

A Mini-Course on Formal and Rigid Geometry

Siegfried Bosch (Münster)

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- 1 Introduction
 - Naive approach to non-Archimedean analytic functions
 - Motivation for a more sophisticated theory
- 2 Classical rigid spaces à la Tate, Grauert, Remmert, Kiehl, ...
- 3 Rigid spaces via Raynaud's formal schemes

Non-Archimedean Fields

Definition

An *absolute value* on a field K is a map $|\cdot|: K \rightarrow \mathbb{R}_{\geq 0}$ such that

- $|\alpha| = 0 \iff \alpha = 0$
- $|\alpha\beta| = |\alpha||\beta|$
- $|\alpha + \beta| \leq |\alpha| + |\beta|$

$|\cdot|$ is called *non-Archimedean* if

- $|\alpha + \beta| \leq \max\{|\alpha|, |\beta|\}$

Fundamental examples: $\mathbb{Q}_p, \mathbb{C}_p$ for $p \in \mathbb{N}$ prime

$$|\cdot|_p: \mathbb{Q} \rightarrow \mathbb{R}_{\geq 0}, \quad |\alpha|_p := \begin{cases} 0 & \text{if } \alpha = 0 \\ \frac{1}{p^n} & \text{if } \alpha = \frac{a}{b}p^n, \ a, b \in \mathbb{Z}, \ p \nmid ab \end{cases}$$

\mathbb{Q}_p = completion of \mathbb{Q} under $|\cdot|_p$ (*p -adic analogue of \mathbb{R}*)

\mathbb{C}_p = completed algebraic closure of \mathbb{Q}_p (*p -adic analogue of \mathbb{C}*)

Non-Archimedean Calculus

K field with complete non-Archimedean absolute value

- $\sum_{\nu=0}^{\infty} \alpha_{\nu}$ converges in $K \iff \lim_{\nu \rightarrow \infty} |\alpha_{\nu}| = 0$
- $|\alpha + \beta| = \max\{|\alpha|, |\beta|\}$ if $|\alpha| \neq |\beta|$.

Proof of 2nd assertion

$$\begin{aligned} |\beta| < |\alpha| \implies |\alpha + \beta| &\leq \max\{|\alpha|, |\beta|\} = |\alpha| = |\alpha + \beta - \beta| \\ &\leq \max\{|\alpha + \beta|, |\beta|\} = |\alpha + \beta| \end{aligned}$$

Geometric implications

- Triangles in K are isosceles
- $D_1, D_2 \subset K$ disks, $D_1 \cap D_2 \neq \emptyset \implies D_1 \subset D_2$ or $D_2 \subset D_1$
- Disk $D = \{x; |x - a| < r\}$ is open **and** closed
- K is totally disconnected

Locally Analytic Functions

Problems

- have no non-trivial continuous paths
- have no line integrals in the usual sense
- have no identity theorems

Example

$$f: K \longrightarrow K, \quad f(x) = \begin{cases} 0 & \text{if } |x| < 1 \\ 1 & \text{if } |x| \geq 1 \end{cases}$$

is locally analytic on K , identically zero in neighborhood of 0,
but is not identically zero globally

Conclusion

Notion of local analyticity is rather weak,
cannot derive meaningful global properties

Motivating Example (J. Tate, 1959)

$K, |\cdot|$ algebraically closed field, complete, non-Archimedean

$$\mathcal{O}(K^*) := \left\{ \sum_{\nu \in \mathbb{Z}} c_\nu \zeta^\nu ; \lim_{|\nu| \rightarrow \infty} |c_\nu| r^\nu = 0 \text{ for all } r > 0 \right\}$$

globally convergent Laurent series on K^*

$$\mathcal{M}(K^*) := \text{Frac}(\mathcal{O}(K^*))$$

field of “meromorphic” functions on K^*

let $q \in K, 0 < |q| < 1$

$$\mathcal{M}^q(K^*) := \{ f \in \mathcal{M}(K^*) ; f(q\zeta) = f(\zeta) \}$$

field of q -periodic meromorphic functions on K^*

Observation

$\mathcal{M}^q(K^*)$ is an elliptic function field with non-integral j -invariant;
i. e. $|j| > 1$. Points of associated Tate elliptic curve E_K :

$$K^*/q^{\mathbb{Z}} \simeq E_K(K) \hookrightarrow \mathbb{P}^2(K)$$

But: Quotient $K^*/q^{\mathbb{Z}}$ does not make sense in algebraic geometry

Comparison with complex elliptic curves

$\Gamma = \mathbb{Z} \oplus \mathbb{Z}\omega$ a lattice in \mathbb{C} with $\omega \notin \mathbb{R}$

Weierstraß \wp -function defines an isomorphism $\mathbb{C}/\Gamma \xrightarrow{\sim} E \subset \mathbb{P}^2(\mathbb{C})$
onto an elliptic curve E in \mathbb{P}^2

use exponential map $\mathbb{C} \rightarrow \mathbb{C}^*$, $z \mapsto e^{2\pi iz}$, get isomorphism
 $\mathbb{C}/\mathbb{Z} \xrightarrow{\sim} \mathbb{C}^*$, thus

$$\mathbb{C}^*/q^{\mathbb{Z}} \simeq \mathbb{C}/\Gamma \simeq E \subset \mathbb{P}^2(\mathbb{C}), \quad q = e^{2\pi i\omega}$$

uniformization of E from multiplicative point of view

Program of Tate

Develop theory of analytic functions over non-Archimedean fields
such that $K^*/q^{\mathbb{Z}}$ makes sense as a Riemann surface, which can be
algebraized to yield an elliptic curve over K

- 1 Introduction
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 - Affinoid algebras
 - Localization of affinoid algebras
 - Affinoid spaces and their subdomains
 - Tate's Acyclicity Theorem
 - Grothendieck topologies
 - Sheaves
 - Construction of global rigid spaces
 - Some advanced results
- 3 Rigid spaces via Raynaud's formal schemes

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Restricted power series

K complete, with non-Archimedean $|\cdot|$

\overline{K} alg. closure, with extension of $|\cdot|$, not necessarily complete
 $B^n(\overline{K}) = \{(x_1, \dots, x_n) \in \overline{K}^n; |x_i| \leq 1 \text{ for all } i\}$

a power series $\sum_{\nu} c_{\nu} \zeta^{\nu}$ with coefficients in K and variables $\zeta = (\zeta_1, \dots, \zeta_n)$ converges globally on $B^n(\overline{K})$ iff $\lim_{\nu} c_{\nu} = 0$

Definition

$T_n = K\langle \zeta_1, \dots, \zeta_n \rangle = \{\sum_{\nu} c_{\nu} \zeta^{\nu}; \lim_{\nu} c_{\nu} = 0\}$ is called the **Tate algebra of restricted power series** in the variables ζ_1, \dots, ζ_n

Properties

T_n is noetherian, factorial, jacobson.

