AUTOMORPHISM GROUPS OF COXETER GROUPS DO NOT HAVE KAZHDAN'S PROPERTY (T)

OLGA VARGHESE

Abstract. We show that for a large class $\mathcal{W}$ of Coxeter groups the following holds: Given a group W_Γ in $\mathcal{W}$, the automorphism group $\mathrm{Aut}(W_\Gamma)$ virtually surjects onto W_Γ. In particular, the group $\mathrm{Aut}(G_\Gamma)$ is virtually indicable and therefore does not satisfy Kazhdan's property (T). Moreover, if W_Γ is not virtually abelian, then the group $\mathrm{Aut}(W_\Gamma)$ is large.

1. Introduction

One fascinating property of a group is property (T). It was defined by Kazhdan for topological groups in terms of unitary representations and was reformulated by Delorme and Guichardet in geometric group theory. A countable group G has Kazhdan's property (T) if every action of G on a real Hilbert space by isometries has a global fixed point ([3, Theorem 2.12.4]). Examples of groups satisfying this property are finite groups ([3, Proposition 1.1.5]), the general linear groups $\mathrm{GL}_n(\mathbb{Z})$ for $n \geq 3$ ([3, Theorem 4.2.5]), the automorphism groups of free groups $\mathrm{Aut}(F_n)$ for $n \geq 5$ ([17], [18]). We investigate the following question:

Under which conditions on the group G does the automorphism group $\mathrm{Aut}(G)$ *not satisfy Kazhdan's property (T)?*

An interesting class of groups is the class consisting of graph products. Given a finite simplicial graph $\Gamma = (V, E)$ and a collection of groups $\mathcal{G} = \{G_u \mid u \in V\}$, the *graph product* G_Γ is defined as the quotient $\left(\underset{u\in V}{*} G_u\right)/\langle\langle [g,h] = 1,\ g \in G_u, h \in G_v, \{u,v\} \in E\rangle\rangle$. There are some results in the literature regarding property (T) of automorphism groups of graph products of finite groups in [11], of infinite cyclic groups in [1], [3], [5], [13] and of arbitrary vertex-groups in [10].

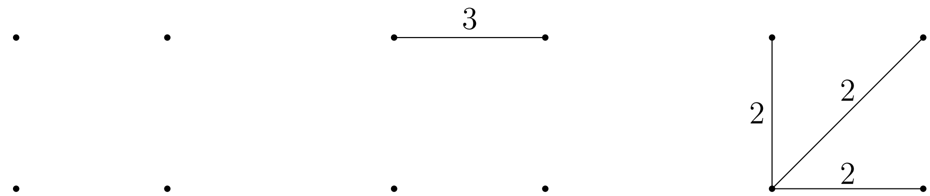
Date: March 9, 2020.

2010 *Mathematics Subject Classification.* Primary: 20F55; secondary: 20F65.

Key words and phrases. Kazhdan's property (T), automorphism groups, Coxeter groups.

The work was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC 2044–390685587, Mathematics Münster: Dynamics-Geometry-Structure.

Here we focus on groups which are also defined in combinatorial way. Given a finite simplicial graph $\Gamma = (V, E)$ with an edge-labelling $\varphi : E \to \mathbb{N}_{\geq 2}$, the Coxeter group W_Γ associated to Γ is the group with the presentation $W_\Gamma = \langle V \mid v^2, (vw)^{\varphi(\{v,w\})} \text{ for all } v \in V \text{ and } \{v,w\} \in E \rangle$. If Γ is disconnected with connected components $\Gamma_1, \ldots, \Gamma_n$, then



$W_\Gamma = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ $W_\Gamma = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathrm{Sym}(3)$ $W_\Gamma = \mathbb{Z}_2 \times (\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2)$

W_Γ is the free product $W_{\Gamma_1} * \ldots * W_{\Gamma_n}$ and if Γ is a join $\Gamma = \Gamma_1 * \Gamma_2$ and $\varphi(\{v,w\}) = 2$ for all $v \in V_1$ and $w \in V_2$, then W_Γ is the direct product $W_{\Gamma_1} \times W_{\Gamma_2}$. Coxeter groups are fundamental, well understood objects in geometric group theory, but there are many open questions concerning their automorphism groups.

