Exercises for Index theory I

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The purpose of this exercise sheet is that you familiarize yourself with some basic notions of Lie theory; these facts will be needed for the theory of characteristic classes. It is useful to consider the literature; I recommend the first pages of: Duistermaat, Kolk: "Lie groups", available on googlebooks and Sharpe: "Differential Geometry". First some definitions. A *Lie algebra* is a vector space $\mathfrak{g}$, together with a bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(X, Y) \mapsto [X, Y]$, the *bracket*, such that

$$[X, Y] = -[Y, X] \text{ and } [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

hold for all $X, Y, Z \in \mathfrak{g}$. The second identity is called *Jacobi identity*, and the first one is equivalent to $[X, X] = 0$, at least in characteristic $\neq 2$. The prime example of a Lie algebra is $\mathfrak{gl}(V)$; this is the space of all endomorphisms of the vector space V, with the commutator as bracket. It is useful to view $\mathrm{GL}(V) \subset \mathfrak{gl}(V)$ as an open subset, thus there is a canonical isomorphism $T_1 \mathrm{GL}(V) \cong \mathfrak{gl}(V)$. The following notations are fixed on the rest of this sheet.

Assumption. *Let G be a Lie group with multiplication map $\mu : G \times G \to G$ and unit $1 \in G$. We let $\mathfrak{g} = T_1 G$. If H is another Lie group, we let $\mathfrak{h} := T_1 H$. Let $\phi : G \to H$ be a smooth group homomorphism and $\varphi : \mathfrak{g} \to \mathfrak{h}$ be the derivative of ϕ at the identity.*

For each $g \in G$, we let $C_g : G \to G$, $h \mapsto ghg^{-1}$ be the conjugation map, which is smooth and satisfies $C_g(1) = 1$ and $C_g \circ C_h = C_{gh}$. The *adjoint representation* of G is the map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$; $\mathrm{Ad}(g) = D_1 C_g$. It is clear that Ad is a smooth group homomorphism $G \to \mathrm{GL}(\mathfrak{g})$. We define $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ as $\mathrm{ad} = D_1 \mathrm{Ad}$ and for $X, Y \in \mathfrak{g}$: $[X, Y] := \mathrm{ad}(X)(Y)$ or

$$[X, Y] = \mathrm{ad}(X)Y = \frac{d}{dt}|_{t=0}(\mathrm{Ad}(x_t)Y) = \frac{d}{dt}|_{t=0}\frac{d}{dt}|_{s=0}(x_t y_s x_t^{-1}).$$

The group G acts on $\mathfrak{g}$ by the adjoint representation and on $\mathfrak{h}$ by the composition $G \stackrel{\phi}{\to} H \stackrel{\mathrm{Ad}_H}{\to} \mathrm{GL}(\mathfrak{h})$.

Exercise 1. Prove that the derivative $\varphi : \mathfrak{g} \to \mathfrak{h}$ is G-equivariant, in other words: $\varphi(\mathrm{Ad}(g)X) = \mathrm{Ad}(\phi(g))\varphi(X)$. Hint: take a curve $x_t \in G$ be a curve with $x_0 = 1$ and $\frac{d}{dt}|_{t=0} x_t = X$. Derive from this by differentiating that $\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$, in other words, the derivative of a homomorphism preserves the bracket.

Exercise 2. Let $G = \mathrm{GL}(V)$. Prove that $\mathrm{Ad}(g)X = gXg^{-1}$ and $\mathrm{ad}(X)Y = XY - YX$.

This proves easily that $\mathfrak{gl}(V)$, with the bracket defined above, is a Lie algebra. The proof that this is a Lie algebra for general G is not easy at all, and I refer to Duistermaat-Kolk for this fact. However, we are only interested in Lie groups which are *linear* in the sense that there is a vector space V and an injective homomorphism $G \to \mathrm{GL}(V)$. All Lie groups that are relevant for us are linear.

Exercise 3. Prove that $\mathfrak{g}$, with the commutator, is a Lie algebra, under the simplifying assumption that G is a linear Lie group. Hint: use the linearity to show that $[X, X] = 0$. Use the first exercise for $\phi : G \to \mathrm{GL}(V)$.