Noether normalization: for any ideal $\mathfrak{a} \subsetneq T_n$ there is a finite monomorphism $T_d \hookrightarrow T_n/\mathfrak{a}$

Corollary to Noether normalization

$\mathfrak{m} \subset T_n$ maximal ideal $\implies [T_n/\mathfrak{m} : K] < \infty$

Proof

Choose $T_d \hookrightarrow T_n/\mathfrak{m}$ finite. Then:

$$T_n/\mathfrak{m} \text{ a field} \implies T_d \text{ a field,}$$

hence, $d = 0$, $T_d = K$, and $[T_n/\mathfrak{m} : K] < \infty$ □

Definition

$\text{Sp } T_n = \{\text{maximal ideals } \subset T_n\}$ is called the **max spectrum** of T_n

Evaluation homomorphisms

For $x = (x_1, \dots, x_n) \in B^n(\overline{K})$, have **evaluation homomorphism**

$$\begin{aligned} \text{eval}_x : T_n &\longrightarrow K(x_1, \dots, x_n) \hookrightarrow \overline{K}, \\ \sum_{\nu} c_{\nu} \zeta^{\nu} &\longmapsto \sum_{\nu} c_{\nu} x^{\nu} \end{aligned}$$

Note:

$K(x_1, \dots, x_n)$ is finite over K , hence **complete**

$\implies \text{eval}_x$ factors surjectively through $K(x_1, \dots, x_n)$,

$\implies \mathfrak{m}_x := \ker(\text{eval}_x)$ is a maximal ideal in T_n

Proposition

The map $B^n(\overline{K}) \longrightarrow \text{Sp } T_n$, $x \longmapsto \mathfrak{m}_x$, induces a bijection

$$B^n(\overline{K}) / \text{Aut}_K(\overline{K}) \xrightarrow{\sim} \text{Sp } T_n$$

T_n as a Banach algebra

View elements $f \in T_n$ as functions on $\text{Sp } T_n$

- for $x \in \text{Sp } T_n$ let $f(x)$ be residue class of f in T_n/\mathfrak{m}_x
- $f(x) = 0 \iff f \in \mathfrak{m}_x$
- $|f(x)|$ is well-defined

Norms on T_n

- **Gauß norm:** $|\sum_{\nu} c_{\nu} \zeta^{\nu}|_{\text{Gauß}} := \max |c_{\nu}|$
- **Supremum norm:** $|f|_{\text{sup}} = \sup_{x \in \text{Sp } T_n} |f(x)|$

Proposition

- $|f|_{\text{Gauß}} = |f|_{\text{sup}}$ for all $f \in T_n$;
- T_n is a Banach K -algebra under $|\cdot| = \text{Gauß or sup norm}$

Note: $|f|_{\text{Gauß}} \geq |f|_{\text{sup}}$ is obvious; for full proof need **reduction**

Method: Reduction of coefficients

$$\begin{aligned} R &:= \{\alpha \in K ; |\alpha| \leq 1\} && \text{valuation ring of } K \\ \mathfrak{m} &:= \{\alpha \in K ; |\alpha| < 1\} && \text{valuation ideal, maximal ideal in } R \\ k &:= R/\mathfrak{m} && \text{residue field} \end{aligned}$$

Canonical projection $R \rightarrow k, \alpha \mapsto \tilde{\alpha}$, induces a **reduction map**

$$\begin{array}{ccc} R\langle\zeta\rangle & \longrightarrow & k[\zeta], & f = \sum_{\nu} c_{\nu}\zeta^{\nu} \longmapsto \tilde{f} = \sum_{\nu} \tilde{c}_{\nu}\zeta^{\nu}, \\ \text{eval}_x \downarrow & & \text{eval}_{\tilde{x}} \downarrow & \\ \overline{R} & \longrightarrow & \overline{k} & \end{array}$$

compatible with evaluation at points $x \in B^n(\overline{K})$

Maximum Principle

Given $f \in T_n$, there is $x \in \text{Sp } T_n$ such that $|f|_{\text{Gau\ss}} = |f(x)|$

Proof of Maximum Principle

let $f \in T_n$, may assume $|f|_{\text{Gauß}} = 1$; then $f \in R\langle \zeta \rangle$ and $\tilde{f} \neq 0$

choose $\tilde{x} \in \bar{k}^n$ such that $\tilde{f}(\tilde{x}) \neq 0$, lift to $x \in B^n(\bar{K})$

get $\widetilde{f(x)} = \tilde{f}(\tilde{x}) \neq 0$, hence $|f(x)| = 1$

in particular, get $|f|_{\text{Gauß}} \leq |f|_{\text{sup}}$, hence, $|f|_{\text{Gauß}} = |f|_{\text{sup}}$

Properties of Gauß resp. sup norm $|\cdot|$

- $|f| = 0 \iff f = 0$
- $|fg| = |f||g|$
- $|f + g| \leq \max\{|f|, |g|\}$

Proposition (non-trivial)

$\mathfrak{a} \subset T_n$ ideal $\implies \mathfrak{a}$ is closed in T_n

Definition

Call a K -algebra A **affinoid** if there is an epimorphism $\pi: T_n \rightarrow A$; call $\text{Sp } A = \{\text{maximal ideals } \subset A\}$ an **affinoid K -space**

Properties

- $\text{Sp } A \xrightarrow{\sim} V(\mathfrak{a}) = \{x \in \text{Sp } T_n; \mathfrak{a} \subset \mathfrak{m}_x\}$ for $\mathfrak{a} = \ker \pi$
- Set $|f| = \inf\{|g|; g \in \pi^{-1}(f)\}$ for $f \in A$
- $|\cdot|$ is a so-called **residue norm** on A , satisfies
 - $|f| = 0 \iff f = 0$ (since \mathfrak{a} is closed in T_n)
 - $|fg| \leq |f||g|$
 - $|f + g| \leq \max\{|f|, |g|\}$
- A is complete with respect to residue norm $|\cdot|$
- also have sup norm $|\cdot|_{\text{sup}}$ on A , is **semi-norm**,
 - $|f|_{\text{sup}} = 0 \iff f$ is nilpotent
 - $|f|_{\text{sup}} \leq |f|$

Fact: Let $A \rightarrow B$ finite homomorphism of affinoid K -algebras, $f \in B$. Then exist $a_1, \dots, a_r \in A$ such that

$$f^r + a_1 f^{r-1} + \dots + a_r = 0, \quad |f|_{\text{sup}} = \max_i |a_i|_{\text{sup}}^{1/i}$$

Proposition (non-trivial)

Let $A, |\cdot|$ be an affinoid K -algebra with a residue norm, $f \in A$. Then:

- $|f|_{\text{sup}} \leq 1 \iff |f^n|$ for $n \in \mathbb{N}$ is bounded (f is **power bounded**)
- $|f|_{\text{sup}} < 1 \iff \lim_n |f^n| = 0$ (f is **topologically nilpotent**)
- $|\cdot|_{\text{sup}}$ equivalent to $|\cdot| \iff A$ reduced

Corollary

Let $\varphi: A \rightarrow B$ be a homomorphism of affinoid K -algebras; fix any residue norms on A and B . Then φ is **continuous**.
Hence, all residue norms on an affinoid K -algebra are equivalent.

