http://www.math.uni-bonn.de/people/ebert/ Dr. Johannes Ebert
Sommersemester 2009

ÜBUNGEN ZUR VORLESUNG TOPOLOGIE II

Aufgabenblatt 12
Abgabe: Mittwoch, 15.7., in der Vorlesung

Exercise 12.1. Let N, M be modules over the principal ideal domain R. Show that there is an isomorphism $\mathrm{Tor}_1^R(M;N) \cong \mathrm{Tor}_1^R(N;M)$.

Exercise 12.2. Let R, S be two rings. We say that a covariant functor $F : R-\mathbf{Mod} \to S-\mathbf{Mod}$ is exact if for all R-modules N, M, the map $F : \mathrm{Hom}_R(M;N) \to \mathrm{Hom}_S(FM;FN)$ is a homomorphism of abelian groups. We say that F is *exact* if for any short exact sequence $0 \to M' \to M \to M'' \to 0$ of R-modules, the induced sequence $0 \to FM' \to FM \to FM'' \to 0$ is exact, too.

Let F be an exact functor and $C_\bullet$ be a chain complex in $R-\mathbf{Mod}$. Show that there is a natural isomorphism $H_*(F(C_\bullet)) \cong F(H_*(C_\bullet))$.

Exercise 12.3. Let (X,Y) be a space pair such that $H_*(X,Y;\mathbb{Z}/p) = 0$ for all prime numbers p and $H_*(X,Y;\mathbb{Q}) = 0$. Show that $H_*(X,Y;\mathbb{Z}) = 0$.

Exercise 12.4. Let $f : X \to Y$ be a map of spaces. The *mapping cylinder* Z_f of f is the topological space $Y \coprod (X \times [0,1]) / \sim$, where $\sim$ is the equivalence relation $(x,0) \sim f(x)$ for all $x \in X$. Show that there exists a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & Z_f \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where $g : X \to Z_f$ is the map $x \mapsto (x,1)$ and the right-hand-side vertical map is a homotopy equivalence. Let A be an abelian group. Show that $f_* : H_*(X;A) \to H_*(Y;A)$ is an isomorphism if and only if $H_*(Z_f, X; A) = 0$. Conclude with the help of exercise 9.3 that the following are equivalent:

(1) $f_* : H_*(X;\mathbb{Z}) \to H_*(Y;\mathbb{Z})$ is an isomorphism.
(2) $f_* : H_*(X;A) \to H_*(Y;A)$ is an isomorphism for $A = \mathbb{Z}/p$ (for all prime numbers p) and $A = \mathbb{Q}$.

Exercise 12.5. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules. Show that there exist free resolutions $F'_\bullet \to M'$, $F_\bullet \to M$, $F''_\bullet \to M''$ and a short exact sequence of chain complexes $0 \to F'_\bullet \to F_\bullet \to F''_\bullet \to 0$. Deduce that there is a long exact sequence $0 \to \mathrm{Hom}(M'', N) \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(M', N) \to \mathrm{Ext}(M'', N) \to \mathrm{Ext}(M, N) \to \mathrm{Ext}(M', N) \to \ldots$.