If M is a manifold and $\mathfrak{g}$ a Lie algebra, we can talk about the space $\mathcal{A}^p(M; \mathfrak{g})$ of p-forms with values in $\mathfrak{g}$. One can combine the wedge product and the Lie bracket:

$$[;] : \mathcal{A}^p(M; \mathfrak{g}) \otimes \mathcal{A}^q(M; \mathfrak{g}) \xrightarrow{\wedge} \mathcal{A}^{p+q}(M; \mathfrak{g} \otimes \mathfrak{g}) \xrightarrow{[,]} \mathcal{A}^{p+q}(M, \mathfrak{g});$$

more concretely, if ω, η are real valued forms and $X, Y \in \mathfrak{g}$, then $[\omega \otimes X, \eta \otimes Y] := \omega \wedge \eta \otimes [X, Y]$.
Let $\mathfrak{g} = \mathfrak{gl}(V)$ and $X, Y \in \mathfrak{gl}(V)$, $\omega \in \mathcal{A}^p(M)$, $\eta \in \mathcal{A}^q(M)$. Then

$$[\omega \otimes X, \eta \otimes Y] = \omega \wedge \eta \otimes XY - \omega \wedge \eta \otimes YX = (\omega \otimes X) \wedge (\eta \otimes Y) - (-1)^{pq} \eta \wedge \omega \otimes YX = (\omega \otimes X) \wedge (\eta \otimes Y) + (-1)^{pq+1} (\eta \otimes Y) \wedge (\omega \otimes X).$$

In other words, if $\omega \in \mathcal{A}^p(M, \mathfrak{gl}(V))$ and $\eta \in \mathcal{A}^q(M, \mathfrak{gl}(V))$, we find that

$$[\omega, \eta] = \omega \wedge \eta - (-1)^{pq} \eta \wedge \omega, \tag{1}$$

where $\wedge$ denotes the combination of the wedge product and the matrix multiplication. In general, one can prove easily that $\mathcal{A}^p(M, \mathfrak{g})$ has the structure of a *differential graded Lie algebra*: Let $\omega \in \mathcal{A}^p(M; \mathfrak{g})$, $\eta \in \mathcal{A}^q(M; \mathfrak{g})$, and $\zeta \in \mathcal{A}^r(M; \mathfrak{g})$. Then

a) $d[\omega, \eta] = [d\omega, \eta] + (-1)^p [\omega, d\eta]$,

b) $[\omega, \eta] = (-1)^{pq+1} [\eta, \omega]$,

c) $(-1)^{pr} [[\omega, \eta], \zeta] + (-1)^{qp} [[\eta, \zeta], \omega] + (-1)^{rq} [[\zeta, \omega], \eta] = 0$.

A Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{h}$ induces, in an obvious manner, a map $\varphi_* : \mathcal{A}^*(M, \mathfrak{g}) \to \mathcal{A}^*(M, \mathfrak{h})$. Smooth maps $g : M \to G$ act on $\mathfrak{g}$-valued differential forms by the adjoint representation.

The mother of all Lie algebra valued forms is a canonical 1-form that exists on every Lie group.

Definition. *Let G be a Lie group and $\pi : TG \to G$ be the tangent bundle. By R_g, L_g, we denote the left and right translations by $g \in G$. The maps $TG \to G \times \mathfrak{g}$, $v \mapsto (\pi(v), L_{\pi(v)^{-1}*}v)$ and $G \times \mathfrak{g} \to TG$, $(g, x) \mapsto L_{g*}x$ are two mutually inverse bundle isomorphisms. The* 1*-form $\omega_G \in \mathcal{A}^1(G; \mathfrak{g})$, $v \mapsto L_{\pi(v)^{-1}*}v$ is called* Maurer-Cartan-form.

Exercise 4. Let G be a linear group, and $\phi : G \to \mathrm{GL}(V)$ be an injective group homomorphism. We denote the function $\phi : G \to \mathrm{End}(V)$ by the letter g. Prove that $\omega_G = g^{-1}dg$.

Exercise 5. Let G and H be Lie groups and $\phi : G \to H$ be a homomorphism with derivative $\varphi : \mathfrak{g} \to \mathfrak{h}$. Let $\mu : G \times G \to G$ be the multiplication. Assume that H and G are linear. Prove:

a) For all $g \in G$: $L_g^*\omega_G = \omega_G$; $R_g^*\omega_G = \mathrm{Ad}(g^{-1})\omega_G$.

b) $\phi^*\omega_H = \varphi_*\omega_G$.

c) $\mu^*\omega_G = p_2^*\omega_G + \mathrm{Ad}(p_2^{-1})p_1^*\omega_G$, where $p_i : G \times G \to G$ are the two projections.

d) (structural equation) $d\omega_G + \frac{1}{2}[\omega_G, \omega_G] = 0$.

e) $d(\mathrm{Ad}(g^{-1})\eta) = \mathrm{Ad}(g^{-1})d\eta - [\omega_G, \mathrm{Ad}(g^{-1}\eta]$.

What do these things say for the homomorphism $\exp : \mathbb{C} \to \mathbb{C}^\times$?