Localization of affinoid K -algebras

Let A affinoid K -algebra, $f_0, \dots, f_r \in A$,
assume $V(f_0, \dots, f_r) = \emptyset$; i. e., f_0, \dots, f_r generate unit ideal in A

set $A\langle \frac{f}{f_0} \rangle = A\langle \frac{f_1}{f_0}, \dots, \frac{f_r}{f_0} \rangle = A\langle \zeta_1, \dots, \zeta_r \rangle / (f_i - f_0 \zeta_i; i = 1, \dots, r)$

Lemma

Canonical homomorphism $\varphi: A \longrightarrow A\langle \frac{f}{f_0} \rangle$ satisfies:

- $\varphi(f_0)$ is a unit
- $\frac{\varphi(f_i)}{\varphi(f_0)}$, $i = 1, \dots, r$, are power bounded

Furthermore, φ is **universal**: any $\varphi': A \longrightarrow B$ satisfying above conditions factors uniquely through $\varphi: A \longrightarrow A\langle \frac{f}{f_0} \rangle$

Remark

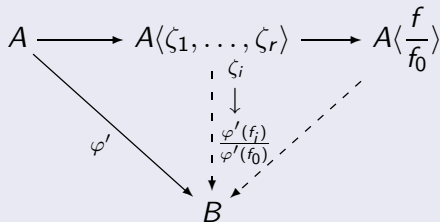
$A\langle \frac{f}{f_0} \rangle$ is (**non-canonical**) completion of localization $A[f_0^{-1}]$

Proof of localization Lemma

Let $x \in \text{Sp } A\langle \frac{f}{f_0} \rangle$, **assume** $f_0(x) = 0$ (drop φ in notation),
have $f_i = f_0 \bar{\zeta}_i$ in $A\langle \frac{f}{f_0} \rangle$, get $f_i(x) = 0$ for all i ,
hence, $(f_0, \dots, f_r) \neq (1)$, contradiction!
Thus: f_0 has no zero on $\text{Sp } A\langle \frac{f}{f_0} \rangle$, hence, f_0 is unit in $A\langle \frac{f}{f_0} \rangle$

have $\bar{\zeta}_i = \frac{f_i}{f_0}$ in $A\langle \frac{f}{f_0} \rangle$; hence, by continuity of $A\langle \zeta \rangle \longrightarrow A\langle \frac{f}{f_0} \rangle$:
all $\frac{f_i}{f_0}$ are power bounded in $A\langle \frac{f}{f_0} \rangle$

Universal property:



Examples of localization

$A = T_1 = K\langle\zeta\rangle$ restricted power series, $\pi \in K^*$

$$\begin{aligned} A\left\langle\frac{\zeta}{\pi}\right\rangle &= K\langle\zeta, \eta\rangle/(\zeta - \pi\eta) \\ &= \begin{cases} \{\sum_{\nu \in \mathbb{N}} c_\nu \zeta^\nu\} \text{ converging for } |\zeta| \leq |\pi| & \text{if } |\pi| \leq 1 \\ K\langle\zeta\rangle & \text{if } |\pi| \geq 1 \end{cases} \end{aligned}$$

$$\begin{aligned} A\left\langle\frac{\pi}{\zeta}\right\rangle &= K\langle\zeta, \eta\rangle/(\pi - \zeta\eta) \\ &= \begin{cases} \{\sum_{\nu \in \mathbb{Z}} c_\nu \zeta^\nu\} \text{ conv. for } |\pi| \leq |\zeta| \leq 1 & \text{if } |\pi| \leq 1 \\ 0 & \text{if } |\pi| > 1 \end{cases} \end{aligned}$$

Lemma

Any $\varphi: A \rightarrow B$ homomorphism of affinoid K -algebras induces a map

$${}^a\varphi: \mathrm{Sp} B \rightarrow \mathrm{Sp} A, \quad \mathfrak{m} \mapsto \varphi^{-1}(\mathfrak{m})$$

Proof

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow & & \downarrow \\ K \hookrightarrow A/\varphi^{-1}(\mathfrak{m}) & \hookrightarrow & B/\mathfrak{m} \end{array}$$

B/\mathfrak{m} is finite over K , hence
 B/\mathfrak{m} is integral over $A/\varphi^{-1}(\mathfrak{m})$;
 B/\mathfrak{m} field $\implies A/\varphi^{-1}(\mathfrak{m})$ field

Definition

${}^a\varphi: \mathrm{Sp} B \rightarrow \mathrm{Sp} A$ as in Lemma is called the **morphism of affinoid K -spaces** associated to $\varphi: A \rightarrow B$

Let A affinoid K -algebra, $f_0, \dots, f_r \in A$ without common zeros

Lemma

The canonical map $\varphi: A \rightarrow A\langle \frac{f}{f_0} \rangle$ induces a bijection

$${}^a\varphi: \mathrm{Sp} A\langle \frac{f}{f_0} \rangle \xrightarrow{\sim} U = \{x \in \mathrm{Sp} A; |f_1(x)|, \dots, |f_r(x)| \leq |f_0(x)|\}$$

Definition

A subset $U \subset \mathrm{Sp} A$ as in Lemma is called a **rational subdomain** of the affinoid K -space $\mathrm{Sp} A$

Consider $X = \mathrm{Sp} A$ together with its rational subdomains as a category, inclusions as morphisms, define a **presheaf** \mathcal{O}_X on it by

$$X \supset U \simeq \mathrm{Sp} A\langle \frac{f}{f_0} \rangle \rightsquigarrow A\langle \frac{f}{f_0} \rangle$$

restriction morphisms by universal property of localizations $A\langle \frac{f}{f_0} \rangle$

