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Classification of actions with the Rokhlin property

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Introduction

In the commutative setting, the nicest results in C^*-dynamics require the action to be free (and minimal). An action $G \curvearrowright X$ is said to be *free* if no non-trivial element of G acts with fixed points.

In the non-commutative setting, and for finite group actions, there are several different notions of freeness; the Rokhlin property being one of the strongest ones. It implies structure preservation results for crossed products, and it is the main assumption in most classification theorems for finite group actions so far available.

The definition of the Rokhlin property for compact group actions was introduced in [4]. Virtually nothing has since then been done in the compact, non-finite, case. Part of my PhD project consists in exploring this notion, and in particular solving the following problems:

Problem 1: Structure of the crossed product. Find classes of C^*-algebras that are preserved under formation of crossed products by Rokhlin actions. Are classifiable classes preserved?

Main technical difficulty: crossed products by non-discrete groups are more difficult to handle. This problem has essentially been solved.

Problem 2: Classification of actions. If classifiable classes are indeed preserved, can one obtain classification results for the actions?

This problem faces a more substantial issue: the invariants considered in the finite group case all vanish for connected groups. We will give an account of the progress so far made on this problem, mostly for $G = \mathbb{T}$.

Definition and comments on examples

Definition 1. *An action $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ has the* Rokhlin property *if for every finite set $F \subseteq A$ and for every $\varepsilon > 0$ there exists a unitary $u \in \mathcal{U}(A)$ such that*

- $\|\alpha_\zeta(u) - \zeta u\| < \varepsilon$ *for all* $\zeta \in \mathbb{T}$,
- $\|ua - au\| < \varepsilon$ *for all* $a \in F$.

The Rokhlin property is rare. No compact group acts with the Rokhlin property on the Cuntz algebra $\mathcal{O}_\infty$ or the Jiang-Su algebra $\mathcal{Z}$. On the other hand, every compact group G acts on $\mathcal{O}_2$ with the Rokhlin property. In fact, most G-actions on $\mathcal{O}_2$ have the Rokhlin property:

Theorem 2. *Let A be a separable unital $\mathcal{O}_2$-absorbing C^*-algebra (for example, $A = \mathcal{O}_2$), and let G be a compact group. Then G-actions on A with the Rokhlin property are generic.*

The Rokhlin property implies severe K-theoretic restrictions. For circle actions, we have the following.

Theorem 3. *Let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then*

1.) $K_0(A) \cong K_1(A)$

2.) α is a dual action, this is, there is $\varphi \in \mathrm{Aut}(A^\alpha)$ such that

$$A^\alpha \rtimes_\varphi \mathbb{Z} \cong A$$

via an isomorphism that sends the dual action φ to α.

Part 2) above is fundamental in the study of circle Rokhlin actions. Its proof uses the theory of partial automorphisms and their crossed products developed by Ruy Exel in [1] and equivariant semiprojectivity.

Classification on Kirchberg algebras

Two actions $\alpha, \beta\colon \mathbb{T} \to \mathrm{Aut}(A)$ are conjugate, written $\alpha \cong \beta$, if there exists $\theta \in \mathrm{Aut}(A)$ approximately inner such that

$$\alpha_\zeta = \theta \circ \beta_\zeta \circ \theta^{-1} \quad \text{for all} \quad \zeta \in \mathbb{T}.$$

Theorem 4. *(Uniqueness) Let A be a unital Kirchberg algebra, and let $\alpha, \beta\colon \mathbb{T} \to \mathrm{Aut}(A)$ be actions with the Rokhlin property such that A^α and A^β satisfy the UCT. Then*

$$\alpha \cong \beta \quad \Leftrightarrow \quad (K_0^\alpha(A), [1_{A^\alpha}], K_1^\alpha(A)) \cong (K_0^\beta(A), [1_{A^\beta}], K_1^\beta(A)).$$

The range of the invariant can be described completely.

Theorem 5. *(Existence) Let A be a unital Kirchberg algebra satisfying the UCT. Then a triple (G_0, g_0, G_1) consisting of two abelian groups G_0, G_1 and an element $g_0 \in G_0$ is the equivariant K-theory of a circle action with the Rokhlin property if and only if there are isomorphisms*

$$G_0 \oplus G_1 \cong K_0(A) \cong K_1(A)$$

such that the first isomorphism sends $(g_0, 0)$ to $[1_A]$.

Question for the reader: Use Theorem 4 and Theorem 5 to answer the following. Suppose A is a UCT Kirchberg algebra with K-theory

$$K_0(A) \cong K_1(A) \cong \mathbb{Z} \oplus \mathbb{Z}_6$$

and $[1_A] = (1, 0)$. How many conjugacy classes of Rokhlin actions on A are there? ‡

Classification on $\mathcal{O}_2$-absorbing algebras

If $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$, then $\mathrm{Prim}(\alpha)\colon \mathbb{T} \to \mathrm{Homeo}(\mathrm{Prim}(A))$ is the induced action on the primitive ideal space.

Theorem 6. *Let A be a unital separable $\mathcal{O}_2$-absorbing algebra, and let $\alpha, \beta\colon \mathbb{T} \to \mathrm{Aut}(A)$ be actions with the Rokhlin property.*

$$\alpha \cong \beta \quad \Leftrightarrow \quad \mathrm{Prim}(\alpha) = \mathrm{Prim}(\beta).$$

The invariant has full range: any circle action on $\mathrm{Prim}(A)$ that comes from *some* circle action on A (Rokhlin or not) *is* $\mathrm{Prim}(\alpha)$ for some Rokhlin action α on A. Describing what circle actions on $\mathrm{Prim}(A)$ come from circle actions on A is very difficult. (Probably easier if A is nuclear; probably even easier if $\mathrm{Prim}(A)$ is Hausdorff.)

Work in progress

Conjecture 7. *Theorem 6 holds for arbitrary compact Lie groups, with the same invariant.*

This has been done for $A = \mathcal{O}_2$.

Conjecture 8. *Let A be a unital AF-algebra and let $\alpha, \beta\colon G \to \mathrm{Aut}(A)$ be actions with the Rokhlin property. Then*

1.) G is totally disconnected (equivalently, zero-dimensional).

2.) $\alpha \cong \beta \quad \Leftrightarrow \quad K_0(\alpha_g) = K_0(\beta_g)$ for all $g \in G$.

We can show 2) assuming 1).

References

[1] R. Exel: *Circle Actions on C^*-Algebras, Partial Automorphisms and a Generalized Pimsner-Voiculescu Exact Sequence*, J. Funct. Anal., 122, no. 2 (1994), 361–401.

[2] E. Gardella: *Circle actions with the Rokhlin property on $\mathcal{O}_2$-absorbing algebras*, in preparation.

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[4] I. Hirshberg, W. Winter: *Rokhlin actions and self-absorbing C^*-algebras*, Pacific J. Math. 233 (2007), 125–143.

‡ The answer is 4.