Exercises for Index theory I

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Exercise 1. Let M be a compact complex manifold, $\dim_{\mathbb{C}} M = n$ and let $E \to M$ be a holomorphic vector bundle. Assume that E is equipped with a hermitian bundle metric h (there is no condition that the metric is "holomorphic" or something alike). There is a $\mathbb{C}$-antilinear isomorphism $\tau : E \to E^*$, $\tau(e)(e') := h(e, e')$ (the metric is complex linear in the second variable). Let $\bar{\star}_E : \mathcal{A}^{p,q}(M; E) \to \mathcal{A}^{n-p,n-q}(M; E^*)$ be defined by $\bar{\star}_E(\omega \otimes e) := \bar{\star}(\omega) \otimes \tau(e)$. Prove that the adjoint of $\bar{\partial}_E : \mathcal{A}^{p,q}(M; E) \to \mathcal{A}^{p,q+1}(M; E)$ is given by $-\bar{\star}_{E^*} \bar{\partial}_{E^*} \bar{\star}_E$. Hint: check out the literature on compact manifolds: Griffiths-Harris, Voisin, Wells.

Exercise 2. Let M and E as before. We let $H^p(M, E)$ be the pth cohomology of the elliptic complex $0 \to \mathcal{A}^{0,0}(M; E) \to \mathcal{A}^{0,1}(M; E) \to \ldots$. Prove the *Serre duality theorem*: there is a conjugate linear isomorphism $H^p(M; E) \cong H^{n-p}(M, \Lambda^{n,0} T^* M \otimes E^*)$. Hint: the operator $\bar{\star}_E$ gives a suitable isomorphisms of elliptic complexes. You might also need a general result on elliptic complexes which follows from the general Hodge theorem. Namely, if $\mathcal{E}$ is an elliptic complex, then the cohomologies of the "adjoint complex" are the same as the cohomologies of the original complex.

Exercise 3. Let V be a real vector bundle of rank n. Give precise statements and proofs of the following statements:

a) A reduction of the structural group of V to $O(n)$ is "the same as" a bundle metric.

b) A reduction of the structural group of V to $\mathrm{GL}_n(\mathbb{R})$ is "the same as" an orientation of V.

c) A reduction of the structural group of V to the group $G = \{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} ; A \in \mathrm{GL}_m(\mathbb{R}), C \in \mathrm{GL}_{n-m}(\mathbb{R}) \}$ is "the same as" a rank m subbundle.

d) A reduction of the structural group of V to the group $H = \{ \begin{pmatrix} 1 & B \\ 0 & C \end{pmatrix} ; C \in \mathrm{GL}_{n-1}(\mathbb{R}) \}$ is "the same as" a a section of V without zeroes.

Exercise 4. Prove that a local trivializations of a G-principal bundle is "the same" as a local section.