Inducing Irreducible Representations

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SFB-Workshop on
Groups, Dynamical Systems and C*-Algebras
23 August 2013

Rieffel Induction

1. Let X be a right Hilbert B-module together with a $*$-homomorphism $\phi : A \to \mathcal{L}(\mathsf{X})$.
2. Then we view X as an $A-B$-bimodule: $a \cdot x := \phi(a)(x)$ so that $\langle a \cdot x \, , \, y \rangle_{_B} = \langle x \, , \, a^* \cdot y \rangle_{_B}$.
3. Then we call (X, ϕ) an $A-B$-correspondence.
4. Let $\pi : B \to B(\mathcal{H})$ be a representation.
5. Then $X \odot \mathcal{H}$ is a pre-Hilbert space with respect to the pre-inner product

$$(x \otimes h \mid y \otimes k) := \big(\pi\big(\langle y \, , \, x\rangle_{_B}\big) h \mid k\big).$$

6. Then the induced representation of A, $\operatorname{Ind}_B^A \pi$ acts on the completion $\mathsf{X} \otimes_B \mathcal{H}$ by

$$(\operatorname{Ind}_B^A \pi)(a)[x \otimes h] := [a \cdot x \otimes h].$$

Motivation: Rieffel '74 + Green '76

1. Recall that a dynamical system (A, G, α) is a strongly continuous homomorphism $\alpha : G \to \operatorname{Aut} A$.
2. This allows us to endow $C_c(G, A)$ with a $*$-algebra structure: $$f * g(s) = \int_G f(r)\alpha_r(g(r^{-1}s))\,dr \text{ and } f^*(s) = \alpha_s(f(s^{-1})^*).$$
3. The crossed product, $A \rtimes_\alpha G$ is the enveloping C^*-algebra of $C_c(G, A)$.
4. In particular, its representations $L := \pi \rtimes U$ are in one-to-one correspondence to covariant pairs (π, U) consisting of a representation $\pi : A \to B(\mathcal{H})$ and $U : G \to U(\mathcal{H})$ such that $\pi(\alpha_s(a)) = U(s)\pi(a)U(s)^*$.
5. If $A = \mathbb{C}$, $\mathbb{C} \rtimes G \cong C^*(G)$. If $G = \{e\}$, then $A \rtimes G = A$ and if $\alpha_s = \mathrm{id}$ for all s, $A \rtimes_\alpha G \cong A \otimes_{\max} C^*(G)$.

The Fundamental Example

Example (Ignoring Modular Functions)

1. Let (A, G, α) be a dynamical system and H a closed subgroup of G so that $(A, H, \alpha|_H)$ is a subsystem.
2. View $\mathsf{X}_0 = C_c(G, A)$ as a pre-Hilbert $A \rtimes_{\alpha|_H} H$-module:

$$\langle f \, , g \rangle_{C_c(H)} = f^* * g|_H \quad \text{and}$$

$$f \cdot b(s) = \int_H f(st^{-1})\alpha_{sh}(b(t))\, d\mu_H(t),$$

   and complete to a Hilbert $A \rtimes_{\alpha|_H} H$-module $\mathsf{X} = \mathsf{X}_H^G$.
3. Then $C_c(G, A) \subset A \rtimes_\alpha G$ acts on X_H^G via "convolution": $f \cdot [g] = [f * g]$ for $f, g \in C_c(G)$.
4. This makes X_H^G into a $A \rtimes_\alpha G - A \rtimes_{\alpha|_H} H$-correspondence, and we can induce representations L of $A \rtimes_{\alpha|_H} H$ to a representation $\operatorname{Ind}_H^G L$ of $A \rtimes_\alpha G$.

Mackey Induction

Example (Rieffel, 1974)

Let H be a closed subgroup of G. Then if we let $A = \mathbb{C}$ in the above and let ω be a representation of H, then the representation $\mathrm{Ind}_H^G \omega$ of G obtained via the correspondence X_H^G is (unitarily equivalent to) Mackey's induced representation.

