ÜBUNGEN ZUR VORLESUNG TOPOLOGIE II

Aufgabenblatt 10
Abgabe: Mittwoch, 1.7., in der Vorlesung

Exercise 10.1. Compute the follwing tensor products:
(1) $\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Z}/m$,
(2) $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n$,
(3) $\mathbb{Q}/\mathbb{Z} \otimes \mathbb{Q}/\mathbb{Z}$,

Exercise 10.2. Let $(C_\bullet, d_C)$ and $(B_\bullet, d_B)$ be two chain complexes. The *tensor product* $C_\bullet \otimes B_\bullet$ is the following chain complex: The chain groups are $(C_\bullet \otimes B_\bullet)_n := \bigoplus_{p+q=n} C_p \otimes B_q$. For $c \in C_p$ and $b \in B_q$, let $d_{C\otimes B}(c \otimes b) := d_C(c) \otimes b + (-1)^p c \otimes d_B(b)$. Show that this defines indeed a chain complex.

Exercise 10.3. We say that a space X is of *finite homological type* if $\bigoplus_{n=0}^{\infty} H_n(X;\mathbb{Z})$ is finitely generated (in particular the Euler number is defined). Let X be a space and $U, V \subset X$ be open with $U \cup V = X$. Show: If $U, V, U \cap V$ are of finite homological type, then so is X and $\chi(X) = \chi(U) + \chi(V) - \chi(U \cap V)$.

Exercise 10.4. Let X, Y be finite CW-complexes. Show that $\chi(X \times Y) = \chi(X)\chi(Y)$. Let $f : Y \to X$ be a covering map of degree k. Show that $\chi(Y) = k\chi(X)$.

Exercise 10.5. Let X be a connected 1-dimensional finite CW-complex (often called a graph). Show that the fundamental group is a free group $F_n = \langle x_1, \ldots, x_n \rangle$ on n generators, where n is related to the Euler number of X by $\chi(X) = 1 - n$. Now let $G \subset F_n$ be a subgroup of index k. Show that G is a free group on $1 - k + kn$ generators. (Hint: the bouquet of n copies of $\mathbb{S}^1$ is a graph X with Euler number $1 - n$. Realize the inclusion $G \to F_n$ by a finite covering $Y \to X$ and use exercise 10.4)