Exercises for Index theory II

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To be discussed: 15.05.14

Exercise 1. Let $V \to X$ be a complex vector bundle of rank n, and $L \to Y$ be a line bundle, with first Chern class $c_1(L) = x \in H^2(Y)$. Prove

$$c_1(\Lambda^n V) = c_1(V);\ c_n(L \boxtimes V) = \sum_{p=0}^{n} x^{n-p} \times c_p(V) \in H^{2n}(Y \times X).$$

Remark: reduce this problem to a computation with invariant polynomials.

Exercise 2. Let G be a Lie group and $g \in G$. Let $c_g : G \to G$ be the automorphism given by conjugation with g; $c_g(h) = ghg^{-1}$. Prove that the induced map $Bc_g : BG \to BG$ (determined up to homotopy) is homotopic to the identity. Hint: show that for each G-principal bundle $P \to X$, the bundle $P \times_{G,c_g} G \to X$ (interprete this notation!) is isomorphic to P, and apply this to the universal bundle $EG \to BG$.

The following exercises prove the following result, which will be important for us (and enters the proof of the general index formula).

Theorem. Let $q : E \to M$ be a smooth fibre bundle of closed manifolds (what you need is that E and M are closed and q is a submersion. Assume for simplicity that M is connected and (essential) that the Euler characteristic of the fibre $q^{-1}(x)$ is nonzero. Then the induced maps

$$q^* : H^*(M) \to H^*(E)$$

is injective.

The proof requires three steps: the case when q is a covering, the case when E and M are oriented, and the general case.

Exercise 3. Let $f : M \to N$ be a k-sheeted covering of closed manifolds. The *transfer* $\mathrm{trf}_f : H^p(M) \to H^p(N)$ is defined by the following procedure. Let $\omega \in \mathcal{A}^p(M)$ and let $U \subset N$ be such that $f^{-1}(U) = \coprod_{i=1}^k U_i$. Define $f_! : \mathcal{A}^p(M) \to \mathcal{A}^p(N)$ by

$$(f_!\omega)|_U = \sum_{i=1}^{k} (f|_{U_i})^{-1,*}\omega \in \mathcal{A}^p(U).$$

Prove that this is a well-defined chain map $\mathcal{A}^*(M) \to \mathcal{A}^*(N)$, which induces the transfer on cohomology. Show that

$$f_!(f^*(x)) = kx$$

for each $x \in H^*(N)$ and conclude that the induced map $f^* : H^*(N) \to H^*(M)$ is injective.

Exercise 4. Let E and M be closed oriented and $q : E \to M$ be a submersion (this is a fibre bundle, by Ehresmann's fibration lemma). Let M be connected and $F := q^{-1}(x)$. Assume that $\chi(F9 \neq 0$. Prove the Theorem under this assumption. Hints: let $T_v E := \ker(dq)$ be the *vertical tangent bundle*. Define the *transfer* $\mathrm{trf}_q : H^p(E) \to H^p(M)$ by

$$\mathrm{trf}_q(\omega) = q_!(e(T_v E)\omega).$$

Show that $\mathrm{trf}_f \circ f^* = \chi(F) \cdot \_$.

Exercise 5. Use the previous two exercises to show the theorem. Hint: orientation cover.