Tate's Acyclicity Theorem

Let $X = \text{Sp } A$ an affinoid K -space with presheaf \mathcal{O}_X
 $\mathfrak{U} = (U_i)_{i \in I}$ a **finite** covering of X by rational subdomains $U_i \subset X$

Theorem

\mathfrak{U} is \mathcal{O}_X -acyclic; i. e., the augmented Čech complex

$$0 \longrightarrow \mathcal{O}_X(X) \longrightarrow \prod_{i_0 \in I} \mathcal{O}_X(U_{i_0}) \longrightarrow \prod_{i_0, i_1 \in I} \mathcal{O}_X(U_{i_0} \cap U_{i_1}) \longrightarrow \dots$$

is exact

Corollary

\mathcal{O}_X is a **sheaf** in the following sense:

given $f_i \in \mathcal{O}_X(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j ,
there is a unique $f \in \mathcal{O}_X(X)$ such that $f|_{U_i} = f_i$ for all i

Proof of Acyclicity Theorem, basic case

$X = \mathbb{B}_K^1 = \text{Sp } K\langle\zeta\rangle$ unit disk, let $\pi \in K^*$, $|\pi| < 1$

$U_1 = \{x \in \mathbb{B}_K^1; |x| \geq |\pi|\}$, ($|x| := |\zeta(x)|$)

$U_2 = \{x \in \mathbb{B}_K^1; |x| \leq |\pi|\}$

$\mathcal{O}_X(U_1) = \{\sum_{\nu \in \mathbb{Z}} c_\nu \zeta^\nu; \text{converging on } |\pi| \leq |\zeta| \leq 1\}$

$\mathcal{O}_X(U_2) = \{\sum_{\nu \in \mathbb{N}} c_\nu \zeta^\nu; \text{converging on } |\zeta| \leq |\pi|\}$

$\mathcal{O}_X(U_1 \cap U_2) = \{\sum_{\nu \in \mathbb{Z}} c_\nu \zeta^\nu; \text{converging on } |\zeta| = |\pi|\}$

$$\begin{array}{ccccc} \mathcal{O}_X(U_1) & \hookrightarrow & \mathcal{O}_X(U_1 \cap U_2) & \hookleftarrow & \mathcal{O}_X(U_2) \\ \sum_{\nu \in \mathbb{Z}} a_\nu \zeta^\nu & \mapsto & \bullet & \longleftarrow & \sum_{\nu \in \mathbb{N}} b_\nu \zeta^\nu \end{array}$$

coincidence on $U_1 \cap U_2$ means $a_\nu = b_\nu$, $\nu \geq 0$ and $a_\nu = 0$, $\nu < 0$,
 $\sum_{\nu \in \mathbb{Z}} a_\nu \zeta^\nu$ converges on U_1 , e. g. at $x = 1$ (max. ideal is $(\zeta - 1)$)
 $\implies \lim |a_\nu| = 0 \implies \sum_{\nu \in \mathbb{N}} b_\nu \zeta^\nu \in K\langle\zeta\rangle = \mathcal{O}_X(\mathbb{B}_K^1)$

Implications from Tate's theorem

Strategy for constructing global analytic spaces

- glue affinoid K -spaces as local parts
- do this under the constraints of Tate's Acyclicity Theorem: glue with respect to **admissible open overlaps** (= rational subdomains in our case), respect **finite coverings** on affinoid pieces
- get so-called **rigid analytic spaces**

Technical tool: Grothendieck topology on a topological space X

- specify certain open sets $U \subset X$ as **admissible open sets**
- specify certain open coverings $U = \bigcup_{i \in I} U_i$, where U and all U_i are admissible open in X , as **admissible open coverings**
- require a minimal amount of compatibility conditions such that Čech cohomology can be done in the **admissible** setting

Definition: Grothendieck topology

A **Grothendieck topology** (short: **G-topology**) consists of

- a category $\text{Cat } T$
- for each object $U \in \text{Cat } T$ a set $\text{Cov } U$ of families $(U_i \xrightarrow{\phi_i} U)_{i \in I}$ of morphisms in $\text{Cat } T$, called **coverings**

such that:

- $\Phi: U' \rightarrow U$ isomorphism $\implies (\Phi) \in \text{Cov } U$
- **composition**: $(U_i \rightarrow U)_i \in \text{Cov } U$ and $(U_{ij} \rightarrow U_i)_j \in \text{Cov } U_i$ for all $i \implies (U_{ij} \rightarrow U_i \rightarrow U)_{ij} \in \text{Cov } U$
- **restriction**: $(U_i \rightarrow U)_i \in \text{Cov } U$ and $V \rightarrow U$ a morphism in $\text{Cat } T \implies (U_i \times_U V \rightarrow V)_i \in \text{Cov } V$; in particular, the fiber products $U_i \times_U V$ must exist in $\text{Cat } T$

Note, for a **G-topology** on a topological space X :

$\text{Cat } T =$ category of admissible open sets in X , with inclusions as morphisms; fiber products exist: $U_i \times_U V = U_i \cap V$

Topologies on an affinoid K -space $X = \mathrm{Sp} A$

Canonical topology

Topology generated by all **rational subdomains** $U \subset X$

Weak G -topology

- **admissible open sets**: all rational subdomains in X
- **admissible coverings**: all **finite** coverings by rational subdomains of X

Strong G -topology

- $U \subset X$ **admissible open** \iff exists covering $U = \bigcup_{i \in I} U_i$ by rational subdomains $U_i \subset X$ such that:
for any morphism of affinoid K -spaces $\varphi: Y \rightarrow X$ with $\varphi(Y) \subset U$ the covering $(\varphi^{-1}(U_i))_i$ of Y admits a finite refinement by rational subdomains of Y
- $U = \bigcup_{i \in I} U_i$ (with U, U_i admissible open) is **admissible**
 $\iff U = \bigcup_{i \in I} U_i$ satisfies same condition as before

Definition

Let X be a set with a G -topology. A **presheaf** (of rings) on X is a contravariant functor

$$\mathfrak{F}: (\text{admissible opens } \subset X) \longrightarrow (\text{rings})$$

It is called a **sheaf** if for any admissible covering $U = \bigcup_{i \in I} U_i$ the sequence

$$\mathfrak{F}(U) \longrightarrow \prod_{i \in I} \mathfrak{F}(U_i) \rightrightarrows \prod_{i, j \in I} \mathfrak{F}(U_i \cap U_j)$$

is exact

Remark

The strong G -topology on affinoid K -spaces X is a best possible refinement of the weak G -topology such that:

- morphisms of affinoid K -spaces remain **continuous**,
- any **sheaf** with respect to the weak G -topology on X extends uniquely to a **sheaf** with respect to the strong G -topology

