SFB-workshop: Groups, dynamical systems and C*-algebras, Münster, August 20-24, 2013

C*-Quantum Groups with Projection

Sutanu Roy
(jointly with Ralf Meyer & Stanisław Lech Woronowicz)
Georg-August-Universität Göttingen, Germany
sutanu@uni-math.gwdg.de

Introduction

We introduce the notion of *braided multiplicative unitaries* over C*-quantum groups.

Every *standard multiplicative unitary* of a C*-quantum group $\mathbb{G}$ acting on a Hilbert space $\mathcal{H}$ and every *braided multiplicative unitary* over $\mathbb{G}$ acting on another Hilbert space $\mathcal{K}$ give rise to a *standard multiplicative unitary* acting on $\mathcal{H}\otimes\mathcal{K}$.

We use this to study *semidirect product of C*-quantum groups* or *C*-quantum groups with projection* in the level of multiplicative unitaries.

Hopf algebras with projection (Radford '85)

Let (H,Δ) be a Hopf-algebra and let $p\colon H\to H$ is an idempotent Hopf algebra homomorphism.

- The image of p is again a Hopf algebra (H_1,Δ_1).
- Define $H_2 := \{h\in H : (p\otimes \mathrm{id}_H)\Delta(h) = 1\otimes h\}$. Then (H_2,Δ_2) is a *braided Hopf algebra* over (H_1,Δ_1):
  - → H_2 is a H_1-*(right right) Yetter-Drinfeld* algebra.
  - → The restriction of Δ on H_2 defines $\Delta_2\colon H_2\to H_2\boxtimes H_2$, where $\boxtimes$ is the *braided tensor* product induced by the H_1-Yetter-Drinfeld structure on H_2.

Conversely, a *Hopf algebra* (H_1,Δ_1) and an H_1-*braided Hopf algebra* (H_2,Δ_2) give rise to a unique *Hopf algebra* (H,Δ).

Multiplicative unitaries (Baaj-Skandalis '93)

A unitary $\mathbb{W}\in\mathbb{B}(\mathcal{H}\otimes\mathcal{H})$ is said to be a *multiplicative unitary* if it satisfies the *pentagon equation*:

$$\mathbb{W}_{23}\mathbb{W}_{12} = \mathbb{W}_{12}\mathbb{W}_{13}\mathbb{W}_{23} \qquad \text{in } \mathcal{U}(\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}).$$

Dual multiplicative unitary is defined by $\widehat{\mathbb{W}} := \Sigma\mathbb{W}^*\Sigma\in\mathcal{U}(\mathcal{H}\otimes\mathcal{H})$, where Σ is the flip operator.

C*-quantum groups and duality (Woronowicz '96)

Let C be a C*-algebra and $\Delta\colon C\to\mathcal{M}(C\otimes C)$ is a nondegenerate *-homomorphism. A pair $\mathbb{G}=(C,\Delta_C)$ is said to be a *C*-quantum group* if it comes from a manageable multiplicative unitary $\mathbb{W}\in\mathcal{U}(\mathcal{H}\otimes\mathcal{H})$:

- $C = \{(\omega\otimes\mathrm{id}_{\mathcal{H}})\mathbb{W} : \omega\in\mathbb{B}(\mathcal{H})_*\}^{\text{norm closure}}$,
- $\Delta_C(c) = \mathbb{W}(c\otimes 1)\mathbb{W}^*$ for all $c\in C$.

Manageability is preserved under duality. Quantum group $\widehat{\mathbb{G}} = (\hat{C},\hat{\Delta}_C)$, *dual* to $\mathbb{G}$, is defined by

- $\hat{C} = \{(\mathrm{id}_{\mathcal{H}}\otimes\omega)\mathbb{W} : \omega\in\mathbb{B}(\mathcal{H})_*\}^{\text{norm closure}}$,
- $\hat{\Delta}_C(\hat{c}) = \sigma\big(\mathbb{W}^*(1\otimes\hat{c})\mathbb{W}\big)$ for all $\hat{c}\in\hat{C}$, where σ is the flip morphism.

$\mathbb{W}\in\mathcal{U}(\mathcal{H}\otimes\mathcal{H})$ is also an unitary element of $\mathcal{M}(\hat{C}\otimes C)$.
Moreover, $\mathrm{W}\in\mathcal{UM}(\hat{C}\otimes C)$ is unique: two different multiplicative unitaries giving rise to $\mathbb{G}$ are same while viewed in $\mathcal{UM}(\hat{C}\otimes C)$. W is known as the *reduced bicharacter* of $\mathbb{G}$.

