SEMINAR "DIFFERENTIAL FORMS IN ALGEBRAIC TOPOLOGY", SUMMER TERM 2010

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The principal method of algebraic topology is to associate with a geometric situation an *algebraic model* which helps to solve geometric problems. Finding good models is always a compromise between two extremes: if the algebraic picture is too close to the geometric reality, then it tends to be as complicated the original geometric situation. In particular, solving the algebraic problem might be as difficult as solving the geometric problem and nothing is gained. On the other hand, if the algebraic invariant is too easy to compute, the reason might be that it discards too much of the original geometry. Again, the algebraic picture fails to make the geometric problem accesible.

Experience shows that homology is a good compromise between these extremes. Another example of an algebraic model is given by the homotopy groups $\pi_n(X)$ of a space. There are two important theorems of Whitehead. The first one says that a map $f : X \to Y$ between connected CW-complexes is a homotopy equivalence if f induces an isomorphism of homotopy groups. The second one says a map $f : X \to Y$ induces an isomorphism in (integral) homology if and only if it induces an isomorphism on homotopy groups (at least if X and Y are simply connected). In a sense, the information given by homology and homotopy groups is equivalent.

Note that these results do not say that a space X is determined (up to homotopy equivalence) by its homotopy groups. It is also not true that the homotopy groups are determined by the homology groups or vice versa. Neither can we recover the homotopy class of a map $f : X \to Y$ from the effect on homotopy or homology. Another drawback is that the homotopy groups $\pi_*(X)$ are tremendously hard to compute. Even for such simple-looking spaces as $\mathbb{S}^2$, the group $\pi_k(\mathbb{S}^2)$ is only known for a finite number of dimensions k. In fact, *there is no finite, simply-connected CW-complex X such that all groups $\pi_n(X)$ are known.*

It was the ingenious insight by two mathematicians, Jean Pierre Serre (around 1953) and Dennis Sullivan (20 years later) that the situation is drastically simplified if one considers the *rational homotopy groups* $\pi_*(X) \otimes \mathbb{Q}$. The process of tensoring the homotopy groups with $\mathbb{Q}$ discards all torsion information and it is of course not true that a map $f : X \to Y$ between simply-connected CW-complexes inducing an isomorphism $\pi_*(X) \otimes \mathbb{Q} \to \pi_*(Y) \otimes \mathbb{Q}$ is an isomorphism.

But Serre discovered that we can save the other of Whiteheads theorems: given two simply-connected spaces X and Y and a map $f : X \to Y$, then $f : H_*(X; \mathbb{Q}) \to H_*(Y; \mathbb{Q})$ is an isomorphism if and only if $f : \pi_*(X) \otimes \mathbb{Q} \to \pi_*(Y) \otimes \mathbb{Q}$ is an isomorphism. Moreover, the groups $\pi_k(\mathbb{S}^n) \otimes \mathbb{Q}$ have a simple structure: they vanish unless $k = n$ or $k = 2n - 1$ and n even, in which case they are one-dimensional vector spaces.

Sullivan pushed these ideas further. There is a construction which associates to any (simply-connected) space X a space $X_{\mathbb{Q}}$ and a map $X \to X_{\mathbb{Q}}$ which induces an isomorphism in rational homology and such that $\pi_*(X_{\mathbb{Q}}) = \pi_*(X) \otimes \mathbb{Q}$; the rationalization of $\mathbb{Q}$. It turns out that the space $X_{\mathbb{Q}}$ is homotopy equivalent to a "twisted product" of Eilenberg Mac-Lane spaces $K(\pi_n(X_{\mathbb{Q}}); n)$. The twisting is described by the so called "Postnikov invariants".

For some spaces, the rational Postnikov tower can be computed by traditional methods of homotopy theory. In these cases the rational cohomology ring contains all information that is needed to determine the rational homotopy type. Examples include spheres, complex projective spaces, compact Lie groups and their classifying spaces; we will see both the computations and the abstract reason why this is possible ("formality"). For the vast majority of spaces, however, the cohomology algebra does not contain enough information and one needs new ideas.