The emphasis of this article is mainly on the fixed point property (T) of automorphism groups of Coxeter groups. We address the following conjecture:

Conjecture. *The automorphism group of an infinite Coxeter group does not have Kazhdan's property (T).*

Results regarding the above conjecture were proved in [11, Corollary 1.3] where it is shown that the automorphism group of an infinite right-angled Coxeter group does not have Kazhdan's property (T). Combining the fact that Kazhdan's property (T) is preserved by taking finite index subgroups with the facts that an infinite Coxeter group does not satisfy property (T) ([4]) and has finite center [14] one obtain: If W_Γ is infinite and $\mathrm{Out}(W_\Gamma)$ is finite, then $\mathrm{Aut}(W_\Gamma)$ does not have Kazhdan's property (T). Examples of Coxeter groups with finite outer automorphism group are Coxeter groups W_Γ where Γ is a complete graph ([15]), affine Coxeter groups ([8, Proposition 4.5]) and hyperbolic Coxeter groups in the sence of Humphreys [16] ([8, Theorem 4.10]).

Our goal here is to verify the above conjecture for a large class of Coxeter groups.

Theorem A. *Let W_Γ be an infinite Coxeter group. If Γ has a maximal complete subgraph Δ such that the center of W_Δ is trivial, then the automorphism group $\mathrm{Aut}(W_\Gamma)$ virtually surjects onto W_Γ. In particular, $\mathrm{Aut}(W_\Gamma)$ does not have Kazhdan's property (T).*

The next large class of Coxeter groups on which we want to focus is the class consisting of *even* Coxeter groups. This class of groups is known as a generalization of right-angled Coxeter groups. A Coxeter group W_Γ is called *even*, if all edge labels are even. A vertex $v \in V$ is called *even*, if all edge labels of $e \in E$ with $v \in e$ are even.

Theorem B. *Let W_Γ be a Coxeter group. If there exist two non-adjacent even vertices $v, w \in V$, then the automorphism group $\mathrm{Aut}(W_\Gamma)$ virtually surjects onto $\mathbb{Z}_2 * \mathbb{Z}_2$. In particular, $\mathrm{Aut}(W_\Gamma)$ does not have Kazhdan's property (T).*

Combining the results of Theorem A and B we obtain:

Corollary C. *Let W_Γ be an infinite Coxeter group.*

(i) *If Γ has no cycle of length 3, then $\mathrm{Aut}(W_\Gamma)$ does not have property (T).*
(ii) *If all edge labels are even, then $\mathrm{Aut}(W_\Gamma)$ does not have property (T).*
(iii) *If all edge labels are at least 3, then $\mathrm{Aut}(W_\Gamma)$ does not have property (T).*

A group G is said to be *virtually indicable* if there exists a finite index subgroup $H \subseteq G$ such that H surjects onto $\mathbb{Z}$. It was proven in [12] that an infinite Coxeter group is virtually indicable. The following corollary is a direct consequence of Theorems A and B.

Corollary D. *Let W_Γ be an infinite Coxeter group. If Γ has a maximal complete subgraph Δ such that the center of W_Δ is trivial or if there exist two non-adjacent even vertices $v, w \in V$, then $\mathrm{Aut}(W_\Gamma)$ is virtually indicable.*

Next, recall that a group G is called *large*, if it virtually surjects onto F_2. It is known that any Coxeter group is either virtually abelian or large ([6, Theorem 14.1.2]). Moreover, these properties can be verified easily on the Coxeter graph ([6, Proposition 17.2.1]). Concerning largeness of automorphism groups of Coxeter groups, it was proven in [21] that the outer automorphism group of a right-angled Coxeter group is either large or virtually abelian. We obtain

Corollary E. *Let W_Γ be a large Coxeter group. If Γ has a maximal complete subgraph Δ such that the center of W_Δ is trivial, then the automorphism group $\mathrm{Aut}(W_\Gamma)$ is large.*

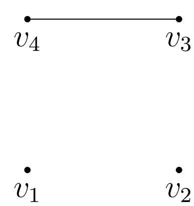
Acknowledgment. The author thanks Anthony Genevois for giving many constructive comments which helped improving the quality of the manuscript.

