Exercises for Index theory I

Sheet 4 J. Ebert / W. Gollinger

Deadline: 15.11.2013

Exercise 1. Give a formula for the adjoint of a differential operator $P : C^\infty(U;\mathbb{C}^{p_0}) \to C^\infty(U;\mathbb{C}^{p_1})$. More precisely, assume that there is *some* Riemann metric on U. The volume form on U can then be written as $a(x)dx_1 \wedge \ldots \wedge dx_n$ for a smooth function $a(x) > 0$. The vector bundles $\underline{\mathbb{C}^{p_i}} = U \times \mathbb{C}^{p_i}$ have the standard hermitian scalar product.

Exercise 2. Assume that $D_p : \mathcal{A}^p(M) \to \mathcal{A}^{p+1}(M)$ is a differential operator, which is given for any manifold. Assume that

a) $D(\omega \wedge \eta) = (D\omega \wedge \eta + (-1)^{|\omega|}\omega \wedge D\eta$, for all forms.

b) $D_0 : \mathcal{A}^0(M) \to \mathcal{A}^1(M)$ is the usual total differential.

c) If $f : M \to N$ is any local diffeomorphism (i.e, a map whose differential is everywhere regular), then $D(f^*\omega) = f^*D\omega$ (naturality).

Prove that $D = d$. Hint: it is enough to show that $D^2 = 0$, according to the characterization of the exterior derivative. To prove this property, use naturality to show that it is enough to consider $M = \mathbb{R}^n$. On $\mathbb{R}^n$, there are special diffeomorphisms: $T_x(y) := x + y$, $x \in \mathbb{R}^n$, and $H_a(y) := ay$, $a \in \mathbb{R} \setminus 0$. Use these diffeomorphisms to prove that D maps each form dx_I to zero.

Exercise 3. Let V be finite-dimensional real vector space with an inner product and let $(e_1, \ldots, e_n)$ be an orthonormal basis. Let e^i be the image of e_i under the musical isomorphism $\sharp : V \to V^*$. On the exterior algebra Λ^*V^*, we introduce an inner product (ω, η) by requiring that $(e^{i_1} \wedge \ldots \wedge e^{i_p})_{1 \le i_1 < \ldots < i_p \le n}$ is an orthonormal basis of Λ^pV^* and that Λ^pV^* and Λ^qV^* are orthogonal for $p \neq q$.

a) Show that this inner product does not depend on the choice of the orthonormal basis. Hint: a direct computation is cumbersome. Here is an alternative suggestion. First show that it is enough to prove the following: if $V = \mathbb{R}^n$ and the standard basis $(e_1, \ldots, e_n)$ is used for the inner product, then $(A^*\omega, A^*\eta) = (\omega, \eta)$ holds for all forms ω, η. Next, prove that for all $B \in \mathfrak{o}(n)$, the space of skew-symmetric matrices, one has $(B^*\omega, \eta) + (\omega, B^*\eta) = 0$. This is simpler, because $\mathfrak{o}(n)$ is a vector space and the relation is linear in all variables! Show that this implies $(\exp(tB)^*\omega, \exp(tB)^*\eta) =$

(ω, η), for all $t \in \mathbb{R}$ and $B \in \mathfrak{o}(n)$. Recall that the matrices $\exp(B)$, $B \in \mathfrak{o}(n)$ generate $SO(n)$. To get the relation for all all $A \in O(n)$, it is enough to consider a single element $T \in O(n) \setminus SO(n)$.

b) Moreover, prove that for $v \in V$, the adjoint of $\omega \mapsto v^\sharp \wedge \omega$ is given by the insertion operator ι_v. Hint: it is useful to use bases here, because we used them for the inner product

Let M be a Riemannian manifold. On the exterior algebra bundle, we use the metric induced from the Riemannian metric fibrewise. The adjoint of d is taken with respect to these metrics. Compute the symbols of d^* and $(d + d^*)^2$.
If $M = \mathbb{R}^n$ with the standard metric, we have the operator $(d + d^*)^2$ and the Laplace operator Δ, acting on functions with values in the exterior algebra of $\mathbb{R}^n$ (which is the same as forms). Show that $(d + d^*)^2 = \Delta$.