Example: Let G be a locally compact group and let $\mathcal{H} = L^2(G,\mu)$, where μ is the right Haar measure.
The unitary $\mathbb{W}\in\mathcal{U}(\mathcal{H}\otimes\mathcal{H})$ defined by $(\mathbb{W}f)(g_1,g_2) := f(g_1g_2,g_2)$ for $f\in L^2(G\times G,\mu\times\mu)$ and $g_1,g_2\in G$ is a manageable multiplicative unitary.
This gives rise to the quantum group $(C_0(G),\Delta_G)$, where $\Delta_G(f)(g_1,g_2) := f(g_1g_2)$ for all $f\in C_0(G)$.
Dual quantum group $\hat{C} = C^*_{\mathrm{red}}(G)$ with $\hat{\Delta}_G(\lambda_g) = \lambda_g\otimes\lambda_g$, where λ_gs are the right regular representations of G.

Semidirect products of groups

A group G is isomorphic to a *semidirect product* of groups Q and K if and only if there is an *idempotent* group homomorphism $p\colon G\to G$ such that $\mathrm{Im}(p)\cong Q$ and $\mathrm{Ker}(p)\cong K$.

- G is homeomorphic to $K\times Q$ via multiplication map $(k,q)\to k\cdot q$.
- K is homeomorphic to G/Q via quotient map $\delta\colon G\to G/Q$.

Generalisation of the Radford's construction to the C*-algebraic framework for the semidirect product of groups identifies H_1 with $C_0(Q)$ and H_2 with $C_0(G/Q)$.

$\mathbb{G}$-Yetter-Drinfeld pair of representations and braiding

A unitary $\mathrm{U}\in\mathcal{M}(\mathbb{K}(\mathcal{K})\otimes C)$ is said to be a *representation* of $\mathbb{G}=(C,\Delta_C)$ on a Hilbert space $\mathcal{H}$ if it is a character in the second leg: $(\mathrm{id}_{\mathcal{K}}\otimes\Delta_C)\mathrm{U} = \mathrm{U}_{12}\mathrm{U}_{13}$.

A pair of representations (U,V) of $\mathbb{G}$ and $\widehat{\mathbb{G}}$ acting on $\mathcal{K}$ is called $\mathbb{G}$-*Yetter-Drinfeld* if and only if

$$\mathrm{W}_{23}\mathrm{U}_{13}\mathrm{V}_{12} = \mathrm{V}_{12}\mathrm{U}_{13}\mathrm{W}_{23} \qquad \text{in } \mathcal{UM}(\mathbb{K}(\mathcal{K})\otimes\hat{C}\otimes C).$$

Every $\mathbb{G}$-Yetter-Drinfeld pair gives rise to a *braiding operator* $\times^{(\mathcal{K},\mathcal{K})}$.

When either of the representations U, V is trivial then the braiding operator $\times^{(\mathcal{K},\mathcal{K})}$ is the standard flip operator Σ.

Braided multiplicative unitaries

Let (U,V) be a $\mathbb{G}$-Yetter-Drinfeld pair of representations acting on $\mathcal{K}$. An element $\mathbb{F}\in\mathcal{U}(\mathcal{K}\otimes\mathcal{K})$ is called a *braided multiplicative unitary over* $\mathbb{G}$ if it satisfies

- $\mathbb{G}$-(co)invariance condition: $\mathrm{U}_{13}\mathrm{U}_{23}\mathbb{F}_{12} = \mathbb{F}_{12}\mathrm{U}_{13}\mathrm{U}_{23}$ in $\mathcal{UM}(\mathbb{K}(\mathcal{K}\otimes\mathcal{K})\otimes C)$,
- $\widehat{\mathbb{G}}$-(co)invariance condition: $\mathrm{V}_{13}\mathrm{V}_{23}\mathbb{F}_{12} = \mathbb{F}_{12}\mathrm{V}_{13}\mathrm{V}_{23}$ in $\mathcal{UM}(\mathbb{K}(\mathcal{K}\otimes\mathcal{K})\otimes\hat{C})$,
- *braided pentagon* equation: $\mathbb{F}_{23}\mathbb{F}_{12} = \mathbb{F}_{12}(\times^{(\mathcal{K},\mathcal{K})}_{23})\mathbb{F}_{12}(\times^{(\mathcal{K},\mathcal{K})})^*_{23}\mathbb{F}_{23}$ in $\mathcal{U}(\mathcal{K}\otimes\mathcal{K}\otimes\mathcal{K})$.

Theorem

The unitary $\mathbb{W}_{1234}\in\mathcal{U}(\mathcal{H}\otimes\mathcal{K}\otimes\mathcal{H}\otimes\mathcal{K})$ defined by $\mathbb{W}_{1234} := \mathbb{W}_{13}\mathbb{U}_{23}\hat{\mathbb{V}}^*_{34}\mathbb{F}_{24}\hat{\mathbb{V}}_{34}$ is a multiplicative unitary, where $\hat{\mathrm{V}} := \sigma(\mathrm{V}^*)\in\mathcal{UM}(\hat{C}\otimes\mathbb{K}(\mathcal{K}))$.