2. Graphs and groups

2.1. **Graphs.** In this section we define simplicial graphs and collect some important properties of these combinatorial objects which we will need to define the groups of our interest. We largely following [7].

A *simplicial graph* $\Gamma = (V, E)$ consists of a set V of *vertices* and a set E of subsets of V of cardinality two which are called *edges*. A graph Γ is said to be *finite* if V is a finite set. Two vertices $v, w \in V$ are called adjacent, if $\{v, w\}$ is an edge of

Γ. In practice, a graph $\Gamma = (V, E)$ is often represented by a diagram. We draw a point for each vertex $v \in V$ of the graph, and a line joining two vertices v and w if $\{v, w\} \in E$. If all the vertices of Γ are pairwise adjacent, then Γ is called *complete*. A graph $\Lambda = (X, Y)$ is a subgraph of Γ if $X \subseteq V$ and $Y \subseteq E$. A *(maximal) clique* in Γ is a (maximal) complete subgraph of Γ. Given a subset $S \subseteq V$, the *graph generated by* S, denoted by $\langle S \rangle$ is the graph with vertex set S and edge set $\{\{v, w\} \in E \mid v, w \in S\}$. A subgraph $\Lambda = (X, Y)$ of Γ is called *full* if $\langle X \rangle = \Lambda$. A graph Γ is called *connected* if for every two vertices $v, w \in V$ there exist vertices $v_1, \ldots, v_n \in V$ such that $\{v, v_1\}, \{v_i, v_{i+1}\}, \{v_n, w\} \in E$ for $i = 1, \ldots, n-1$. A graph $P = (\{v_1, \ldots, v_n\}, \{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\})$ is called a *path from* v_1 *to* v_n. Let us discuss some properties of the following graph in detail.



$$\Gamma = (\{v_1, v_2, v_3, v_4\}, \{\{v_3, v_4\}\})$$

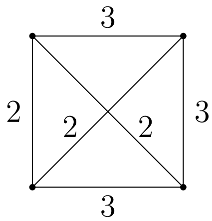
The subgraphs $\Delta_1 = (\{v_1\}, \varnothing)$ and $\Delta_2 = (\{v_3, v_4\}, \{\{v_3, v_4\}\})$ are cliques in Γ and Δ_2 is a maximal clique. The graph $\Lambda = (\{v_3, v_4\}, \varnothing)$ is a subgraph of Γ, but Λ is not a full subgraph, because v_3 and v_4 are adjacent in Γ but not in Λ. For a set $S = \{v_1, v_3, v_4\}$ the graph $\langle S \rangle$ is equal to $(\{v_1, v_3, v_4\}, \{\{v_3, v_4\}\})$.

A special class of graphs which will be important here is the class consisting of trees. A graph Γ is called a *simplicial tree*, if Γ is not empty, connected and has no full cycles. A cycle is a graph $C = (X, Y)$ where $X = \{x_1, \ldots, x_n\}$ with $n \geq 3$ and $Y = \{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n-1}, x_n\}, \{x_n, x_1\}\}$.

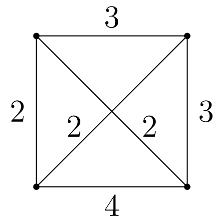
2.2. **Coxeter groups.** In this section we recall the basic definitions and notations concerning Coxeter groups. For further details we refer to [6].

Definition 2.1. Let $\Gamma = (V, E)$ be a finite simplicial graph with an edge-labelling $\varphi : E \to \mathbb{N}_{\geq 2}$. We call such a labelled graph a *Coxeter graph.* The Coxeter group W_Γ is defined as follows: $W_\Gamma = \langle V \mid v^2, (vw)^{\varphi(\{v,w\})}$ for all $v \in V$ and $\{v, w\} \in E\rangle$.

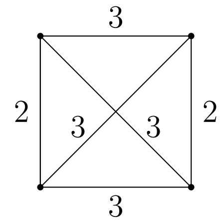
Let us observe the following examples of Coxeter graphs.