The new idea, due to Sullivan, is to import a concept from differential geometry into homotopy theory. Let M be a manifolds and $\mathcal{A}^*(M)$ be the differential graded algebra (d.g.a.) of differential forms. The famous theorem of de Rham asserts that there is a natural isomorphism of algebras $H^*(\mathcal{A}^*(M)) \cong H^*(M; \mathbb{R})$.

There are three special features of differential forms. The first is that they are really local, which makes the proof of results like the Mayer-Vietoris sequence or Poincaré duality much easier. The second property is that they are sensitive to geometric information, for example symmetries (Lie group actions) or Kähler metrics.

The most important feature of differential forms, however, is that $\mathcal{A}^*(M)$ is commutative, whereas the singular cochains on a space $C^*(X)$ are not commutative.

To use differential forms in homotopy theory, two hurdles must be taken. First of all, not any space is a manifold. Surprisingly, this is not too much a serious issue. We could consider *simplicial smooth forms*. Such a simplicial p-form associates to any singular simplex $\sigma : \Delta^n \to X$ a smooth form $\omega_\sigma \in \mathcal{A}^p(\Delta^n)$. The different ω_σ have to be compatible and the collection of these simplicial smooth forms form a commutative d.g.a. that computes $H^*(X; \mathbb{R})$.

The other, more subtle hurdle is that we need a d.g.a. *over* $\mathbb{Q}$ that computes $H^*(X; \mathbb{Q})$ in order to come in contact with homotopy theory (in the above construction $X \mapsto X_{\mathbb{Q}}$, one cannot replace $\mathbb{Q}$ by $\mathbb{R}$. This problem is overcome by the introduction of *rational polynomial forms*, which form a commutative d.g.a. $\mathcal{A}^*_{PL}(X)$ such that $H^*(\mathcal{A}^*_{PL}(X)) \cong H^*(X; \mathbb{Q})$.

The next step is to simplify $\mathcal{A}^*_{PL}(X)$. The result is that there exists a *minimal model* $f : \mathcal{M}_X \to \mathcal{A}^*_{PL}(X)$. Here $\mathcal{M}_X$ is a "minimal" d.g.a (essentially, it is the product of a polynomial algebra with an exterior algebra and there is a condition on the differential) and f induces an isomorphism in cohomology. The minimal model is unique up to isomorphism.

In the next step, is turns out that the minimal model completely characterizes the rational homotopy type of X! Moreover, each minimal d.g.a is the minimal model of some space and we can also see all homotopy classes of maps $X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ in this algebraic model.

In short, the theory gives a completely algebraic picure of rational homotopy theory.

De Rham cohomology of smooth manifolds.

Talk 1. (Öznur Albayrak)

Background material from global analysis. This talk should give an overview of several topics from the theory of smooth manifolds: Vector fields, differential forms, exterior derivative d and the equation $d \circ d = 0$, Lie derivative, insertion operators and the infinitesimal homotopy formula, wedge products. Integration of differential forms and Stokes' theorem. This material is covered in [13], chapters 1.4, 2.1, 2.2, 3.2. Other recommendable sources are [17] and [10].

Prerequisites/Comments: The material is straighforward and only a little background knowledge on manifolds and multilinear algebra is necessary. Nevertheless this talk requires a good organization. Due to time reasons, you can only sketch the proofs. In order to give a convincing sketch of a proof, you have to separate the idea of the proof from routine calculations.

