Exercises for Index theory I

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Exercise 1. Which of the following differential operators are elliptic (we take complex valued functions)?

a) $\Delta : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$, $\Delta f := -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} f$ (Laplace operator).

b) $P : C^\infty(\mathbb{R} \times \mathbb{R}^n) \to C^\infty(\mathbb{R} \times \mathbb{R}^n)$; $Pf = \frac{\partial^2}{\partial t^2} f - \Delta_x f$ (Wave operator).

c) $P : C^\infty(\mathbb{R} \times \mathbb{R}^n) \to C^\infty(\mathbb{R} \times \mathbb{R}^n)$; $Pf = \frac{\partial}{\partial t} f - \Delta_x f$ (Heat operator).

d) $\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} : C^\infty(\mathbb{C}) \to C^\infty(\mathbb{C})$, $\frac{\partial}{\partial z} f := \frac{1}{2}(\frac{\partial}{\partial x} f - i \frac{\partial}{\partial y} f)$, $\frac{\partial}{\partial \bar{z}} f := \frac{1}{2}(\frac{\partial}{\partial x} f + i \frac{\partial}{\partial y} f)$ ($\frac{\partial}{\partial \bar{z}}$ is the *Cauchy-Riemann operator*, and a function is holomorphic iff $\frac{\partial}{\partial \bar{z}} f = 0$).

e) If $\mu : \mathbb{C} \to \mathbb{C}$ is smooth and $|\mu(z)| < 1$, then $\frac{\partial}{\partial \bar{z}} + \mu \frac{\partial}{\partial z}$ is elliptic (this plays a crucial role in the theory of Riemann surfaces).

Exercise 2. Let M be a manifold and X a vector field. Recall the *insertion operator* $\iota_X : \mathcal{A}^p(M) \to \mathcal{A}^{p-1}(M)$, defined by $(\iota_X \omega)(X_1, \ldots, X_{p-1}) := \omega(X, X_1, \ldots X_{p-1})$ for vector fields X_i on M (This is an operator of order 0) and the operator $\epsilon_\eta : \mathcal{A}^p(M) \to \mathcal{A}^{p+1}(M)$, $\epsilon_\eta(\omega) := \eta \wedge \omega$ for a given 1-form η. It is a fact that $\iota_X(\omega \wedge \varphi) = (\iota_X \omega) \wedge \varphi + (-1)^{|\omega|} \omega \wedge \iota_X \varphi$. The *Lie derivative* is $L_X : \mathcal{A}^p \to \mathcal{A}^p$, $L_X = d\iota_X + \iota_X d$. Prove:

a) $\epsilon_\eta \iota_X + \iota_X \epsilon_\eta = \eta(X)$ (this is a purely linear algebraic identity).

b) L_X commutes with d and ι_X and satisfies $L_X(\omega \wedge \varphi) = (L_X \omega) \wedge \varphi + \omega \wedge L_X \varphi$. Compute the symbol of L_X.

c) Let M be a Riemannian manifold with volume form vol. For a vector field X on M, let $\mathrm{div}(X)$ be the unique function such that $\mathrm{div}(X)\mathrm{vol} = L_X \mathrm{vol}$. Prove that for $M = \mathbb{R}^n$ with the standard metric, you get the classical divergence operator.

Exercise 3. Denote by $\mathbb{C}[\xi_1, \ldots, \xi_n]^{\leq k} \subset \mathbb{C}[\xi_1, \ldots, \xi_n]$ the space of polynomials of degree $\leq k$. Let $U \subset \mathbb{R}^n$ be open. Consider the vector space $C^\infty(U; \mathbb{C}[\xi_1, \ldots, \xi]^{\leq k} \otimes \mathrm{Mat}_{q,p}(\mathbb{C}))$. Elements in this vector space are functions $p(x, \xi)$ which are matrix-valued, smooth in the x-variable and polynomial of degree $\leq k$ in the ξ-variables. Prove that this is isomorphic to the space of differential operators $D : C^\infty(U; \mathbb{C}^p) \to C^\infty(U; \mathbb{C}^q)$ of order k; the isomorphism given by sending $\xi_j \mapsto D^j := (-\sqrt{-1}) \frac{\partial}{\partial x_j}$.