K-theory and 2-Cocycles on Transformation Groups

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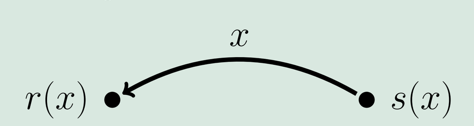
Groupoids, Cocycles, and Homotopy

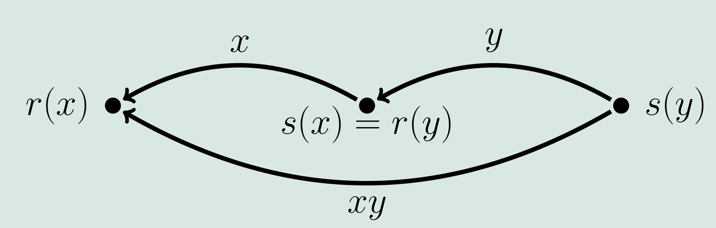
The class of *groupoids* includes many familiar mathematical objects — groups, (topological) spaces, equivalence relations, and group actions, for example. Roughly speaking, a groupoid $\mathcal{G}$ is a set with a partially defined multiplication. We write

$$\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G} = \{(x, y) : \text{ the product } xy \in \mathcal{G} \text{ is defined.}\}$$

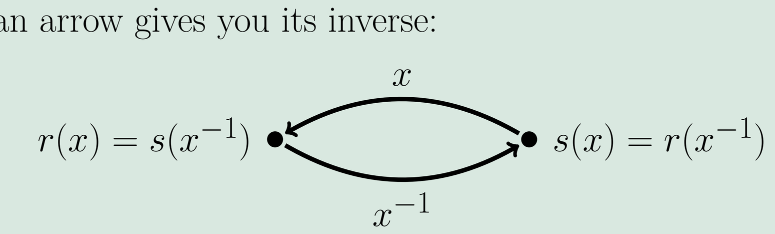
We can think of elements of $\mathcal{G}$ as arrows:



Then, the product xy is defined iff $s(x) = r(y)$:



Reversing an arrow gives you its inverse:



Let

$$\mathcal{G}^{(0)} = \{u \in \mathcal{G} : u = s(u) = r(u)\}.$$

These are the *units* of $\mathcal{G}$. Note that

$$\forall\, x \in \mathcal{G},\ s(x), r(x) \in \mathcal{G}^{(0)}$$

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Groupoid Cocycles

Definition: Let $\mathcal{G}$ be a groupoid. A *2-cocycle* on $\mathcal{G}$ is a function $\omega : \mathcal{G}^{(2)} \to \mathbb{T}$ such that

$$\omega(x, yz)\omega(y, z) = \omega(xy, z)\omega(x, y)$$

whenever this makes sense.

A *homotopy of 2-cocycles* on $\mathcal{G}$ is a 2-cocycle ω on the groupoid $\mathcal{G} \times [0, 1]$ such that for each composable pair $(x, y) \in \mathcal{G}^{(2)}$, the function

$$t \mapsto \omega((x, t), (y, t))$$

is continuous.

Groupoid C^*-Algebras

Given a groupoid $\mathcal{G}$ with a locally compact Hausdorff topology, a Haar system $\{\lambda^u\}_{u\in\mathcal{G}^{(0)}}$ and a continuous 2-cocycle ω, we can make $C_c(\mathcal{G})$ into a convolution algebra:

$$f *_\omega g(x) = \int f(y)g(y^{-1}x)\omega(y, y^{-1}x)\, d\lambda^{s(x)}(y).$$

By taking different completions of $C_c(\mathcal{G})$ we get the *full* and *reduced twisted groupoid C^*-algebras*

$$C^*(\mathcal{G}, \omega), \quad C^*_r(\mathcal{G}, \omega).$$

Motivation

- **Noncommutative Tori** One way to think of the irrational rotation algebra A_θ is as a twisted group C^*-algebra:
$$A_\theta = C^*(\mathbb{Z}^2, c_\theta) \text{ where } c_\theta((m, n), (j, k)) = e^{2\pi i \theta jn}.$$
Note that the map $\theta \mapsto c_\theta((m, n), (j, k))$ is continuous, so $\{c_\theta\}_{\theta\in[0,1]}$ gives us a homotopy of 2-cocycles on $\mathbb{Z}^2$.
In 1980, Pimsner and Voiculescu proved in [3] that
$$\forall\, \theta,\ K_0(A_\theta) = \mathbb{Z} \oplus \mathbb{Z} = K_1(A_\theta).$$