Talk 2. (Benjamin Schmidt)

The de Rham cohomology and de Rham's theorem. The de Rham complex; functoriality. Homotopy invariance of the de Rham cohomology (a short proof can be found in [10], p. 199 f.); Poincaré lemma. The de Rham homomorphism and the proof of the de Rham theorem, which asserts that $H^*_{dR}(M) \to H^*_{sing}(M;\mathbb{R})$ is an isomorphism. References: [2], section V.5, V.9 is probably the most accesible reference, but you can also follow the proof in [3] or [5]. ([13] is a bit clumsy; [17] offers a more conceptual proof in the realm of sheaf theory). For enthusiastic students: prove that the de Rham isomorphism is also an isomorphism of rings, i.e. the wedge product of forms corresponds to the cup product, e.g.[5]. If time permits, you can present the proof of Poincaré duality from [3], p.44 f.

Prerequisites: This talk is rather straightforward to prepare once you know singular cohomology theory and the contents of talk 1. Multiplicativity is more difficult, though.

Some applications.

Talk 3. (Viktoria Ozornova)

Cohomology of Lie groups and their homogeneous spaces. The first goal is to recall the definition of Lie groups, homogeneous spaces and the Lie algebra of a Lie group. This is covered in [17], chapter 3.

The second goal is to show how symmetries of a manifold (i.e, Lie groups acting on M) can be used to effectively cut down the size of the de Rham complex and allow explicit calculations. More precisely, if G is a connected compact Lie group that acts on a manifold M, then the inclusion $\mathcal{A}^*(M)^G \to \mathcal{A}^*(M)$ is a quasiisomorphism. This is shown in [2], section V.12, especially Theorem 12.3.

Then some special cases should be discussed: if G acts on itself by translations, then $\mathcal{A}^*(G)^G \cong \Lambda^*(\mathfrak{g}^\vee)$. Slightly more generally, if $H \subset G$ a closed subgroup, $M = G/H$, then $\mathcal{A}^*(G/H)^G$ is the *Chevalley-Eilenberg complex* $C^*(\mathfrak{g}, \mathfrak{h})$ that computes the Lie algebra cohomology ([14], p. 60). If moreover G/H is a symmetric space, then all invariant forms are closed, which simplifies the computations even further and leads to the isomorphism $H^*(G;\mathbb{R}) \cong \Lambda^*(\mathfrak{g}^\vee)^G$. The classical reference for this is [4]

Prerequisites/Comments: The first part should be a survey without proofs, but precise definitions and theorems. You should only present the material that you need later on. The second part is straightforward, however: do not get lost in routine calculations. Unfortunately, I did not find a coherent reference for the material of part 3, so you have to browse through the literature and discover some of this material for yourself.

Talk 4. The Chern-Weil isomorphism.(Lars Borutzky)

The purpose of this talk is the Chern-Weil theorem. Let $P \to M$ be a G-principal bundle on the manifold M. Then there is a natural ring homomorphism $CW_P : \mathrm{Sym}^*(\mathfrak{g}\vee)^G \to H^*(M)$, where $\mathrm{Sym}^*(\mathfrak{g}\vee)^G$ is the graded algebra of G-invariant polynomial functions on $\mathfrak{g}$. In the universal case $EG \to BG$ and if G is compact, then CW_{EG} is an isomorphism.

Topics: Connections in principal bundles and their curvature. The Weil algebra $W(\mathfrak{g})$ and the fundamental homomorphism $W^*(\mathfrak{g}) \to \mathcal{A}^*(P)$ induced by any connection on P. From this, one constructs the homomorphism CW. Reference: [13], 265-282, but take the definition of a connection from [5], p.46. Then: state the isomorphism theorem. Then either sketch the proof ([5], section 8 for one proof, [9] for another and [1] for a sketch of yet another proof) or give some credibility to the theorem by discussing examples ($G = U(n), SO(n), Sp(n)$, [5], chapter 7).