Tate's theorem, revisited

Let $X = \text{Sp } A$ be an affinoid K -space, with weak G -topology, have **presheaf of affinoid functions** on X :

$$\mathcal{O}_X: \text{Sp } A\langle \frac{f}{f_0} \rangle \rightsquigarrow A\langle \frac{f}{f_0} \rangle$$

Tate's Theorem

- The presheaf of affinoid functions \mathcal{O}_X is a sheaf.
- \mathcal{O}_X extends uniquely to a sheaf with respect to the strong G -topology, again denoted by \mathcal{O}_X .
- $H^q(X, \mathcal{O}_X) = 0$ for $q > 0$

Definition of affinoid K -spaces, revisited

An **affinoid K -space** is a pair (X, \mathcal{O}_X) , where $X = \text{Sp } A$, for an affinoid K -algebra A , is equipped with the **strong** G -topology, and where \mathcal{O}_X is (the extension of) the sheaf of affinoid functions on X

Completeness conditions for strong G -topology

The **strong G -topology** on $X = \mathrm{Sp} A$ satisfies:

(G_0) \emptyset, X are admissible open

(G_1) let $U = \bigcup_{i \in I} U_i$ be an admissible covering, let $V \subset U$ be a subset such that $V \cap U_i$ is admissible open for all i .

Then: V is admissible open in X

(G_2) let $U = \bigcup_{i \in I} U_i$ be a covering with U, U_i admissible open for all i , and assume that $(U_i)_i$ has an admissible refinement.

Then: $(U_i)_i$ is an admissible covering of U

Fact

Let X be a set with a G -topology which satisfies (G_0), (G_1), (G_2). Let $X = \bigcup_{i \in I} X_i$ be an admissible covering. Then the G -topology on X can be **uniquely recovered** from

- the restriction of the G -topology to each X_i ,
- knowing the overlaps $X_i \cap X_j$ as admissible opens in X_i, X_j

Definition

A **rigid K -space** is a pair $X = (X, \mathcal{O}_X)$ where

- X is a set with a G -topology on it, satisfying (G_0) , (G_1) , (G_2) ,
- \mathcal{O}_X is a sheaf of K -algebras on X such that there exists an admissible covering $X = \bigcup_{i \in I} X_i$ where each $(X_i, \mathcal{O}_X|_{X_i})$ is an affinoid K -space.

A **morphism of rigid K -spaces** is a morphism in the sense of locally ringed spaces over K

Proposition

Let A, B be affinoid K -algebras, and let X, Y be the associated affinoid K -spaces (in the sense of rigid K -spaces as above). Then the canonical map

$$\mathrm{Hom}_K(A, B) \longrightarrow \mathrm{Hom}_K(Y, X)$$

is bijective

Remark

Rigid K -spaces can be constructed by the usual glueing techniques.
Need:

- a set X with a covering $X = \bigcup_{i \in I} X_i$,
- each X_i with structure of affinoid (or rigid) K -space, compatible with overlaps $X_i \cap X_j$ (admissible open in X_i, X_j).

Then, thanks to (G_0) , (G_1) , (G_2) , there is a unique structure as a rigid K -space on X such that $X = \bigcup_{i \in I} X_i$ is an admissible covering (**structural admissible covering of X**)

Example: Affine n -space \mathbb{A}_K^n

Choose $\pi \in K$, $0 < |\pi| < 1$. Let $\zeta = (\zeta_1, \dots, \zeta_n)$. Then

$$\mathrm{Sp} K\langle \zeta \rangle \hookrightarrow \mathrm{Sp} K\langle \pi \zeta \rangle \hookrightarrow \mathrm{Sp} K\langle \pi^2 \zeta \rangle \hookrightarrow \dots$$

is increasing sequence of balls with radii $1, |\pi|^{-1}, |\pi|^{-2}, \dots$

gives structural admissible covering, get the rigid version of \mathbb{A}_K^n

- $\mathbb{B}_K^1 = \text{Sp } T_1$ (affinoid K -space, as usual, connected)
- $\tilde{\mathbb{B}}_K^1 = \{x; |x| < 1\} \cup \{x; |x| = 1\}$ as structural covering (not connected)
- K algebraically closed, not spherically complete (e. g., $K = \mathbb{C}_p$)
 \implies exists decreasing sequence of disks $\mathbb{B}_K^1 \supset D_1 \supset D_2 \supset \dots$
with empty intersection,
take $\tilde{\tilde{\mathbb{B}}}_K^1 = \bigcup_i (\mathbb{B}_K^1 - D_i)$ as structural covering, get connected rigid K -space
have canonical morphism $\tilde{\tilde{\mathbb{B}}}_K^1 \longrightarrow \mathbb{B}_K^1$, becoming a
(non-bijective) open immersion after enlarging K suitably

Serre's GAGA functor

The construction of the rigid affine n -space works more generally for Zariski closed subschemes of \mathbb{A}_K^n . Thereby get functor

$$\begin{array}{ccc} (K\text{-schemes of locally finite type}) & \longrightarrow & (\text{rigid } K\text{-spaces}) \\ \mathfrak{X} & \longmapsto & \mathfrak{X}^{\text{rig}} \end{array}$$

In particular, the rigid version of projective n -space \mathbb{P}_K^n is defined (can be covered by $n + 1$ unit balls)

Tate elliptic curves

Let $q \in K$, $0 < |q| < 1$. Then the quotient

$$E = \mathbb{G}_{m,K}/q^{\mathbb{Z}}$$

makes sense as a rigid K -space; is obtained by glueing the annuli $\{x \in \mathbb{B}_K^1; |q| \leq |x| \leq |q|^{1/2}\}$ and $\{x \in \mathbb{B}_K^1; |q|^{1/2} \leq |x| \leq 1\}$ via multiplication by q . As in complex analytic geometry, E can be algebraized to yield an algebraic curve in \mathbb{P}_K^2

Some advanced results

Notions to translate from algebraic geometry

- open, resp. closed immersions
- separated, resp. proper morphisms
- coherent modules

Grothendieck's Proper Mapping Theorem (Kiehl)

Let $\varphi: X \rightarrow Y$ be a proper morphism of rigid K -spaces and \mathcal{F} a coherent \mathcal{O}_X -module. Then the higher direct images $R^i\varphi_*(\mathcal{F})$, $i \geq 0$, are coherent \mathcal{O}_Y -modules

Applications

- proper \implies closed
- Stein factorization
- Serre's GAGA theorems (Köpf)
- Chow's Theorem

- 1 Introduction
- 2 Classical rigid spaces à la Tate, Grauert, Remmert, Kiehl, ...
- 3 Rigid spaces via Raynaud's formal schemes
 - Motivating example
 - Admissible formal schemes
 - Generic fiber of a formal scheme
 - Admissible formal blowing-up
 - An equivalence of categories
 - Relative rigid spaces
 - Zariski-Riemann space
 - Some advanced results

Motivating example

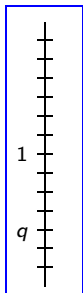
Tate elliptic curve $\mathbb{G}_{m,K}/q^{\mathbb{Z}}$, Mumford style

R valuation ring of K , field with complete non-Archimedean val.