- **Symplectic Vector Bundles** Let $V \to M$ be a smooth even-dimensional vector bundle. A *symplectic form* ω on V is a skew-symmetric, nondegenerate map $\omega : V \times V \to \mathbb{R}$; if V admits a symplectic form then we say V is a *symplectic vector bundle.*
  Example: For any smooth manifold M, let $X = T^*M$. Then $TX \to X$ is a symplectic vector bundle.
Note that we can think of $V := V \to M$ as a groupoid;
$$(v, w) \in V^{(2)} \Leftrightarrow \pi(v) = \pi(w); \quad vw = v + w.$$
Moreover, the symplectic form ω gives us a homotopy σ of 2-cocycles on V:
$$\sigma((v, t), (w, t)) = e^{2\pi i t\omega(v,w)}.$$
Here, σ_0 is the trivial cocycle.
Invoking Bott periodicity, and the dual Dirac element in $KK(\mathbb{C}, C^*(V))$, we can construct a KK-equivalence between $C^*(V, \omega) = C^*(V, \sigma_1)$ and $C^*(V) = C^*(V, \sigma_0)$. In particular, this implies that
$$K_*(C^*(V, \sigma_1)) \cong K_*(C^*(V, \sigma_0)).$$

- **Groups Satisfying the Baum-Connes Conjecture** In a 2010 paper [1], Echterhoff, Lück, Phillips, and Walters proved a far-reaching generalization of Pimsner and Voiculescu's result:
  Theorem: [ELPW, 2010] Let G be a LCH group that satisfies the Baum-Connes conjecture with coefficients $\mathcal{K}$ and $C([0, 1], \mathcal{K})$. Let ω be a homotopy of 2-cocycles on $\mathcal{G}$. Then
$$K_*(C^*_r(G, \omega_0)) \cong K_*(C^*_r(G, \omega_1)).$$
Our main Theorem is an extension of this result to the case of transformation groups; the outline of the proof and some of the main technical lemmas are the same as in ELPW's proof.

Homotopies & $C([0, 1])$-Algebras

Definition: A C^*-algebra A is a *$C_0(X)$-algebra* if A admits a $*$-homomorphism

$$\Psi : C_0(X) \to ZM(A)$$

such that $\overline{\text{span}}\{\Psi(f) \cdot a\} = A$.

Writing

$$I_x = \overline{\text{span}}\{\Psi(f) \cdot a : f \in C_0(X\backslash x), a \in A\},$$

we see that I_x is an ideal in A, so we can define the *fiber algebra A_x of A at x* by

$$A_x := A/I_x.$$

Proposition: [G.] Let $\mathcal{G}$ be a locally compact Hausdorff groupoid and let ω be a homotopy of continuous 2-cocycles on $\mathcal{G}$. Then $C^*(\mathcal{G} \times [0, 1], \omega)$ is a $C([0, 1])$-algebra, with fiber algebra $C^*(\mathcal{G}, \omega_t)$.

Proposition: [G.] Let $G \ltimes X$ be a LCH transformation group and let ω be a homotopy of *continuous* 2-cocycles on $G \ltimes X$. Then $C^*(G \ltimes X \times [0, 1], \omega)$ is a $C([0, 1])$-algebra, with fiber algebra $C^*(G \ltimes X, \omega_t)$.
If G is compact then

$$C^*(G \ltimes X \times [0, 1], \omega) \cong C^*(G \ltimes X, \omega_t) \otimes C([0, 1]).$$