There is also a notion of manageability for $\mathbb{F}$ which yields the manageability for $\mathbb{W}_{1234}$.

Projections on C*-quantum groups

Let $\mathbb{W}\in\mathcal{U}(\mathcal{H}\otimes\mathcal{H})$ is a multiplicative unitary gives rise to $\mathbb{G}$.

A unitary $\mathrm{P}\in\mathcal{U}(\hat{C}\otimes C)$ is said to be a *projection* on $\mathbb{G}$ if the corresponding concrete unitary $\mathbb{P}\in\mathcal{U}(\mathcal{H}\otimes\mathcal{H})$ satisfies:

- bicharacter conditions: $\mathbb{W}_{23}\mathbb{P}_{12} = \mathbb{P}_{12}\mathbb{P}_{13}\mathbb{W}_{23}$ and $\mathbb{P}_{23}\mathbb{W}_{12} = \mathbb{W}_{12}\mathbb{P}_{13}\mathbb{P}_{23}$ in $\mathcal{U}(\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H})$,
- idempotent condition: $\mathbb{P}_{23}\mathbb{P}_{12} = \mathbb{P}_{12}\mathbb{P}_{13}\mathbb{P}_{23}$ in $\mathcal{U}(\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H})$.

Proposition

$\mathbb{P}\in\mathcal{U}(\mathcal{H}\otimes\mathcal{H})$ is a manageable multiplicative unitary.

The quantum group $\mathbb{H} = (A,\Delta_A)$ generated by $\mathbb{P}$ is called the *image of the projection* P.

The reduced bicharacter $\mathrm{W}^A\in\mathcal{UM}(\hat{A}\otimes A)$ of $\mathbb{H}$ is same as P.

Braided m.u. on C*-quantum groups with projection

Let $\mathbb{G}$ be a C*-quantum group with projection $\mathrm{P}\in\mathcal{UM}(\hat{C}\otimes C)$ and let $\mathbb{H}$ be its image.
P defines a unique left quantum group homomorphism $\Delta_L\colon C\to C\otimes C$:

$$\begin{array}{ccc} C & \xrightarrow{\Delta_L} & C\otimes C \\ {\scriptstyle\Delta_C}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_C\otimes\Delta_C} \\ C\otimes C & \xrightarrow[\Delta_L\otimes\mathrm{id}_C]{} & C\otimes C\otimes C, \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\Delta_L} & C\otimes C \\ {\scriptstyle\Delta_L}\downarrow & & \downarrow{\scriptstyle \Delta_C\otimes\mathrm{id}_C} \\ C\otimes C & \xrightarrow[\mathrm{id}_C\otimes\Delta_L]{} & C\otimes C\otimes C, \end{array}$$

such that $(\mathrm{id}_C\otimes\Delta_L)\mathrm{W} = \mathrm{P}_{12}\mathrm{W}_{13}$.

Second leg of the unitary $\mathrm{F} := \mathrm{P}^*\mathrm{W}\in\mathcal{UM}(\hat{C}\otimes C)$ is (co)invariant under Δ_L, or equivalently, $(\mathrm{id}_C\otimes\Delta_L)\mathrm{F} = \mathrm{F}_{13}$.

Theorem

There are representations $\rho\colon C\to\mathbb{B}(\mathcal{K})$ and $\hat{\rho}\colon\hat{C}\to\mathbb{B}(\mathcal{K})$, where $\mathcal{K} = \overline{\mathcal{H}}\otimes\mathcal{H}$, such that

- (U,V) is a $\mathbb{H}$-Yetter-Drinfeld pair acting on $\mathcal{K}$, where $\mathrm{U} := (\hat{\rho}\otimes\mathrm{id}_A)\mathrm{P}\in\mathcal{UM}(\mathbb{K}(\mathcal{K})\otimes A)$ and $\mathrm{V} := (\mathrm{id}_{\hat{A}}\otimes\rho)\mathrm{P}\in\mathcal{UM}(\hat{A}\otimes\mathbb{K}(\mathcal{K}))$,
- the braiding operator is defined by $\times^{(\mathcal{K},\mathcal{K})} := (\hat{\rho}\otimes\rho)\mathrm{P}\circ\Sigma$,
- $\mathbb{F} := (\hat{\rho}\otimes\rho)\mathrm{F}\in\mathcal{U}(\mathcal{K}\otimes\mathcal{K})$ is a braided multiplicative unitary over $\mathbb{H}$.

Moreover, $\mathbb{W}_{1234}\in\mathcal{U}(\mathcal{H}\otimes\mathcal{K}\otimes\mathcal{H}\otimes\mathcal{K})$ a multiplicative unitary of $\mathbb{G}$.