$\text{Spec } R = \{\eta, s\}$

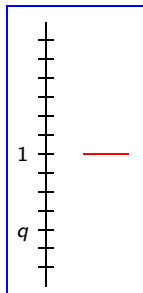
η generic point (\simeq zero ideal $\subset R$)

s **special** point (\simeq maximal ideal $\subset R$)



η

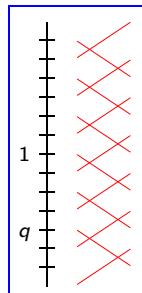
$\mathbb{G}_{m,K}$



η

s

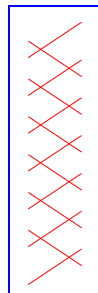
$\mathbb{G}_{m,R}$



η

s

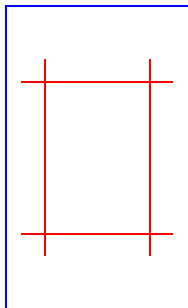
$q^{\mathbb{Z}}$ -action



s

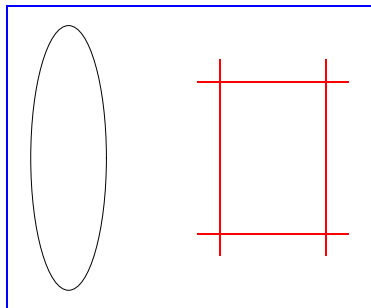
formal completion

Tate elliptic curve, continued



S

quotient by lattice $q^{\mathbb{Z}}$



η

S

algebraization

Tate elliptic curve $\mathbb{G}_{m,K}/q^{\mathbb{Z}}$ comes equipped with R -model

Program

Make generic fibre of a formal scheme visible as rigid space!

R ring, assume:

- R **complete** and **separated** with respect to I -adic topology, for some finitely generated ideal $I = (g_1, \dots, g_s) \subset R$

- R has **no I -torsion**; i. e.:

$$(I\text{-torsion})_R = \{r \in R; I^n r = 0 \text{ for some } n \in \mathbb{N}\} = 0$$

- no I -torsion is equivalent to: $R \hookrightarrow \prod_{i=1}^s R_{g_i}$

- consider only following cases:

(N) R is noetherian, or

(V) R is an adic valuation ring with a finitely generated ideal of definition I

- **Main example**: R valuation ring of a field K with a complete non-Archimedean absolute value $|\cdot|$, with $I = (t)$ for some $t \in R$, $0 < |t| < 1$

Definition

An R -algebra A is called

- of **tf (= topologically finite) type** if $A = R\langle\zeta_1, \dots, \zeta_n\rangle/\mathfrak{a}$
- of **tf presentation** if $A = R\langle\zeta_1, \dots, \zeta_n\rangle/(a_1, \dots, a_s)$
- **admissible** if A is of tf presentation and has no I -torsion

Theorem (based on flattening techniques of Raynaud-Gruson)

Let A an R -algebra of tf presentation. Then A is a **coherent** ring. In particular, any A -module M of finite presentation is **coherent**; i. e., M is finitely generated, and each finite submodule of M is of finite presentation

Corollary

Let A an R -algebra of tf type. If A has no I -torsion, A is of **tf presentation** and, hence, **admissible**

Corollary

Any R -algebra A of tf type is I -adically complete and separated

For $\lambda \in \mathbb{N}$ set: $R_\lambda = R/I^{\lambda+1}$, $A_\lambda = A \otimes_R R_\lambda$; then $A = \varprojlim_\lambda A_\lambda$

Construction of formal scheme $\mathrm{Spf} A$

$$\mathrm{Spf} A = \varinjlim \mathrm{Spec} A_\lambda$$

more precisely: $\mathrm{Spf} A$ is a locally ringed space (X, \mathcal{O}_X) where

- $X = \mathrm{Spec} A_0 \subset \mathrm{Spec} A$ with Zariski topology
- $\mathcal{O}_X: D(f) \rightsquigarrow A\langle f^{-1} \rangle := \varprojlim A_\lambda[f^{-1}]$, for $f \in A$

Definition

A **formal R -scheme of locally tf type** is a locally ringed space which is locally of type $\mathrm{Spf} A$ with A an R -algebra of tf type.

Same for **tf presentation**, **admissible**

Proposition

Let $X = \mathrm{Spf} A$ be a formal R -scheme, where A is I -adically complete and separated. Equivalent:

- X is **locally of tf type**
- A is of **tf type**

Same assertion for **tf presentation**, **admissible**

Lemma

Let A be an R -algebra of tf type; let $f_0, \dots, f_r \in A$ generate the unit ideal. Then

$$A \longrightarrow \prod_{i=0}^r A\langle f_i^{-1} \rangle$$

is faithfully flat

Generic fiber of formal R -schemes (classical case)

Assume for a moment: R valuation ring of field K with complete non-Archimedean absolute value $|\cdot|$

Proposition

There is a canonical functor

$$\text{rig} : \left(\begin{array}{l} \text{formal } R\text{-schemes} \\ \text{of locally tf type} \end{array} \right) \longrightarrow (\text{rigid } K\text{-spaces})$$

which on affine formal R -schemes is given by

$$\text{Spf } A \rightsquigarrow \text{Sp } A_{\text{rig}}, \quad A_{\text{rig}} := A \otimes_R K$$

Proof

$A = R\langle\zeta_1, \dots, \zeta_n\rangle/\mathfrak{a} \implies A_{\text{rig}} = A \otimes_R K = K\langle\zeta_1, \dots, \zeta_n\rangle/(\mathfrak{a})$
is an affinoid K -algebra; globalization:

$$A\langle f^{-1}\rangle_{\text{rig}} = A\langle\zeta\rangle/(1 - f\zeta) \otimes_R K = A_{\text{rig}}\langle\zeta\rangle/(1 - f\zeta) = A_{\text{rig}}\langle f^{-1}\rangle$$

Question

How far is **rig** from being an equivalence of categories?

