Exercises for Index theory I

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Exercise 1. (Invariance of Sobolev spaces under coordinate changes) Let $\phi : U \to U'$ be a diffeomorphism of open subsets of $\mathbb{R}^n$. Let $V \subset U$ be relatively compact and $\phi(V) =: V'$. Consider the induced map $\phi^* : C^\infty_{cpt}(V') \to C^\infty_{cpt}(V)$, $\phi^* f := f \circ \phi$. Prove that for all $k \in \mathbb{Z}$, there is a constant C such that for all $u \in C^\infty_{cpt}(V')$, one has $\|\phi^* u\| \leq C\|u\|$ holds. Hint: that V is relatively compact is essential. First treat the case of nonnegative k. The chain rule for higher derivatives goes under the name "Faá di Bruno's formula" (wikipedia). You need only a very rough qualitative version. The case of negative k is done by duality, but be aware that some care is needed for the application of duality, due to the compact support condition.

Exercise 2. Let M be a closed manifold and $E \to M$ be a complex vector bundle. Use the previous exercise to prove that the equivalence class of the norm $\|\|_k$ on $\Gamma(M, E)$ is independent of the choice of the atlases.

Exercise 3. Let V^n be an oriented real euclidean vector space of dimension n with volume form $\mathrm{vol} \in \Lambda^n V^*$. Prove that there is a unique linear map $* : \Lambda^p V^* \to \Lambda^{n-p} V^*$ (called the *star operator*) such that

$$\langle \omega, \eta \rangle \mathrm{vol} = \omega \wedge *\eta$$

holds for all forms ω, η (using the scalar product from sheet 4). Show moreover that

$$* * \omega = (-1)^{p(n-p)} \omega$$

holds for all $\omega \in \Lambda^p V^*$.

Exercise 4. (Back to geometry) Let M^n be an oriented Riemann manifold. We use the inner product on forms defined by the Riemannian metric. The star operator from the previous exercise induces a bundle homomorphism $* : \Lambda^p T^* M \to \Lambda^{n-p} T^* M$ satisfying the identities from that exercise. Prove that the adjoint of the exterior derivative $d^* : \mathcal{A}^p(M) \to \mathcal{A}^{p-1}(M)$ is given by

$$d^* = (-1)^{pn+n+1} * d * .$$

Hint: Stokes' Theorem. Prove moreover that $*$ commutes with $\Delta = (d + d^*)^2$ (this is a stupid calculation).