Morita Equivalence

1. A particularly friendly example of Rieffel induction occurs when X is an A– B-correspondence with $\langle \cdot \,,\, \cdot \rangle_{B}$ full and $\phi : A \to \mathcal{L}(\mathsf{X})$ is an isomorphism onto the generalized compact operators $\mathcal{K}(\mathsf{X})$ on X. (Recall that $\mathcal{K}(X)$ is a closed span of the rank-one operators $\Theta_{x,y}$ where $\Theta_{x,y}(z) := x \cdot \langle y \,,\, z \rangle_{B}$.)
2. In this case, the situation is symmetric. The bimodule X is also a full left Hilbert A-module with respect to the inner product ${}_{A}\langle x \,,\, y \rangle = \phi^{-1}(\Theta_{x,y})$.
3. Then induction provides an "isomorphism of the representation theories" of A and B, and we usually write X–Ind in place of $\operatorname{Ind}_{B}^{A}$.
4. In particular, X–Ind π is irreducible if and only if π is irreducible.

Mackey's Imprimitivity Theorem

1. Recall that representations of crossed products $A \rtimes_\alpha G$ are in one-to-one correspondence with covariant pairs (π, U) where $\pi : A \to B(\mathcal{H})$ is a representation and $U : G \to U(\mathcal{H})$ is a unitary representation such that $\pi(\alpha_s(a)) = U(s)\pi(a)U(s)^*$.
2. In particular, representations of $C_0(G/H) \rtimes_{\mathrm{lt}} G$ are in one-to-one correspondence with "systems of imprimitivity" for representations U of G. That is, with covariant pairs (M, U) of $(C_0(G/H), G, \mathrm{lt})$: $M(\mathrm{lt}_s(\phi)) = U(s)M(\phi)U(s)^*$ where $\mathrm{lt}_s(\phi)(rH) = \phi(s^{-1}rH)$.
3. Then we obtain Mackey's Imprimitivity Theorem from the observation that $\mathcal{K}(\mathsf{X}_H^G)$ is isomorphic to $C_0(G/H) \rtimes_{\mathrm{lt}} G$: untangling gives us the result that a representation of U of G is induced from a representation π of H exactly when there is a system of imprimitivity M such that (M, U) is convariant and therefore a representation of $C_0(G/H) \rtimes_{\mathrm{lt}} G$.

Inducing Irreducible Representations — Base Case

1. Consider a dynamical system (A, G, α) with $A = C_0(X)$ and $\alpha_s(f)(x) = f(s^{-1} \cdot x)$.
2. For $x \in X$, let $G_x = \{ s \in G : s \cdot x = x \}$ and let ω be a representation of G_x.
3. If $\mathrm{ev}_x : C_0(X) \to \mathbb{C}$ is evaluation at x, then (ev_x, ω) is a covariant representation of $C_0(X) \rtimes_{\alpha|_{G_x}} G_x$.

Theorem (Mackey '49, Glimm '62)

For each $x \in X$ and every irreducible representation ω of G_x, the representation $L = \mathrm{Ind}_{G_x}^{G}(\mathrm{ev}_x \rtimes \omega)$ induced from the stability group G_x is an irreducible representation of $C_0(X) \rtimes_\alpha G$.

Proof

Sketch of the Proof: [W '79].

We easily see that ω irreducible implies $\mathrm{ev}_x \rtimes \omega$ is irreducible. Hence $\text{X–Ind}(\mathrm{ev}_x \rtimes \omega) \cong (M \otimes N) \rtimes U$ is an irreducible representation of $C_0(G/G_x) \otimes C_0(X) \rtimes_{\mathrm{lt}\otimes\alpha} G \cong^{\text{Green}} \mathcal{K}(\mathsf{X}^G_{G_x})$ on $\mathcal{H}_L$ for suitable representations M of $C_0(G/G_x)$, N of $C_0(X)$ and U of G. However $L := \mathrm{Ind}_{G_x}^G(\mathrm{ev}_x \rtimes \omega) \cong N \rtimes U$ for the same N and U.

We want to see that any operator on $\mathcal{H}_L$ commuting with the image of L is a scalar. Therefore it will suffice to show that if T computes with the image of N (and U), then it also commutes with the image of M. (This will force T to commute with the image of the irreducible representation $\text{X–Ind}(\mathrm{ev}_x \rtimes \omega)$.) This is easy if $G \cdot x = \{\, s \cdot x : s \in G \,\}$ is closed and homeomorphic to G/G_x. The general case follows via some topological gymnastics and playing around in the weak operator topology. □