Observation

- Let A an R -alg. of tf type, let $f_0, \dots, f_r \in A$, $(f_0, \dots, f_r) = A$
- Zariski covering $\mathrm{Spf} A = \bigcup_i \mathrm{Spf} A \langle f_i^{-1} \rangle$ yields via **rig** a formal covering $\mathrm{Sp} A_{\mathrm{rig}} = \bigcup_i \mathrm{Sp} A_{\mathrm{rig}} \langle f_i^{-1} \rangle$ of the associated affinoid K -space; recall

$$\mathrm{Sp} A_{\mathrm{rig}} \langle f_i^{-1} \rangle = \{x \in \mathrm{Sp} A_{\mathrm{rig}} ; |f_i(x)| \geq 1 = |f_i|_{\mathrm{sup}}\}$$

- Exist more general coverings on $\mathrm{Sp} A_{\mathrm{rig}}$: let $f_0, \dots, f_r \in A_{\mathrm{rig}}$ such that $(f_0, \dots, f_r) = A_{\mathrm{rig}}$. Then $\mathrm{Sp} A_{\mathrm{rig}} = \bigcup_i \mathrm{Sp} A_{\mathrm{rig}} \langle \frac{f}{f_i} \rangle$ is admissible (**rational**) covering, generally not induced via **rig**
- Note: $A_{\mathrm{rig}} \langle \frac{f}{f_i} \rangle = A_{\mathrm{rig}} \langle \zeta_0, \dots, \hat{\zeta}_i, \dots, \zeta_r \rangle / (f_j - f_i \zeta_j ; j \neq i)$ alludes to **blow-up**; recall

$$\mathrm{Sp} A_{\mathrm{rig}} \langle \frac{f}{f_i} \rangle = \{x \in \mathrm{Sp} A_{\mathrm{rig}} ; |f_j(x)| \leq |f_i(x)| \text{ for all } j\}$$

Admissible formal blowing-up

- Let R base ring, general, of type (N) or (V), $I \subset R$ ideal of definition; let X a formal R -scheme, locally of tf presentation.
- An ideal $\mathcal{A} \subset \mathcal{O}_X$ is **coherent** if, locally on affine open pieces $\text{Spf } A \subset X$, it is associated to an ideal of finite type $\mathfrak{a} \subset A$.
- An ideal $\mathcal{A} \subset \mathcal{O}_X$ is called **open** if, locally on X , it contains powers of type $I^n \mathcal{O}_X$

Definition

Let $\mathcal{A} \subset \mathcal{O}_X$ be a coherent open ideal. Then

$$X_{\mathcal{A}} = \varinjlim_{\lambda \in \mathbb{N}} \text{Proj} \left(\bigoplus_{d=0}^{\infty} \mathcal{A}^d \otimes (\mathcal{O}_X / I^{\lambda} \mathcal{O}_X) \right)$$

together with canonical projection $X_{\mathcal{A}} \longrightarrow X$ is called the **formal blowing-up of \mathcal{A} on X**

$$X_{\mathcal{A}} = \varinjlim_{\lambda \in \mathbb{N}} \text{Proj} \left(\bigoplus_{d=0}^{\infty} \mathcal{A}^d \otimes (\mathcal{O}_X / I^\lambda \mathcal{O}_X) \right)$$

Proposition

- $X = \text{Spf } A$ affine, $\mathcal{A} \subset \mathcal{O}_X$ associated to the coherent open ideal $\mathfrak{a} \subset A$. Then $X_{\mathcal{A}}$ equals the I -adic completion of the scheme theoretic blowing-up of \mathfrak{a} on $\text{Spec } A$
- X admissible \implies
 - $X_{\mathcal{A}}$ admissible
 - $\mathcal{A}\mathcal{O}_{X_{\mathcal{A}}}$ is invertible on $X_{\mathcal{A}}$
 - $X_{\mathcal{A}} \longrightarrow X$ satisfies universal property:
Any morphism of formal schemes $\varphi: Y \longrightarrow X$ such that $\mathcal{A}\mathcal{O}_Y$ is invertible on Y factors uniquely through $X_{\mathcal{A}} \longrightarrow X$

Basic Lemma

Let $X = \mathrm{Spf} A$ be **admissible**, assume that the coherent open ideal $\mathcal{A} \subset \mathcal{O}_X$ is associated to the ideal $\mathfrak{a} = (f_0, \dots, f_r) \subset A$. Then:

- Let $U_i \subset X_{\mathcal{A}}$ be the locus where $\mathcal{A}\mathcal{O}_{X_{\mathcal{A}}}$ is generated by f_i , $i = 0, \dots, r$. Then the U_i define an affine open covering of $X_{\mathcal{A}}$.
- Let

$$C_i = A\langle \frac{f}{f_i} \rangle = A\langle \zeta_0, \dots, \hat{\zeta}_i, \dots, \zeta_r \rangle / (f_i \zeta_j - f_j; j \neq i)$$
$$A_i = C_i / (I\text{-torsion})_{C_i}$$

The I -torsion of C_i coincides with its f_i -torsion, and

$$U_i = \mathrm{Spf} A_i$$

Method of Proof

Start with scheme theoretic blowing-up of \mathfrak{a} on $\mathrm{Spec} A$, and use a flatness lemma of **Gabber** for I -adic completion

Observation

Let the situation be as in Basic Lemma, but assume R to be the valuation ring of a complete non-Archimedean field K . Then the functor **rig** (which “is” tensoring over R with K) transforms the open affine covering

$$X_A = \bigcup_{i=0}^r U_i = \bigcup_{i=0}^r \mathrm{Spf} \left(A \langle \frac{f}{f_i} \rangle / (I\text{-torsion}) \right)$$

into the admissible rational covering

$$X_{\mathrm{rig}} = \bigcup_{i=0}^r U_{i,\mathrm{rig}} = \bigcup_{i=0}^r \mathrm{Sp} A_{\mathrm{rig}} \langle \frac{f}{f_i} \rangle$$

of the rigid K -space X_{rig}

In particular, **rig** transforms admissible formal blow-ups into isomorphisms.