$W_{\Gamma_1} = \mathrm{Sym}(5)$



$W_{\Gamma_2} = \mathbb{Z}_2^4 \rtimes \mathrm{Sym}(4)$



$W_{\Gamma_3} = \mathbb{Z}^3 \rtimes \mathrm{Sym}(4)$

The next proposition shows that each subset $X \subseteq V$ generates a Coxeter group. For a proof we refer to [6, Theorem 4.1.6].

Proposition 2.2. *Let $\Gamma = (V, E)$ be a Coxeter graph. For each $X \subseteq V$, the subgroup which is generated by X is canonically isomorph to the Coxeter group W_Λ where Λ is the subgraph of Γ generated by X. For a full subgraph $\Lambda \subseteq \Gamma$, the subgroup W_Λ is called special.*

In the above example, the groups $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathrm{Sym}(3)$ and $\mathrm{Sym}(4)$ are special subgroups of W_{Γ_1}.

We are now going to recall the centralizer and normalizer of a subgroup H in a group G. The *centralizer* of H in G, denoted by $Z_G(H)$, is defined as follows: $Z_G(H) := \{g \in G \mid gh = hg \text{ for all } h \in H\}$. The *center* of G is the group $Z(G) := Z_G(G)$. The normalizer of a subgroup H in G is the following group $\mathrm{Nor}(H) := \{g \in G \mid gHg^{-1} = H\}$. In particular, we have $Z_G(H) \subseteq \mathrm{Nor}(H)$.

We will need the following facts about special subgroups of Coxeter groups which are direct consequences of [19, Corollary 13]).

Proposition 2.3.

(i) *Let W_Γ be a Coxeter group and let Δ and Λ be maximal cliques in Γ. If the special subgroups W_Δ and W_Λ are conjugate, then $\Delta = \Lambda$.*
(ii) *Let W_Γ be a Coxeter group and let Δ be a maximal clique in Γ. The normalizer of W_Δ in W_Γ is equal to the special subgroup W_Δ. In particular, $Z_{W_\Gamma}(W_\Delta) = Z(W_\Delta)$.*

Let us define some important classes of Coxeter groups.

Definition 2.4. Let $\Gamma = (V, E)$ be a Coxeter graph with an edge labelling $\varphi : E \to \mathbb{N}_{\geq 2}$.

(i) The Coxeter group W_Γ is called *right-angled* if $\varphi(E) = \{2\}$.
(ii) The Coxeter group W_Γ is called *even* if $\varphi(e)$ is even for all $e \in E$.
(iii) A vertex $v \in V$ is called even, if all edges $e \in E$ with $v \in e$ are even.

It is well known that for a right-angled Coxeter group W_Γ and a special subgroup W_Λ there exists a retraction homomorphism $r : W_\Gamma \twoheadrightarrow W_\Lambda$ such that $r(v) = v$ for all $v \in \Lambda$ and $r(v) = 1$ otherwise. This kind of retraction map exists also for other Coxeter groups. For the proof of Theorem B we will need the following result which is a special case of [9, Proposition 2.1].

Proposition 2.5. *Let $\Gamma = (V, E)$ be a Coxeter graph. If there exist two non-adjacent even vertices $v, w \in V$, then*

(i) *there exists a well-defined homomorphism $r : W_\Gamma \twoheadrightarrow \langle v \rangle * \langle w \rangle$ such that $r(v) = v$, $r(w) = w$ and $r(x) = 1$ for all $x \in V - \{v, w\}$.*
(ii) *the kernel of r is the normal closure of the special subgroup $W_{\langle V - \{v,w\} \rangle}$.*

3. Fixed point properties

In this section we introduce Serre's fixed point property F$\mathcal{A}$ and Kazhdan's property (T) and provide some basic observations and examples concerning these fixed point properties. Let us start with the following definition.

Definition 3.1.

(i) Let $\mathcal{A}$ be the class of simplicial trees. A group G is said to have *Serre's property* F$\mathcal{A}$ if any simplicial action, without inversions, of G on any member of $\mathcal{A}$ has a fixed point. A group G is called an F$\mathcal{A}$ *group* if G has property F$\mathcal{A}$.
(ii) Let $\mathcal{H}$ be the class of real Hilbert spaces. A countable group G is said to have *Kazhdan's property (T)* if any action of G by isometries on any member of $\mathcal{H}$ has a fixed point.

