Exercises for Index theory I

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Exercise 1. Let M^{2m} be an oriented closed Riemann manifold, and we let $\mathcal{A}^*(M)$ be the space of *complex* valued forms. There are, as in the lecture, two involutions on $\mathcal{A}^*(M)$. Let $\omega \in \mathcal{A}^p(M)$. We put $I\omega := (-1)^p \omega$ and $\tau\omega := i^{p(p-1)+m} * \omega$. Further, $D = d + d^*$. Show that $D\tau = -\tau D$, $I\tau = \tau I$ (and $DI = -ID$ is already known).

a) Consider the graded elliptic operator (D, τ). If m is even, then $\operatorname{ind}(D, \tau) = \operatorname{sign}(M)$ as in the lecture. Prove that for odd m, $\operatorname{ind}(D, \tau) = 0$. Hint: let κ be the complex conjugation map on $\mathcal{A}^*(M)$ and observe that $D\kappa = \kappa D$ and $\tau\kappa = -\kappa\tau$. This extra symmetry forces the index to be zero!

b) Regardless of whether m is even or odd, the gradings I and τ commute, so we can decompose $\mathcal{A}^* = \mathcal{A}^{ev,+} \oplus \mathcal{A}^{ev,-} \oplus \mathcal{A}^{odd,+} \oplus \mathcal{A}^{odd,-}$ into *simultaneous* eigenspaces of τ and I. The operator D decomposes accordingly. Define two elliptic operators P and Q on M so that $\operatorname{ind}(P) + \operatorname{ind}(Q) = \chi(M)$ and $\operatorname{ind}(P) - \operatorname{ind}(Q) = \operatorname{sign}(M)$ (for m even) and $\operatorname{ind}(P) - \operatorname{ind}(Q) = 0$ (for m odd).

c) Show that on S^{2n}, there is a "canonical" elliptic operator of order 1 with index 1.

Exercise 2. You probably have heard about the fact that if a manifold M admits a vector field without zeroes, then the Euler number of M is 0 (we will prove this later, with other and arguably easier methods). This exercise gives an analytical proof of this result and begins with some linear algebra: Let V be a euclidean vector space. Recall that for each v, we defined operators ι_v and $\epsilon_v(\eta) := v^\flat \wedge \eta$ on $\Lambda^* V^*$. We use them to define operators $\lambda_v := \epsilon_v - \iota_v$ and $\rho_v := \epsilon_v + \iota_v$. Prove that

a) $\lambda_v \lambda_w + \lambda_w \lambda_v = -2\langle v, w\rangle$.

b) $\rho_v \rho_w + \rho_w \rho_v = 2\langle v, w\rangle$.

c) ρ_v and λ_w commute.

Recall that if M is a closed oriented Riemann manifold, then the symbol of $D = d + d^*$ is given by $\sigma_1(D)(\xi) = i\lambda_{\xi^\#}$. If X is any vector field on M, then we have an induced operator $R := \rho_X$. Show that $DR - RD$ is an operator of order 0. Moreover, prove that R anticommutes with the involution I from the previous exercise.

Now assume that X is a vector field without zeroes. We can therefore write

$$R^{-1}DR = D + Q$$

with Q an operator of order zero. Now observe that R switches even and odd forms and conclude that the index of the selfadjoint graded operator $(D; I)$ is zero (in the lecture, we saw that it is the Euler characteristic).

This proof is due to Atiyah: "Vector fields on manifolds", see his collected works.