Effros-Hahn Conjecture

1. If the action of G on X is nice — so that, orbits are locally closed — then every irreducible representation of $C_0(X) \rtimes_\alpha G$ is induced from a stability group as above.
2. In their 1967 *Memoir*, E. Effros and F. Hahn conjectured that if G was *amenable*, then every primitive ideal is induced from a stability group. (That is, every primitive ideal is the kernel of an irreducible representation induced from a stability group.)
3. In the early 70s, P. Green and others formulated the *Generalized Effros-Hahn Conjecture*: Given a dynamical system (A, G, α) with G amenable and a primitive ideal $J \in \operatorname{Prim} A \rtimes_\alpha G$, then there is a primitive ideal $P \in \operatorname{Prim} A$ and an irreducible representation $\pi \rtimes U$ of $A \rtimes_{\alpha|_{G_P}} G_P$ with $\ker \pi = P$ such that $J = \ker(\operatorname{Ind}_{G_P}^{G} \pi \rtimes U)$.
4. If the action of G on $\operatorname{Prim} A$ is nice, then it is not hard to see that all primitive ideals are induced, as above, from stability groups.

The Solution and the another Problem

1. In 1979, building on work of J.-L. Sauvagoet, E. Gootman and J. Rosenberg verified the Effros-Hahn conjecture for separable systems.
2. Then, combined with the result on inducing irreducible representations from stability groups, we get a very simple picture of the primitive ideal space of $C_0(X) \rtimes_\alpha G$.
3. But the GRS-Theorem does not say that if $\pi \rtimes U$ is an irreducible representation of $A \rtimes_{\alpha|_{G_P}} G_P$ with $P = \ker \pi$, then $\mathrm{Ind}_{G_P}^{G}(\pi \rtimes U)$ is irreducible — even if G is amenable.
4. This is (yet another) serious impediment to employing the GRS-Theorem to obtain a global description of the primitive ideal space of crossed products $A \rtimes_\alpha G$ with A non-commutative.

Definition

We say that (A, G, α) satisfies the strong Effros-Hahn Induction property (strong-EHI) if given $P \in \operatorname{Prim} A$ and an irreducible representation $\pi \rtimes U$ of $A \rtimes_{\alpha|_{G_P}} G_P$ with $\ker \pi = P$, then $\operatorname{Ind}_{G_p}^{G}(\pi \rtimes U)$ is irreducible. (We say that (A, G, α) statisfies the Effros-Hahn Induction property (EHI) if the above is true at the level of primitive ideals.)

Conjecture (Echterhoff & W, 2008)

Every separable dynamical system (A, G, α) satisfies EHI.

Remark

In any case were we can prove that EHI holds, we can also show that strong-EHI holds.

What is True

1. Recall that a representation $\pi : A \to B(\mathcal{H})$ is called *homogeneous* if every non-zero sub-representation of π has the same kernel as π.

Theorem (Echterhoff & W)

Suppose that (A, G, α) is separable, $P \in \operatorname{Prim} A$ and $\pi \rtimes U$ is an irreducible representation of $A \rtimes_{\alpha|_{G_P}} G_P$ with $\ker \pi = P$. If π is homogeneous, then $\operatorname{Ind}_{G_P}^{G}(\pi \rtimes U)$ is irreducible.

Sketch of the Proof.

Morita theory implies that $\text{X–Ind}(\pi \rtimes U) \cong (M \otimes \rho) \rtimes U$ is an irreducible representation of $\mathcal{K}(\mathsf{X}_{G_P}^{G}) \cong C_0(G/G_P) \otimes A \rtimes_{\mathrm{lt} \otimes \alpha} G$. Moreover, $\operatorname{Ind}_{G_P}^{G} \pi \rtimes U \cong \rho \rtimes U$. Homogeneity is used to invoke a 1963 result of Effros to produce an ideal center decomposition of ρ which implies that the range of M is in the center of $\rho(A)$. Now the proof proceeds as in the transformation group case. □

Some Cases Where Strong-EHI Holds

Remark

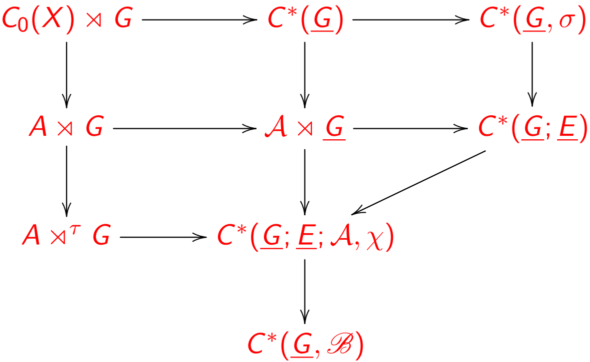
Unfortunately, examples show that $\pi \rtimes U$ irreducible does not always imply that π is homogeneous. Nevertheless, there are some very general situations where our strong-EHI follows from our theorem.