We make the following useful remarks on the fixed point properties F$\mathcal{A}$ and (T).

Remarks 3.2.

(i) A Coxeter group W_Γ is an F$\mathcal{A}$ group if and only if Γ is a complete graph ([22, I §6.1 Theorem 5, §6.5 Corollary 2]).
(ii) Maximal F$\mathcal{A}$ subgroups of a Coxeter group W_Γ are exactly conjugates of special subgroups W_Δ where Δ is a maximal clique in Γ ([19, Theorem 26]).
    In particular, a Coxeter group W_Γ has only finitely many conjugacy classes of maximal F$\mathcal{A}$ subgroups.
(iii) Infinite Coxeter groups do not have Kazhdan's property (T)([4]).
(iv) Property (T) is preserved under taking finite index subgroups ([3, Proposition 2.5.7]) and is inherited by quotients ([3, Proposition 2.5.1]).

4. Proofs of main theorems

Let us start with a puzzle piece of the proof of Theorem A.

Proposition 4.1. *Let Γ be a Coxeter graph and $\Delta_1, \dots, \Delta_n$ their maximal cliques. Define $[W_{\Delta_i}] := \{wW_{\Delta_i}w^{-1} \mid w \in W_\Gamma\}$ for $i = 1, \dots, n$ and $\mathcal{C} := \{[W_{\Delta_1}], \dots, [W_{\Delta_n}]\}$. Then the map*

$$\Phi : \mathrm{Aut}(W_\Gamma) \to \mathrm{Sym}(\mathcal{C})$$
$$f \mapsto \Phi(f) : [W_{\Delta_i}] \mapsto [f(W_{\Delta_i})]$$

is a well-defined homomorphism.

Proof. First of all note that for W_{Δ_i} the group $f(W_{\Delta_i}) \subseteq W_\Gamma$ is a maximal F$\mathcal{A}$ subgroup of W_Γ, thus there exists a unique $j \in \{1, \dots, n\}$ such that $f(W_{\Delta_i}) \in [W_{\Delta_j}]$, see Remarks 3.2(iii) and Proposition 2.3(i).

Now we have to check the injectivity of $\Phi(f)$. Suppose that $\Phi(f)([W_{\Delta_i}]) = \Phi(f)([W_{\Delta_j}])$. Then there exist elements $a, b \in W_\Gamma$ and a unique $k \in \{1, \dots, n\}$ such that $f(W_{\Delta_i}) = aW_{\Delta_k}a^{-1}$ and $f(W_{\Delta_j}) = bW_{\Delta_k}b^{-1}$. Thus

$$W_{\Delta_i} = f^{-1}(a)f^{-1}(W_{\Delta_k})f^{-1}(a^{-1}) = f^{-1}(a)f^{-1}(b^{-1})W_{\Delta_j}f^{-1}(b)f^{-1}(a^{-1})$$

and by Proposition 2.3(i) we get the equality $W_{\Delta_i} = W_{\Delta_j}$. Since the set $\mathcal{C}$ is finite, the map $\Phi(f)$ is a bijection. So far we have verified that the constructed map Φ is well-defined. In fact, the map Φ is a homomorphism. □

Theorem A. *Let W_Γ be an infinite Coxeter group. If Γ has a maximal complete subgraph Δ such that the center of W_Δ is trivial, then the automorphism group $\mathrm{Aut}(W_\Gamma)$ virtually surjects onto W_Γ. In particular, $\mathrm{Aut}(W_\Gamma)$ does not have Kazhdan's property (T).*

