Exercises for Index theory II

Sheet 4 J. Ebert

To be discussed: 29.05.14

Exercise 1. Derive the *Newton formula* relating the Chern classes and the Chern character:

$$\sum_{k=0}^{n}(-1)^{n-k}s_{n-k}c_k = 0 \in I(U(n)).$$

This is done by setting $t = x_i$ in the defining identity

$$\prod_{i=1}^{n}(t+x_i) = \sum_{k=0}^{n} t^{n-k}\sigma_k(x_1,\ldots,x_n).$$

and summing over $i = 1,\ldots,n$.

Exercise 2. Let $V \to S^{2n}$ be a complex vector bundle, $n > 0$. Prove that

$$c_n(V) = \frac{(-1)^{n-1}}{(n-1)!}\mathrm{ch}_n(V) \in H^{2n}(S^{2n}).$$

Hint: you need to distinguish the cases $\mathrm{rank}(V) = n$; $\ldots < n$; $\ldots > n$. For $\mathrm{rank}(V) = n$, use the Newton formula. For $\mathrm{rank}(V) = n-r$, replace V by $V \oplus \mathbb{R}^r$. If $\mathrm{rank}(V) > n$, use that V has a section without zeroes to write $V = V_0 \oplus \mathbb{C}^k$ with $\mathrm{rank}(V_0) = n$ and reduce to the case of $\mathrm{rank}(V) = n$.

Exercise 3. Recall that $\int_{S^{2n}} \mathrm{ch}(V) \in \mathbb{Z}$ (a corollary of Bott periodicity). Use this fact (and the previous exercise, and Gauß-Bonnet) to deduce that TS^{2n} does not have the structure of a complex vector bundle if $n \geq 4$.

Exercise 4. The spheres are stably parallelizable, and therefore $p(TS^{2n}) = 1$. Use the relation between Euler and Pontrjagin classes to show that TS^{4k}, $k \geq 1$, does not have the structure of a complex vector bundle. Hint: a rank $2k$ complex vector bundle $V \to S^{4k}$ satisfies $p_k(V) = (-1)^k 2c_{2k}(V)$.

Exercise 5. Look up the proof that $T\mathbb{CP}^n \oplus \mathbb{C} \cong H^{n+1}$ (as usual, $H \to \mathbb{CP}^n$ is the dual tautological line bundle). You find a proof in Milnor-Stasheff's book and another proof in my bordism theory lecture notes, p. 31. Compute the Chern classes and Pontrjagin classes of $T\mathbb{CP}^n$. Hint: if $L \to X$ is a complex line bundle with Chern class $c-1(L) = x$, then $p_1(L) = x^2$. The relation between Chern classes and Pontrjagin classes of complex vector bundles can be found in Milnor-Stasheff, p. 177.

A very interesting class of manifolds are *algebraic hypersurfaces*. Let q be a homogeneous polynomial function $\mathbb{C}^{n+2} \to \mathbb{C}$ of degree d. Then q determines a section s_q of the line bundle $H^{\otimes d} \to \mathbb{CP}^{n+1}$ (hint: for each line ℓ, consider $s_q(\ell) = q|_\ell$). Assume that $s_q \pitchfork 0$ (such polynomials exist, but that is not the point here). Let $V = V_{n,d} := s_q^{-1}(0)$ (this is of course a complex manifold of dimension n and it can be shown that the diffeomorphism type of this manifold only depends on n and d). An important piece of information about these manifolds is the *Lefschetz hyperplane theorem* which asserts that the inclusion $V \to \mathbb{CP}^{n+1}$ is n-connected. The proof can be given using Morse theory and can be found in Milnor: Morse Theory.

Exercise 6. Show that the normal bundle of $V_{n,d}$ is $H|_{V_{n,d}}$. Let $z := c_1(H)|_{V_{n,d}} \in H^2(V_{n,d})$. Prove that

$$c(TV)(1 + dz) = (1 + z)^{n+2};\ p(TV)(1 + d^2z^2) = (1 + z^2)^{n+2}.$$

Moreover, compute that

$$\int_V z^n = d.$$

Hint: use Theorem 6.3.7 of the lecture notes for the first term.

Exercise 7. Specialize the previous formulae to the case $n = 1$ (so V is a connected Riemann surface). Prove that the genus of V is given by $g(V_{1,d}) = 1 + \frac{d}{2}(d - 3)$. Perform a sanity check for low values of d (this needs some Riemann surface theory).

Exercise 8. Compute the signature and the Euler characteristic of the hypersurfaces $V_{2,d}$ (assuming the Hirzebruch signature formula). Show that if d is even, then $c_1(TV)$ is an even multiple of z and that $\int_V p_1(TV)$ is divisible by 48 in this case.