Theorem (Echterhoff & W)

Let (A, G, α) be separable. Then it satisfies strong-EHI in the following cases.

1. *A is type I or more generally points in $\operatorname{Prim} A$ are locally closed.*
2. *A is a sub-quotient of the group C^*-algebra of an almost connected locally compact group.*
3. *G_P is normal in G for all $P \in \operatorname{Prim} A$ (for example, if G is abelian).*

One Construction to Rule Them All



Transformation Group C^*-Algebras Groupoid C^*-Algebras Crossed Product C^*-Algebras Groupoid Crossed Product C^*-Algebras Twists of various sorts Combine Fell Bundle C^*-Algebras

Fell Bundles

Definition

A Fell bundle over a groupoid G is an upper semicontinuous Banach bundle $p : \mathscr{B} \to G$ equipped with a partial multiplication $(a, b) \mapsto ab$ from $\mathscr{B}^{(2)} := \{ (a, b) : (p(a), p(b)) \in G^{(2)} \}$ and an involution $a \mapsto a^*$, both compatible with the groupoid structure, such that

1. For all $u \in \underline{G}^{(0)}$, $B(u)$ is a C^*-algebra with respect to the inherited operations and
2. For all $x \in G$, $B(x)$ is a $B(r(x))-B(s(x))$-imprimitivity bimodule with respect to the inherited module actions and inner products

$$ {}_{B(r(x))}\langle a \, , \, b\rangle = ab^* \quad \text{and} \quad \langle a \, , \, b\rangle_{B(s(x))} = a^* b. $$

Fell Bundle C^*-Algebras

- Provided G has a Haar system, we make $\Gamma_c(G, \mathscr{B})$ into a $*$-algebra:

$$f * g(x) := \int_G f(y)g(y^{-1}x)\, d\lambda^{r(x)}(y) \quad \text{and}$$
$$f^*(x) = f(x^{-1})^*.$$

  Which only makes sense since $B(y)B(y^{-1}x) = B(x)$ and $B(x^{-1})^* = B(x)$.

- Just as for groupoids, we have a universal norm:

  $$\|f\| := \sup\{\, \|L(f)\| : L \text{ is a suitably continuous representation}\,\}.$$

  and we can complete to get the associated C^*-algebra $C^*(G, \mathscr{B})$.

- Note that $A := \Gamma_0(\underline{G}^{(0)}, \mathscr{B}|_{\underline{G}^{(0)}})$ is a C^*-algebra.
- One should think of $C^*(G, \mathscr{B})$ as a generalized crossed product of A by the groupoid G.

Motivating Example

1. Let (A, G, α) be a dynamical system (with G a group).
2. Let $\mathscr{B} = A \times G$ be the trivial bundle over G.
3. Then $\mathscr{B}$ is naturally a Fell bundle: $(a, s)(b, t) := (a\alpha_s(b), st)$ and $(a, s)^* = (\alpha_s^{-1}(a^*), s^{-1})$.
4. If $g \in \Gamma_c(G, \mathscr{B})$, then $g(s) = (\check{g}(s), s)$ where $g \in C_c(G, A)$.
5. $f * g(s) = (\check{f} * \check{g}(s), s)$ where
$$\check{f} * \check{g}(s) = \int_G \check{f}(r)\alpha_r(g(r^{-1}s))\, dr \text{ and } f^*(s) = (\check{f}^*(s), s)$$
where $\check{f}^*(s) = \alpha_s^{-1}(f(s^{-1})^*)$.
6. Now it is an easy matter to check that $C^*(G, \mathscr{B})$ is isomorphic to $A \rtimes_\alpha G$.

