SEMINAR ON H-PRINCIPLES, SUMMER TERM 2019

CHRISTOPH BÖHM, JOHANNES EBERT, MICHAEL WIEMELER

- The seminar takes place Monday, 10–12, in SR 1C. 10-12. We begin on April 8th.

The Gromov-Phillips h-principle on open manifolds. In this first part of the seminar, we give discuss the Gromov-Phillips h-principle for differential relations on open manifolds.

Talk 1 (Immersed curves in the plane. Julian Kranz)**.** This talk is a warm-up before we delve into more technical material, and discussed the simplest special case of the h-principle: the Whitney-Graustein theorem on curves in the plane. The elementary proof, due to Whitney, is given in [1, Chapter 0].

Talk 2 (Jet bundles, differential relations and the formulation of the h-principle, Michael Wiemeler)**.** The goal of this talk is to formulate the Gromov-h-principle for open, invariant differential relations which is [5, Theorem 3.3]. Along the way, the necessary concepts should be introduced: Jet bundles provide a formalism to treat higher derivatives of maps between manifolds. This basic material is not treated in [5]; it can be found in e.g. [9, §2.4], [6, §II.2], [1, §1.3] or [4, Chapter 1]. The other ingredient for the formulation is the weak C^r-topology on mapping spaces. For detailed definitions and the relevant properties, you are referred to [9, §2.1], [6, §II.3].

Talk 3 (Applications of the h-principle, Maximilian Tönies)**.** In this talk, some examples of open, invariant differential relations should be discussed, to illustrate some applications of the h-principle. Submersion theorem [5, §3.3.2], existence of symplectic structures [5, §3.3.3], existence of arbitrarily pinched metrics on open manifolds [16]. Other examples might be found in [8] or [14]. The case of *immersions* is more subtle since it also holds for closed manifolds and is done in a later talk.

Talk 4 (Proof of the h-principle I, Kevin Poljsak, Dennis Wulle oder Marcel Wunderlich)**.** The proof of the h-principle is divided into three talks, and the speakers need to communicate closely. We use [5, §3.4] as the main reference for the proof of the h-principle. Alternative sources for the same proof are [8], [14], [1], and we recommend to consult those as well. Some background material needs to be introduced: firstly Morse functions and handlebody decompositions ([5] refers to [10, VII 1.6 and VII 6.1] which is a good source), and secondly some methods from homotopy theory (homotopy groups, weak homotopy equivalences, Whiteheads theorem. Fibrations and fibre bundles and the long exact homotopy sequence). The strategy for the proof (induction over a handlebody decoposition) should be explained fairly early. The most interesting and most difficult part of the whole story is [5, Proposition 3.14], which should come in the third talk of this sequence. This is elementary, but very subtle ([14, p. 113] gives a good explanation of what is going wrong for closed manifolds), and it is here where the hypothesis that the manifolds have no closed components is used. *Do not rush over it*, but plan the whole sequence so that there is enough time to discuss this crucial step in detail.

Talk 5 (Proof of the h-principle II, Kevin Poljsak, Dennis Wulle oder Marcel Wunderlich)**.** See description of previous talk.

Talk 6 (Proof of the h-principle III, Kevin Poljsak, Dennis Wulle oder Marcel Wunderlich)**.** See description of previous talk.

Talk 7 (The Smale-Hirsch immersion theorem, Achim Krause)**.** The interesting feature of the immersion case is that it also holds for closed manifolds. Give the proof of the immersion theorem. References: [5, 3.3.1], [1, §3.9], [13, Theorem 8.4]. After the proof of the immersion theorem is

Date: March 26, 2019.

given, you have the opportunity to discuss the famous sphere eversion. There is a video on the web under the title "turning the sphere inside out", which you can present during the seminar. A systematic classification of the immersions from S^n to $\mathbb{R}^m$ in terms of homotopy groups of Stiefel manifolds using the immersion theorem was given by Smale [15]. A question you might like thinking about: in which other dimensions is the sphere eversable? Also, a proof of the *Whitney immersion theorem* that each n-manifolds admits an immersion into $\mathbb{R}^{2n-1}$ can be given by the h-principle.

H-principles on closed manifolds. In this part of the seminar, we follow the paper [11] which gives a uniform treatment of h-principles on closed manifolds.