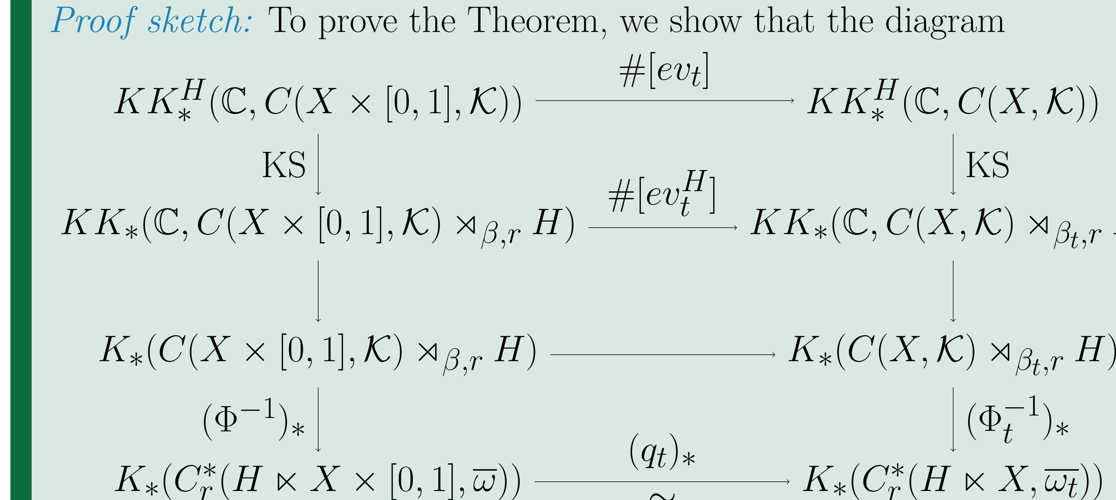
The Main Theorem

Theorem: [G, 2013] Let ω be a homotopy of continuous 2-cocycles on a LCH transformation group $G \ltimes X$ such that X is compact and G satisfies the Baum-Connes conjecture with coefficients. Then

$$K_*(C^*_r(G \ltimes X, \omega_0)) \cong K_*(C^*_r(G \ltimes X, \omega_1)).$$

Moreover, the isomorphism is induced by the homotopy.

Proof sketch: To prove the Theorem, we show that the diagram



commutes for any compact subgroup H of G and any $t \in [0, 1]$. This tells us that the element

$$[ev_t] = [(C(X, \mathcal{K}), ev_t, 0)] \in KK^H(C(X \times [0, 1], \mathcal{K}), C(X, \mathcal{K}))$$

generated by the $*$-homomorphism $ev_t : C(X \times [0, 1], \mathcal{K}) \to C(X, \mathcal{K})$ satisfies the hypotheses of Proposition 1.6 in [1]. To finish the proof of the Theorem, we then follow the same arguments used in [1] to prove Theorem 1.9.

New Directions

The techniques used in [1] and in the proof of our Main Theorem seem unlikely to be applicable to a larger class of groupoids. A very different approach was used in [2] by Kumjian, Pask, and Sims to prove the following:

Theorem: [KPS, 2012] If a 2-cocycle ω on a higher-rank graph Λ is given by

$$\omega(\lambda, \mu) = e^{2\pi i\sigma(\lambda,\mu)}$$

for some $\mathbb{R}$-valued 2-cocycle σ, then

$$K_*(C^*(\Lambda, \omega)) \cong K_*(C^*(\Lambda)).$$

We have recently extended this result:

Theorem: [G, 2013] Let ω_0, ω_1 be homotopic cocycles on a higher-rank graph Λ. Then

$$K_*(C^*(\Lambda, \omega_0)) \cong K_*(C^*(\Lambda, \omega_1)).$$

To connect these results with groupoids, recall that from the space of infinite paths Λ^∞ in a higher-rank graph Λ of rank k, we can construct a groupoid $\mathcal{G}_\Lambda$:

$$\mathcal{G}_\Lambda = \{(x, n, y) : x, y \in \Lambda^\infty, n = m - \ell \in \mathbb{Z}^k, \sigma^m(x) = \sigma^\ell(y)\}.$$

A related class of groupoids, the *Deaconu-Renault groupoids*, are built out of a LCH space X and a local homeomorphism $\phi : X \to X$. The associated Deaconu-Renault groupoid $\mathcal{G}_\phi$ is

$$\mathcal{G}_\phi = \{(x, n, y) : x, y \in X, n = m - \ell \in \mathbb{Z}, \phi^m(x) = \phi^\ell(y)\}.$$

We hope that similar proof techniques to those used in the k-graph case will allow us to prove the following conjecture for Deaconu-Renault groupoids:

Conjecture: If $\mathcal{G}$ is a Deaconu-Renault groupoid, and ω_0, ω_1 are homotopic cocycles on $\mathcal{G}$, then

$$K_*(C^*(\mathcal{G}, \omega_0)) \cong K_*(C^*(\mathcal{G}, \omega_1)).$$

References

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[2] Alexander Kumjian, David Pask, and Aidan Sims, *On the K-theory of twisted higher-rank-graph C^*-algebras*, arXiv:1211.1445v1 (2012).

[3] M. Pimsner and D. Voiculescu, *Exact sequences for K-groups and Ext-groups of certain crossed-product C^*-algebras*, Journal of Operator Theory **4** (1980), 93–118.

[4] Dana P. Williams, *Crossed products of C^*-algebras*, Mathematical Surveys & Monographs, vol. 134, AMS, 2007.