1. If $\underline{G}$ is a groupoid (with a Haar system), a twist over $\underline{G}$ is a groupoid extension $\underline{G}^{(0)} \times \mathbf{T} \longrightarrow \underline{E} \xrightarrow{j} G$ such that $\underline{E}$ becomes a principal $\mathbf{T}$-bundle over G. (Think of $\underline{E}$ as given by a 2-cocycle on $\underline{G}$.)
2. We let $\mathscr{B} = (\underline{E} \times \mathbb{C})/\mathbf{T}$ — where $(e, \lambda) \cdot z := (z \cdot e, \bar{z}\lambda)$ — be the associated complex line bundle over $\underline{G}$.
3. Then $\mathscr{B}$ is a Fell bundle: $[e, \lambda][f, \mu] = [ef, \lambda\mu]$.
4. If $g \in \Gamma_c(\underline{G}, \mathscr{B})$, then $g(j(e)) = [e, \check{g}(e)]$ where $g \in C_c(\underline{E})$ with $g(z \cdot e) = \bar{z}\check{g}(e)$.
5. Then $f * g(j(e)) = [e, \check{f} * \check{g}(e)]$ where
$$\check{f} * \check{g}(e) := \int_G f(e_1)g(e_1^{-1}e)\, d\lambda^{r(e)}(j(e_1)).$$
6. Now we can see that $C^*(\underline{G}, \mathscr{B})$ is the C^*-algebra $C^*(\underline{G}; \underline{E})$ of the twist introduced by Kumjian.
7. Note that if $\underline{E}$ is given by a continuous 2-cocycle σ, then $C^*(\underline{G}; \underline{E})$ is Renault's $C^*(\underline{G}, \sigma)$.

First Result

Theorem (Ionescu & W, 13)

Let $p : \mathscr{B} \to \underline{G}$ be a separable Fell bundle over a locally compact groupoid $\underline{G}$. Suppose that $u \in \underline{G}^{(0)}$, $\underline{G}(u) := \{\, x \in G : r(x) = u = s(x) \,\}$ and that L is an irreducible representation of $C^*(\underline{G}(u), \mathscr{B}|_{\underline{G}(u)})$. Then $\mathrm{Ind}_{\underline{G}(u)}^{\underline{G}} L$ is an irreducible representation of $C^*(\underline{G}, \mathscr{B})$.

Remark

As we'll see on the next slide, when $A = \Gamma_0(\underline{G}^{(0)}, \mathscr{B}|_{\underline{G}^{(0)}})$ is non-commutative, this is not quite the "right" result. But it is just what is needed in the special case where $\mathscr{B}$ is the trivial bundle $\mathscr{B} = \underline{G} \times \mathbb{C}$. Then $C^*(\underline{G}, \mathscr{B})$ is the just the usual groupoid algebra $C^*(\underline{G})$. In particular, it gives another proof of strong-EHI for transformation group C^*-algebras.

Moreover, for groupoid C^*-algebras, we can finish the job and prove a complete Effros-Hahn result.

Theorem (Ionescu & W, 2009)

Suppose that $\underline{G}$ is a second countable locally compact groupoid with a Haar system. Assume that $\underline{G}$ is amenable and that J is a primitive ideal in $C^*(G)$. Then there is a $u \in \underline{G}^{(0)}$ and an irreducible representation L of $C^*(\underline{G}(u))$ such that $J = \ker(\operatorname{Ind}_{\underline{G}(u)}^{\underline{G}} L)$.

The Main Result

Theorem (Ionescu & W, 2013)

Let $p : \mathscr{B} \to \underline{G}$ be a Fell bundle over a locally compact groupoid with Haar system and let $A = \Gamma_0(\underline{G}^{(0)}, \mathscr{B}|_{\underline{G}^{(0)}})$ be the associated C^*-algebra. Let $P \in \operatorname{Prim} A$. Then $\underline{G}_P \subset \underline{G}(u)$ for a unique $u \in \underline{G}^{(0)}$. Suppose that L is an irreducible representation of $C^*(\underline{G}_P, \mathscr{B}|_{\underline{G}_P})$ which is the integrated form of $\pi : \mathscr{B}|_{\underline{G}_P} \to B(\mathcal{H})$ with $\pi|_{A(u)}$ homogeneous with kernel P. Then $\operatorname{Ind}_{\underline{G}_P}^{\underline{G}} L$ is irreducible.

Remark

Just as in the crossed product case, the homogeneity condition is satisfied automatically if A is type I (or more generally if points in $\operatorname{Prim} A$ are locally closed). A true Effros-Hahn result is just a bit out of reach. So far.

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