Hall polynomials and Hall algebras for posets of finite prinjective type

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Let $I = (I, \preceq)$ be a finite poset (partially ordered set) and let K be a finite field. We denote by KI the incidence K-algebra of the poset I and by $\mathrm{mod}(KI)$ the category of finite dimensional right KI-modules. Consider the full subcategory $\mathrm{prin}(KI)$ of the category $\mathrm{mod}(KI)$ consisting of prinjective KI-modules in the sense of the following definition. A KI-module X is said to be *prinjective* if there exists a short exact sequence

$$0 \to P_1 \to P_0 \to X \to 0,$$

where P_0, P_1 are projective KI-modules and P_1 is, in addition, semisimple. We call the poset I of *finite prinjective type* if there exist only finitely many isomorphism classes of indecomposable prinjective KI-modules. In this case the Auslander-Reiten quiver $\Gamma_{prin}(I)$ of the category $\mathrm{prin}(KI)$ is directed and does not depend on the field K. Given a vertex $x \in \Gamma_{prin(I)}$, we denote by $M(x)$ the corresponding indecomposable prinjective KI-module. Similarly, given a map $a : \Gamma_{prin}(I)_0 \to \mathbb{N}$, we denote by $M(a)$ the prinjective KI-module $M(a) = \bigoplus_x M(x)^{a(x)}$. For KI-modules X, Y, Z, by F^Y_{ZX} we denote the number of submodules $U \subseteq Y$ such that $U \simeq X$ and $Y/U \simeq Z$.

The main aim of our talk is to present an idea of the proof of the following theorem.

THEOREM. *Let I be a poset of finite prinjective type and let*

$$a, b, c : \Gamma_{prin}(I)_0 \to \mathbb{N}$$

be maps. There exist polynomials $\varphi^b_{ca} \in \mathbb{Z}[T]$ such that for any finite field K:

$$\varphi^b_{ca}(|K|) = F^{M(b)}_{M(c)M(a)}.$$

Polynomials φ^b_{ca} are called *Hall polynomials*. Moreover we give some consequences of the existence of Hall polynomials and results concerning generators and relations in corresponding Hall algebras.