Prerequisites/Comments: you need some aquaintance with Lie groups, Lie algebras, smooth principal bundles and some virtuosity in multilinear algebra, either in coordinates or coordinate-free. Again, a good organization is required and the ability to give convincing sketches of computations. If you are ambitious and prove the isomorphism, this will be a difficult talk. Warning: there is one seemingly minor detail in the proof from [5] whose solution requires the whole structure theory of compact Lie groups. Don't waste your time with looking for your own simple proof of Proposition 8.3. loc. cit. The proof from [1] is easier, at least if you know spectral sequences and the results from talk 3; however, [1] only give a sketch.

From spaces to commutative differential graded algebras.

Talk 5. (Valentin Krasontovitsch)

Introduction to simplicial technology Simplicial objects in a category. The first aim of the talk is to explain the relationship between topological spaces and simplicial sets: the singular simplicial set functor goes from spaces to simplicial sets and the geometric realization functor goes in the other direction. These functors are adjoint; and the realization of the singular simplicial set of a space is homotopy equivalent to the space. One reference is the beginning of [12], but there are hundreds of other places in the literature. Then introduce the algebra of simplicial differential forms $\mathcal{A}^*_{PL}(X)$ on a simplicial set X. This is a functor from simplicial sets to differential graded algebras over $\mathbb{Q}$, [6], p.116-121.

Prerequisites: sympathy for category theory. If you don't know what a simplicial set is, this is a good opportunity to learn it.

Talk 6. (Ruth Joachimi)

The simplicial de Rham theorem The goal of this talk is the proof of the simplicial de Rham theorem. Also, the multiplicativity of both, the simplicial and the smooth de Rham homomorphism is proved.

Prerequisites: a detailed understanding of talk 5. Needless to say: the speaker of this talk has to communicate with the speaker of talk 5.

From commutative graded algebras to minimal commutative graded algebras.

Talk 7. (Søren Boldsen)

Homotopy theory of differential graded algebras I. Definitions, minimal d.g.a's. Definition of minimal models and construction of a minimal model. [14], 1.2.

Prerequisites/Comment: the arguments in this talk are entirely algebraic and rather elementary. In a technical sense, this is one of the easiest talks in this seminar. However, the challenge is to motivate the constructions and not to get lost in details of computations. Communicate with the speaker of talk 8

Talk 8. (N.N.)

Homotopy theory of differential graded algebras II. Homotopy of morphisms of d.g.a's and uniqueness of the minimal model up to isomorphism. Examples of minimal models for spaces like spheres, projective spaces...

Prerequisites/Comments: Same as for talk 7. Communicate with the speaker of talk 7.

Back to spaces.

Talk 9. (Nikolas Kulke)

An overview of homotopy theory. Fibrations and homotopy fibres, long exact homotopy sequence, Hurewicz and Whitehead theorems, Eilenberg-MacLane spaces $K(G,n)$ and the isomorphism $H^n(X,G) \cong [X, K(G,n)]$, elementary obstruction theory, Postnikov decomposition of a space. Quite a lot of this material is known from the lecture Topology III. You should cover the material which is surveyed in [14], p. 1-10. You can find more details in the relevant chapters of [8].

Prerequisites/Commment: if you do not already know these parts of homotopy theory, this is a good opportunity to learn them. This should be rather a survey talk, with few details of proofs (but qualified sketches)

Talk 10. (Roland Birth)

Rationalization of simply-connected spaces. The goal of this talk is to develop homotopy theory with coefficients in $\mathbb{Q}$ for *simply-connected* spaces, [14], p. 10-17, [8], ch. VII. Main results: $f : X \to Y$ a map of simply-connected CW-complexes. Then $f : H_*(X;\mathbb{Q}) \to H_*(Y;\mathbb{Q})$ is an isomorphism iff $f : \pi_*(X) \otimes \mathbb{Q} \to \pi_*(Y) \otimes \mathbb{Q}$ is an isomorphism (Whitehead-Serre). Definition of rational spaces, universal property and construction of the rationalization of a space.

Prerequisites/Comments: this relies heavily on the material from talk 9. Apart from that, the Leray-Serre spectral sequence is used again and again.