Talk 8 (Microflexible sheaves and the fomulation of Kupers' h-principle, Johannes Ebert)**.** In this talk, a more abstract version of Gromov's h-principle should be introduced, which uses the terminology of *microflexible sheaves*. This abstract version is also due to Gromov [7, §2.2.1 and §2.2.3], in particular the theorem in the middle of p.79. It is sketched in [7, p. 76, Remarks A' and A"] how the h-principle for microflexible sheaves implies the h-principle for differential relations. The proof of the sheaf-theoretic h-principle won't be discussed in the seminar; it is parallel to the proof for differential relations. The formulation of the theory given in [11, §2.1, §3.1 and Theorem 28] might be easier to follow. The second goal of the talk is to give the formulation of Kupers's h-principle, which is [11, Theorem 27], and the conditions H and W are defined in [11, §3.2 and §3.3]. Remarks: [11] also talks about topological and PL-manifolds, which we will leave aside. Hence only talk about smooth manifolds. Hopefully, the reference [2] can provide some help. Keep in mind that now everyone in the audience is familiar with abstract homotopy-theoretic language.

Talk 9 (Application I: Vassiliev's theorem on functions with moderate singularities, Jens Gönner)**.** Vassiliev's Theorem is [17, "The first main theorem" on p. 74], which is reproven as a special case of [11, Corollary 58]. Indicate the derivation in [11, §5.2]. It might also be helpful to give some concrete examples such as those on [17, p. 74f. and §III.2], [11, Example 60] and [12, §4].

Talk 10 (Application II: Generalized Morse functions and the framed function theorem, Lukas Stöveken)**.** References: [11, §5.3 and §6]. Do not forget to define the notion of a generalized Morse function.

Talk 11 (Proof of Kupers' Theorem I, Felix Janssen oder Jannes Bantje)**.** The proof is given in [11, §4], and is based on the technique of *semisimplicial spaces*. An introductory reference for this theory is [3]. The first talk should present the general strategy for the proof (which in [11] is only revealed towards the end), and the second one should give more details.

Talk 12 (Proof of Kupers' Theorem II, Jannes Bantje oder Felix Janssen)**.** See description of the previous talk.

References

[1] Masahisa Adachi. *Embeddings and immersions*, volume 124 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1993. Translated from the 1984 Japanese original by Kiki Hudson.

[2] J. Ebert. Private communication.

[3] Johannes Ebert and Oscar Randal-Williams. Semi-simplicial spaces. *arXiv e-prints*, page arXiv:1705.03774, May 2017.

[4] Y. Eliashberg and N. Mishachev. *Introduction to the h-principle*, volume 48 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[5] Hansjörg Geiges. *h*-principles and flexibility in geometry. *Mem. Amer. Math. Soc.*, 164(779):viii+58, 2003.

[6] M. Golubitsky and V. Guillemin. *Stable mappings and their singularities*. Springer-Verlag, New York-Heidelberg, 1973. Graduate Texts in Mathematics, Vol. 14.

[7] Mikhael Gromov. *Partial differential relations*, volume 9 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1986.

[8] A. Haefliger. Lectures on the theorem of Gromov. pages 128–141. Lecture Notes in Math., Vol. 209, 1971.

[9] Morris W. Hirsch. *Differential topology*, volume 33 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1976 original.

[10] Antoni A. Kosinski. *Differential manifolds*, volume 138 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1993.

[11] Sander Kupers. Three applications of delooping to h-principles. *Geometriae Dedicata, https://doi.org/10.1007/s10711-018-0405-7*, 2019.
[12] Ib Madsen and Michael Weiss. The stable moduli space of Riemann surfaces: Mumford's conjecture. *Ann. of Math. (2)*, 165(3):843–941, 2007.
[13] Anthony Phillips. Submersions of open manifolds. *Topology*, 6:171–206, 1967.
[14] Valentin Poénaru. Homotopy theory and differentiable singularities. In *Manifolds–Amsterdam 1970 (Proc. Nuffic Summer School)*, Lecture Notes in Mathematics, Vol. 197, pages 106–132. Springer, Berlin, 1971.
[15] Stephen Smale. The classification of immersions of spheres in Euclidean spaces. *Ann. of Math. (2)*, 69:327–344, 1959.
[16] Manuel Streil. Pinching the sectional curvature on open manifolds. *J. Geom. Anal.*, 27(3):2224–2234, 2017.
[17] V. A. Vassiliev. *Complements of discriminants of smooth maps: topology and applications*, volume 98 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1992. Translated from the Russian by B. Goldfarb.

Mathematisches Institut, Universität Münster, Einsteinstrasse 62, 48149 Münster, Bundesrepublik Deutschland