Proof. Let W_Γ be an infinite Coxeter group. By construction the kernel of the homomorphism Φ defined in Proposition 4.1 has finite index in $\mathrm{Aut}(W_\Gamma)$. By our assumption, the Coxeter graph Γ has a maximal clique Δ in Γ such that the center of the special subgroup W_Δ is trivial. We define a map $\Psi : \ker(\Phi) \to \mathrm{Out}(W_\Delta)$ in the following way: For an automorphism f in $\ker(\Phi)$ there exists $a \in W_\Gamma$ such that $f(W_\Delta) = aW_\Delta a^{-1}$. We set $\Psi(f) = [c_{a^{-1}} \circ f]$ where $c_{a^{-1}} : W_\Gamma \to W_\Gamma, w \mapsto a^{-1}wa$ for all $w \in W_\Gamma$. First of all we have to confirm that this map is well-defined. Let $b \in W_\Gamma$ be an another element such that $f(W_\Delta) = bW_\Delta b^{-1}$. We claim that $f^{-1} \circ c_a \circ c_{b^{-1}} \circ f \in \mathrm{Inn}(W_\Delta)$. It is easy to verify that $f^{-1} \circ c_a \circ c_{b^{-1}} \circ f \in \mathrm{Aut}(W_\Delta)$. For $x \in W_\Delta$ we have $f^{-1} \circ c_a \circ c_{b^{-1}} \circ f(x) = f^{-1}(ab^{-1})xf^{-1}(ba^{-1})$. The special subgroup W_Δ is by Proposition 2.3 self-normalizing, hence the element $f^{-1}(ab^{-1}) \in W_\Delta$ and $f^{-1} \circ c_a \circ c_{b^{-1}} \circ f \in \mathrm{Inn}(W_\Delta)$. It remains to show that Ψ is a homomorphism. Let f and g be two automorphisms in $\ker(\Phi)$ and we fix $x, y \in W_\Gamma$ such that $f(W_\Delta) = xW_\Delta x^{-1}$ and $g(W_\Delta) = yW_\Delta y^{-1}$. Thus $g \circ f(W_\Delta) = g(xW_\Delta x^{-1}) = g(x)yW_\Delta y^{-1}g(x^{-1})$. We have to check the equality: $\Psi(g \circ f) = \Psi(g) \cdot \Psi(f)$ which is equivalent to $[c_{y^{-1}g(x)^{-1}} \circ g \circ f] = [c_{y^{-1}} \circ g \circ c_{x^{-1}} \circ f]$.

For $x \in W_\Delta$ we have:

$$f^{-1} \circ c_x \circ g^{-1} \circ c_y \circ c_{y^{-1}g(x)^{-1}} \circ g \circ f(x) = f^{-1} \circ c_x \circ g^{-1}(g(x^{-1})g(f(x))g(x))$$
$$= f^{-1} \circ c_x(x^{-1}f(x)x) = x$$

Hence $[c_{y^{-1}g(x)^{-1}} \circ g \circ f] = [c_{y^{-1}} \circ g \circ c_{x^{-1}} \circ f]$.

It was proven in [15] that the group $\mathrm{Out}(W_\Delta)$ is finite, thus the kernel of Ψ has finite index in $\ker(\Phi)$ and in $\mathrm{Aut}(W_\Gamma)$. We claim that $\ker(\Psi)$ surjects onto W_Γ. We define a map $\Xi : \ker(\Psi) \to W_\Gamma$ in the following way: For $f \in \ker(\Psi)$ there exists $a \in W_\Gamma$ such that $f(w) = awa^{-1}$ for all $w \in W_\Delta$. We set: $\Xi(f) = a$. The map Ξ is well-defined since the element a is unique with the above property. More precisely, assume that there exists $b \in W_\Gamma$ such that $f(w) = bwb^{-1}$ for all $w \in W_\Delta$. Hence $a^{-1}b \in Z_{W_\Gamma}(W_\Delta) = Z(W_\Delta) = \{1\}$ and therefore $a = b$. In fact, the map Ξ is a homomorphism. Moreover, Ξ is an epimorphism since $W_\Gamma \cong \mathrm{Inn}(W_\Gamma) \subseteq \ker(\Psi)$. Thus, we have proved that $\mathrm{Aut}(W_\Gamma)$ has a finite index subgroup which maps onto W_Γ. Since W_Γ does not satisfy property (T), the group $\mathrm{Aut}(W_\Gamma)$ does not have property (T) neither. □

Theorem B. *Let W_Γ be a Coxeter group. If there exist two non-adjacent even vertices $v, w \in V$, then the automorphism group $\mathrm{Aut}(W_\Gamma)$ virtually surjects onto $\mathbb{Z}_2 * \mathbb{Z}_2$. In particular, $\mathrm{Aut}(W_\Gamma)$ does not have Kazhdan's property (T).*