Talk 11. (Irakli Patchkoria)

The comparison theorem. To a simply-connected space X, we have can associate a minimal c.d.g.a $\mathcal{M}_X$, namely the minimal model of the algebra of simplicial differential forms. On the other hand, we get a Postnikov tower of rational spaces via homotopy theory (talk 10). The main goal of this talk is to show that both data contain the same information in the precise sense that $X \mapsto \mathcal{M}_X$ induces an equivalences of the homotopy category $\mathbf{Ho}\mathcal{TOP}_{\mathbb{Q}}$ of rational spaces with the category of minimal c.d.g.a's with homotopy classes as morphisms. As a consequence, one can show how to compute the rational homotopy groups of X out of the minimal model.

A good sketch of the proof can be found in [14], p.38-42. For more details, see [8], ch. XI or [6], ch. 15. There is another, more conceptual path to this result via the "spatial realization" functor [6], ch. 17 or [16], section 8, just in case you are interested.

Prerequisites/Comments: it is a good idea to start from the sketch from the sketch in [14]. You should concentrate on the simply-connected case throughout.

Examples and applications.

Talk 12. (Hanno Becker)

Examples. The goal of this talk is twofold. First you should present examples. Second goal: formality of d.g.a.'s. Browse through the literature [8], [6], [7], [14], [16] to choose the examples you prefer.

Talk 13. (N.N.)

What happens for non-simply-connected spaces? The title is self-explanatory. Sources: [16], [14], p. 43-49 (both references suppress a lot of details).

There are several points in the theory where the simple connectivity of the spaces was used in an essential way. However, some pieces of the theory can be saved if one concentrates on "nilpotent spaces", which lie between simply-connected and general spaces. A space X is nilpotent if its fundamental group is nilpotent in the sense of group theory and if it acts 'nilpotently" on the higher homotopy groups.

A consequence is that the theory of Postnikov towers can be refined to capture nilpotent spaces. Moreover, for any nilpotent group Γ, there is a rationalization $\Gamma \otimes \mathbb{Q}$.

Prerequisites/Comments: this will be a challenging talk. You'll need a good overview of the previous theory, especially talks 10, 9, 7, 7, 11. Apart from that, some pieces of the theory of nilpotent Lie groups and group cohomology is used. There seems to be an interesting relation of this theory to the results of [15] which might be interesting to look at.

References

[1] M. Atiyah, R. Bott: *Equivariant cohomology and the moment map*, Topology 23 (1984), 1-28.
[2] G. Bredon: *Topology and Geometry.*
[3] R. Bott, L. Tu: *Differential forms in algebraic topology*

[4] C. Chevalley, S. Eilenberg: *Cohomology theory of Lie groups and Lie algebras.* Trans. Amer. Math. Soc. 63, (1948). 85-124.
[5] J. Dupont: *Curvature and characteristic classes*
[6] Y. Felix, S. Halperin, J.-C. Thomas: *Rational homotopy theory.*
[7] Y. Felix, J. Oprea, D. Tanré: *Algebraic models in Geometry.*
[8] P. Griffiths, J. Morgan: *Rational homotopy theory and differential forms*
[9] Guillemin, Sternberg: *Supersymmetry and equivariant de Rham cohomology*
[10] K. Jänich: *Vektoranalysis*
[11] J. P. May: *A concise course in algebraic topology*
[12] J. P. May: *Simplicial objects in algebraic topology*
[13] S. Morita: *Geometry of differential forms*
[14] S. Morita: *Geometry of characteristic classes*
[15] K. Nomizu: *On the cohomology of compact homogeneous spaces nilpotent Lie groups*, Ann. Math. 59, (1954), p. 531-538.
[16] D. Sullivan: *Infinitesimal computations in topology*, Publ. IHES 47 (1977), 269-331 (1978).
[17] F. Warner: *Foundation of differentiable manifolds and Lie groups*

Note: all texts without bibliographic data are textbooks.