Exercises for Index theory I

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Remark. We use the same notation for forms and their cohomology classes throughout, implicitly justifying that all operations are well-defined on the cohomological level.

Exercise 1. (The Künneth formula)
Let M and N be two manifolds, $p_N : M \times N \to N$ the projection onto N, p_M is defined analogously. Let $\omega \in \mathcal{A}^*(M)$ and $\eta \in \mathcal{A}^*(N)$. We define the exterior product

$$\omega \times \eta := p_M^* \omega \wedge p_N^* \eta.$$

Prove that $(\omega, \eta) \mapsto \omega \times \eta$ defines a map

$$\bigoplus_{p+q=k} H_c^p(M) \otimes H_c^q(N) \to H_c^k(M \times N).$$

Prove that this map is an isomorphism, for all manifolds M and N. Hints:

- Use the bootstrap lemma.
- First prove the result when $N = \mathbb{R}^n$.
- In a second step, generalize to arbitrary N.

Exercise 2. Let $\pi : V \to M$ be a vector bundle. Let $\mathcal{A}_{cv}^*$ be the complex of forms with vertically compact support, i.e. the complex of all forms ω such that $\pi : \operatorname{supp}(\omega) \to M$ is a proper map. Let $H_{cv}^*(V) := H^*(\mathcal{A}_{cv}^*)$. Prove:

a) If $W \to N$ is another bundle and $f : W \to V$ a bundle map (recall that f is fibrewise an isomorphism), then f induces a map $f^* : \mathcal{A}_{cv}^*(V) \to \mathcal{A}_{cv}^*(W)$.

b) If $M = U_0 \cup U_1$ is a union of two open subsets, then there is a Mayer-Vietoris sequence relating $H_{cv}^*(V)$, $H_{cv}^*(V|_{U_i})$ and $H_{cv}^*(V)|_{U_0 \cap U_1}$.

c) If f_i, $i = 0, 1$ are bundle maps which are homotopic (through bundle maps), then the induced homomorphisms f_0^* and f_1^* from part (1) are equal.

Exercise 3. Let M^m and N^n be oriented manifolds and $f : M \to N$ be a proper smooth map. We define $f_! : H^p(M) \to H^p(M)$ as the composition

$$H^p(M) \overset{D_M}{\to} (H_c^{m-p}(M))^* \overset{(f^*)^*}{\to} (H_c^{m-p}(N))^* \overset{(D_N)^{-1}}{\to} H^{n-m+p}(N).$$

Prove

a) $f_!$ satisfies $\int_N f_!(\omega) \wedge \eta = \int_M \omega \wedge f^*\eta$ for all compactly supported forms η on N

b) For all $\eta \in H^*(N)$, we have $f_!(\omega \wedge f^*\eta) = f_!(\omega) \wedge \eta$.