm{lf}}(M)$ with $\operatorname{Lip} c < \infty$ and $\|c\|_1 < \infty$, we obtain a finite measure on $C^1(\Delta^*, M)$

$$\operatorname{smear}_c := \sum_\sigma a_\sigma \cdot \operatorname{smear}_{\widetilde{\sigma}}$$

with Lipschitz determination. Notice that the Lipschitz condition on c is needed to ensure that the smeared chain smear_c has nice finiteness properties (for instance, this enables us to integrate smear_c over forms on M).

If c is a fundamental cycle of N, then integrating smear_c over the volume form shows that smear_c represents (in Lipschitz measure homology) $\operatorname{vol} N / \operatorname{vol} M$ times the Lipschitz measure fundamental class of M.

Furthermore, because $\mu_{\Gamma\setminus G}$ is normalised, the total variation of smear_c is at most $\|c\|_1$. A priori it is not clear that the semi-norm on Lipschitz measure homology given by total variation coincides with the ℓ^1-semi-norm on locally finite Lipschitz homology; however, in the presence of non-positive sectional curvature one can use a suitable form of straightening to show that these semi-norms indeed coincide.

Therefore, we obtain

$$\|M\|_{\mathrm{Lip}} \leq \frac{\operatorname{vol} M}{\operatorname{vol} N} \cdot \|N\|_{\mathrm{Lip}} .$$

By symmetry, the reverse inequality also holds, which proves the proportionality principle. □

Locally symmetric spaces. In the following, we consider locally symmetric spaces M of non-compact type with finite volume; we assume that M ist given in the form $M = \Gamma\backslash G/K$, where G is a connected, non-compact, semi-simple Lie group, K is a maximal compact subgroup of G and Γ is a lattice in G. Whenever we speak about the $\mathbf{Q}$-rank of Γ it is understood that in this case the lattice Γ is arithmetic (so that the $\mathbf{Q}$-rank of Γ is defined).

Combining our results on the (Lipschitz) simplicial volume of non-compact manifolds with the positivity theorem of Lafont and Schmidt, we obtain the picture in Figure (3.8).

Conditions	*Behaviour of (Lipschitz) simplicial volume*
M closed	$\|M\| > 0$ by the result of Lafont and Schmidt
–	$\|M\|_{\mathrm{Lip}} > 0$ by proportionality and the previous line
M a Hilbert modular variety	$\|M\| = \|M\|_{\mathrm{Lip}} > 0$; the first equality is a consequence of the fact that Hilbert modular varieties have sufficiently nice boundary components and Gromov's equivalence theorem [8]. Notice that Hilbert modular varieties are examples where $\mathrm{rk}_{\mathbf{Q}} \Gamma = 1$.
M hyperbolic	$\|M\| = \|M\|_{\mathrm{Lip}} = \mathrm{vol}\, M / v_{\dim M}$
$\mathrm{rk}_{\mathbf{Q}} \Gamma \geq 3$	$\|M\| = 0$ by the vanishing result

Figure (3.8): Simplicial volumes of locally symmetric spaces of non-compact type

Volume estimate. If M is an oriented, connected, complete Riemannian manifold whose sectional curvature lies between -1 and 1, then [4, 9]

$$\|M\|_{\mathrm{Lip}} \leq \mathrm{const}_{\dim M} \cdot \mathrm{vol}\, M.$$

Caveat. The left hand side depends on the chosen Riemannian metric; therefore, in general, this inequality does not give a lower bound for the minimal volume!

4 Applications – a degree theorem

Finally, we come back to the original problem of deriving degree theorems for manifolds of finite volume.

In view of the results presented in the previous sections, we obtain the following degree theorem [9]:

Theorem (4.1) (Degree theorem – locally symmetric spaces)**.** *For every $n \in \mathbf{N}$ there is a constant $C_n > 0$ with the following property: Let M be an n-dimensional locally symmetric space of non-compact type with finite volume. Let N be an n-dimensional complete Riemannian manifold of finite volume with $|\sec(N)| \leq 1$, and let $f \colon N \longrightarrow M$ be a proper Lipschitz map. Then*

$$|\deg(f)| \leq C_n \cdot \frac{\mathrm{vol}\, N}{\mathrm{vol}\, M}.$$

In particular, we recover the result of Connell and Farb [2] that the minimal volume of the Lipschitz class of the standard metric on a locally symmetric space of non-compact type with finite volume is non-zero.

Theorem (4.2) (Degree theorem – products)**.** *For every $n \in \mathbf{N}$ there is a constant $C_n > 0$ with the following property: Let M be a Riemannian n-manifold of finite volume that decomposes as a product $M = M_1 \times \cdots \times M_m$ of Riemannian manifolds, where for every $i \in \{1, \dots, m\}$ the manifold M_j is either negatively curved with $-\infty < -k < \sec(M_j) \leq -1$ or a locally symmetric space of non-compact type. Let N be an n-dimensional, complete Riemannian manifold of finite volume with $|\sec N| \leq 1$. Then for every proper Lipschitz map $f \colon N \longrightarrow M$ we have*

$$|\deg f| \leq C_n \cdot \frac{\operatorname{vol} N}{\operatorname{vol} M}.$$

References

[1] R. Benedetti, C. Petronio. *Lectures on Hyperbolic Geometry*. Universitext, Springer, 1992. Cited on page: 3

[2] C. Connell, B. Farb. The degree theorem in higher rank. *J. Differential Geom.*, 65, No. 1, p. 19–59, 2003. Cited on page: 2, 10

[3] S. Francaviglia. Hyperbolic volume of representations of fundamental groups of cusped 3-manifolds. *Int. Math. Res. Not.*, 9, p. 425–459, 2004. Cited on page: 6

[4] M. Gromov. Volume and bounded cohomology. *Publ. Math. IHES*, 56, pp. 5–99, 1982. Cited on page: 1, 2, 3, 4, 5, 7, 10

[5] H. Inoue, K. Yano. The Gromov invariant of negatively curved manifolds. *Topology*, 21, No. 1, pp. 83–89, 1981. Cited on page: 4

[6] J.-F. Lafont, B. Schmidt. Simplicial volume of closed locally symmetric spaces of non-compact type. *Acta Math.*, 197, No. 1, pp. 129–143, 2006. Cited on page: 2, 4

[7] C. Löh. *ℓ^1-Homology and Simplicial Volume*, PhD thesis, WWU Münster, 2007. Available online at http://nbn-resolving.de/urn:nbn:de:hbz:6-37549578216 Cited on page: 3, 5

[8] C. Löh, R. Sauer. Simplicial volume of Hilbert modular varieties. Preprint, 2007. Available online at http://arxiv.org/abs/0706.3904. To appear in Commentarii Mathematici Helvetici. Cited on page: 1, 10

[9] C. Löh, R. Sauer. Degree theorems and Lipschitz simplicial volume for non-positively curved manifolds of finite volume. Preprint, 2007.
Available online at http://arxiv.org/abs/0710.1635 Cited on page: 1, 6, 8, 10

[10] C. Strohm (= C. Löh). *The Proportionality Principle of Simplicial Volume*. Diploma thesis, WWU Münster, 2004. Available online at: arXiv:math.AT/0504106. Cited on page: 4, 8

[11] W.P. Thurston. *Geometry and Topology of 3-Manifolds*. Lecture notes, Princeton, 1978. Available at http://www.msri.org/publications/books/gt3m. Cited on page: 3, 4, 6, 8