Proof. It was proven in [20] that the cardinality of conjugacy classes of subgroups of order 2 in W_Γ is finite. The group $\mathrm{Aut}(W_\Gamma)$ acts on these conjugacy classes in the canonical way. The kernel of this action is called the group of special automorphisms and will be denoted by $\mathrm{Spe}(W_\Gamma)$. Note that $\mathrm{Spe}(W_\Gamma)$ has finite index in $\mathrm{Aut}(W_\Gamma)$. We claim that $\mathrm{Spe}(W_\Gamma)$ surjects onto $\mathbb{Z}_2 * \mathbb{Z}_2$. By assumption, there exist two even vertices $v, w \in V$ which are not connected by an edge. Let $r : W_\Gamma \twoheadrightarrow \langle v \rangle * \langle w \rangle$ be the retraction map which exists by Proposition 2.5. The kernel of r is by Proposition 2.5 the normal closure of the special subgroup $W_{V-\{v,w\}}$. Since the elements in $V - \{v, w\}$ can not be conjugate to even vertices ([2, p. 65]), this kernel is characteristic under $\mathrm{Spe}(W_\Gamma)$. Hence we get the following canonical homomorphism $\Psi : \ker(\Phi) \to \mathrm{Aut}(\mathbb{Z}_2 * \mathbb{Z}_2)$. Indeed, the image of Ψ is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$. In particular, the group $\mathrm{Aut}(W_\Gamma)$ does not have property (T). □

Theorem B implies the following

Corollary 4.2. *Let W_Γ be an infinite Coxeter group. If W_Γ is an even Coxeter group, then $\mathrm{Aut}(W_\Gamma)$ does not have property (T).*

It is well known that the center of the Coxeter group W_Γ, where $\Gamma = (\{v, w\}, \{\{v, w\}\})$ and the edge-label is odd, is trivial. Thus, applying the results of Theorems A and B we obtain:

Corollary 4.3. *Let W_Γ be an infinite Coxeter group. If Γ has no cycle of length 3, then $\mathrm{Aut}(W_\Gamma)$ does not have property (T).*

A characterization of finite irreducible Coxeter groups in terms of Dynkin diagrams is given in [16, §2]. Hence, if a Coxeter group W_Γ is finite or not can be checked easily on the Coxeter graph Γ. In particular, it follows from this characterization that if Γ has at least 3 vertices and all edge labels are at least 3, then W_Γ is an infinite irreducible Coxeter group. Recall, a Coxeter group W_Γ is called *irreducible* if the vertex set of Γ cannot be partitioned into two non-empty disjoint subsets V_1 and V_2 such that $\{x,y\} \in E$ for all $x \in V_1$, $y \in V_2$ and $\varphi(\{x,y\}) = 2$ for all $x \in V_1$ and $y \in V_2$. Combining the fact that an irreducible infinite Coxeter group has trivial center ([6, Theorem D.2.10]) with the result of Theorem A we obtain:

Corollary 4.4. *Let W_Γ be an infinite Coxeter group. If all edge labels are at least 3, then $\mathrm{Aut}(W_\Gamma)$ does not have property (T).*

We finish this article with the following remark.

Remark 4.5. Let W_Γ be a Coxeter group. Then there exist irreducible special subgroups $W_{\Gamma_1}, \ldots, W_{\Gamma_n}$ such that $W_\Gamma = W_{\Gamma_1} \times \ldots \times W_{\Gamma_n}$. Hence, the center of W_Γ is equal to $Z(W_{\Gamma_1}) \times \ldots \times Z(W_{\Gamma_n})$. We define $J := \{i \in \{1, \ldots, n\} \mid W_{\Gamma_i} \text{ is finite}\}$. Since the center of an irreducible infinite Coxeter group is trivial we obtain $Z(W_\Gamma) = \prod_{j \in J} Z(W_{\Gamma_j})$. Hence, it can be checked easily on the graph Γ if there exist a maximal clique Δ such that the center of W_Δ is trivial.

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Olga Varghese, Department of Mathematics, University of Münster, Einsteinstrasse 62, 48149 Münster (Germany)

E-mail address: `olga.varghese@uni-muenster.de`