An equivalence of categories

Raynaud's Theorem

Let R be the valuation ring of a field K with a complete non-Archimedean absolute value. Then the functor

$$\text{rig} : \left(\begin{array}{l} \text{formal } R\text{-schemes} \\ \text{of locally tf type} \end{array} \right) \longrightarrow (\text{rigid } K\text{-spaces})$$

induces an equivalence between

- 1 the category of all **quasi-paracompact admissible formal R -schemes**, localized by the class of **admissible formal blowing-ups**, and
- 2 the category of all **quasi-separated quasi-paracompact rigid K -spaces**

X **quasi-paracompact** means: X admits an (admissible) covering of finite type by quasi-compact (admissible) open subspaces $U_i \subset X$; i. e., such that each U_i is disjoint from almost all other U_j

Base schemes S

- (V') S an admissible formal R -scheme, where R is an adic valuation ring of type (V), as considered above
- (N') S is a noetherian formal scheme (of quite general type) such that the topology of \mathcal{O}_S is generated by a coherent ideal $\mathcal{I} \subset \mathcal{O}_S$, where \mathcal{O}_S does not admit \mathcal{I} -torsion

Definition

Let S be a formal scheme of type (V') or (N'), let (FSch/S) be the category of admissible formal S -schemes. Then the category (Rig/S) of rigid S -spaces is obtained from (FSch/S) by localization via admissible formal blowing-ups

Remark

Essentially, we take the category (1) in Raynaud's Theorem as definition for rigid spaces over non-classical bases

Idea

Relative rigid space X is a family of classical rigid spaces X_s over suitable rigid points $s \in S$

Example: Universal Tate elliptic curve

$R = \mathbb{Z}[[Q]]$, Q a variable, with Q -adic topology

$S = \text{Spf } \mathbb{Z}[[Q]]$ formal base scheme

$\mathbb{B}_S^1 = \text{Spf } R\langle \zeta \rangle$ unit disk over S

blow up ideal $\mathcal{A} = (\zeta^3, Q\zeta, Q^2) \subset \mathcal{O}_{\mathbb{B}_S^1}$ on \mathbb{B}_S^1 , get $X_{\mathcal{A}} \rightarrow \mathbb{B}_S^1$

$X_1 = \text{Spf } R\langle \zeta, Q\zeta^{-2} \rangle$ part in $X_{\mathcal{A}}$ where ζ^3 generates \mathcal{A}

$X_2 = \text{Spf } R\langle \zeta, Q\zeta^{-1}, Q^{-1}\zeta^2 \rangle$ part in $X_{\mathcal{A}}$ where $Q\zeta$ generates \mathcal{A}

identify $X_1(\zeta^{-1})$ with $X_2(Q^{-1}\zeta)$ via multiplication with Q

associated rigid S -space E is universal Tate elliptic curve:

Any Tate elliptic curve $E_q = \mathbb{G}_{m,K}/q^{\mathbb{Z}}$ over complete

non-Archimedean field K with valuation ring R_K is pull-back of E

with respect to point given by $\mathbb{Z}[[Q]] \rightarrow R_K, Q \mapsto q$

Definition

Let S be a base scheme of type (V') or (N') . Let X be an admissible formal S -scheme, which is quasi-separated and quasi-paracompact. Then

$$[X] = \varprojlim_{\substack{\mathcal{A} \subset \mathcal{O}_X \\ \text{coherent open ideal}}} X_{\mathcal{A}}$$

is called the **Zariski-Riemann space** associated to X

Proposition

- $[X]$ is non-empty if X is non-empty
- $[X]$ is a T_0 -space, but not necessarily Hausdorff
- $[X]$ is quasi-compact if X is quasi-compact

Proposition

Let X be a formal model of a classical rigid K -space X_K . There is a canonical specialization map

$$sp: X_K \longrightarrow [X]$$

where

- sp is injective
- the image of sp is dense in $[X]$ with respect to the constructible topology
- sp induces an equivalence between the category of abelian sheaves on X_K and the one on $[X]$

Advanced results on relative rigid spaces

Notion

Let X be an admissible formal S -scheme with ideal of definition $\mathcal{I} \subset \mathcal{O}_S$ (assume X quasi-paracompact and quasi-separated)

say that associated rigid space X_{rig} satisfies certain property (P) if there exists an open affine covering $(\text{Spf } A_i)_i$ of X such that (P) is satisfied on all **schemes $\text{Spec } A_i - V(\mathcal{I}A_i)$**

in above situation, say **rig-(P)** satisfied on X

Of course, this makes sense only if rig-(P) is independent from the chosen covering $(\text{Spf } A_i)_i$ and invariant under admissible formal blowing-up of X

General problem

Start with rigid S -space X_{rig} enjoying certain property (P)

Find formal S -model X such that rig-(P) extends to (P) on X

Theorem (Raynaud-Gruson, Bosch-Lütkebohmert)

Let $\varphi: X \rightarrow Y$ be quasi-compact morphism of admissible formal S -schemes, which is **rig-flat**. Then there exists commutative diagram of admissible formal S -schemes

$$\begin{array}{ccc} X & \longleftarrow & X' \\ \varphi \downarrow & & \downarrow \varphi' \\ Y & \longleftarrow & Y' \end{array}$$

where φ' is **flat**, $Y' \rightarrow Y$ is formal blowing-up of some coherent open ideal $\mathcal{A} \subset \mathcal{O}_Y$, and where $X' \rightarrow X$ is formal blowing-up of $\mathcal{A}\mathcal{O}_X \subset \mathcal{O}_X$ on X .

Theorem (Bosch-Lütkebohmert-Raynaud)

X/S quasi-compact admissible, flat, fibers $\neq \emptyset$ equidimensional.
 $f \in \Gamma(X, \mathcal{O}_X)$ not nilpotent on fibers of $X_{\text{rig}}/S_{\text{rig}}$. Then:

$$\begin{array}{ccc} & & Y' \\ & & \downarrow \\ X & \longleftarrow & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & S' \end{array}$$

- $X' = X \times_S S'$,
- $S' \rightarrow S$ rig-flat, rig-quasi-finite, rig-surjective
- $Y'_{\text{rig}} \rightarrow X'_{\text{rig}}$ an isomorphism,
- exists $\beta \in \Gamma(S', \mathcal{O}_{S'})$ rig-invertible, $g \in \Gamma(Y', \mathcal{O}_{Y'})$ invertible on an open part of Y' covering S' such that $f = \beta g$ on Y'

Theorem (Bosch-Lütkebohmert-Raynaud)

X/S quasi-compact admissible formal scheme, flat, $X_{\text{rig}}/S_{\text{rig}}$ with reduced geometric fibers, equidimensional. Then:

$$\begin{array}{ccc} & & Y' \\ & & \downarrow \\ X & \longleftarrow & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & S' \end{array}$$

- $X' = X \times_S S'$,
- $S' \rightarrow S$ surjective and $S'_{\text{rig}} \rightarrow S_{\text{rig}}$ étale,
- $Y' \rightarrow X'$ finite and $Y'_{\text{rig}} \rightarrow X'_{\text{rig}}$ an isomorphism,
- $Y' \rightarrow S'$ flat and has reduced geometric fibres.