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# ÜBUNGEN ZUR VORLESUNG TOPOLOGIE II

Aufgabenblatt 3
Abgabe: Mittwoch, 6.5.2009 in der Vorlesung.

**Exercise 3.1.** Let $R$ be a ring and $0 \to V \xrightarrow{f} U \xrightarrow{g} W \to 0$ be a short exact sequence of left $R$-modules, i.e.: $\mathrm{Ker}(g) = \mathrm{Im}(f)$. Show that the following conditions are equivalent:

(1) There is a homomorphism of $R$-modules $s : W \to U$ such that $g \circ s = \mathrm{id}_W$.
(2) There is a homomorphism of $R$-modules $t : U \to V$ such that $t \circ f = \mathrm{id}_V$.
(3) There is an isomorphism of $R$-modules $\phi : U \to V \oplus W$ such that $\phi(f(v)) = (v, 0)$ und $g(\phi^{-1}(v, w)) = w$ for all $u \in U$, $v \in V$ und $w \in W$.

A short exact sequence that satisfies the above conditions is called *split-exact* or one says that the sequence *splits*.

Show furthermore: If $W$ is a free $R$-module, then any short exact sequence $0 \to V \to U \to W \to 0$ splits. What happens if $R$ is a field? Give an example of a short exact sequence of $\mathbb{Z}$-modules that is not split-exact.

**Exercise 3.2.** (The Five Lemma) Let $R$ be a ring and let

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\epsilon} \\ B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5 \end{array}$$

be a commutative diagram of left $R$-modules. Assume that both rows are exact (i.e., the rows are exact at $A_2$, $A_3$, $A_4$, $B_2$, $B_3$, $B_4$). Show the following implications:

- If $\beta$ and $\delta$ are surjective and $\epsilon$ is injective, then $\gamma$ is surjective.
- If $\beta$ and $\delta$ are injective and $\alpha$ is surjective, then $\gamma$ is injective.

Show, by examples, that none of the assumptions on $\delta$, $\alpha$, $\beta$ and $\epsilon$ can be removed.

Background: this is the important *Five Lemma*, which is used very often in homological algebra.

**Exercise 3.3.** This exercise deals with a detail of the proof of the homotopy-invariance of singular homology in the lecture. Let $n \geq 0$. Let $e_i \in \mathbb{R}^{n+2}$, $i = 0, \ldots n+1$ be the standard basis vector. We denote $v_i := e_i$, $i = 0, \ldots n$ and $w_i := e_i + e_{n+1}$, $i = 0, \ldots, n$. Show: the convex hull of $v_0, \ldots v_n, w_0 \ldots w_n$ is equal to $\Delta^n \times [0, 1]$. Show: the union of the $(n+1)$ different $(n+1)$-simplices $[v_0, w_0, \ldots w_n], [v_0, v_1, w_1 \ldots w_n] \ldots [v_0, \ldots v_n, w_n]$ is equal to $\Delta^n \times [0, 1]$ and the interiors of these $(n+1)$-simplices are disjoint.

**Exercise 3.4.** (Long exact homology sequence of a triple) Let $(X, Y, Z)$ be a triple of topological spaces, i.e., $X$ is a topological space and $Z \subset Y \subset X$ are subspaces. Show that there exists a long exact sequence of singular homology groups

$$\ldots \to H_{n+1}(X, Y) \xrightarrow{\delta} H_n(Y, Z) \to H_n(X, Z) \to H_n(X; Y) \xrightarrow{\delta} H_{n-1}(Y, Z) \ldots$$

Formulate and prove the statement that this sequence is natural with respect to the triple.

**Exercise 3.5.** (Retracts and reduced homology) Let $X$ be a topological space and $A \subset X$ be a retract, i.e., there exists a continuous map $r : X \to A$ such that $r|_A = \mathrm{id}_A$. Show that for each $n \in \mathbb{N}$, there is a split short exact sequence in homology:

$$0 \to H_n(A) \to H_n(X) \to H_n(X, A) \to 0.$$

A simple situation in which this statement can be applied is the following. Let $(X, x)$ be a pointed space. We define the *reduced homology* of $X$ by $\tilde{H}_n(X) := H_n(X, \{x\})$. Show the following things:

(1) Let $\epsilon : X \to *$ be the constant map. There is a natural isomorphism $\tilde{H}_n(X) \cong \operatorname{Ker} H_n(\epsilon)$. In particular, the reduced homology groups with respect to two different basepoints are naturally isomorphic. Also, we see that $\tilde{H}_n(X) \cong H_n(X)$ for all $n \geq 1$.

(2) Let $(X, A)$ be a pair of spaces and let $x \in A$ be a basepoint. Show that there is a long exact sequence relating the groups $\tilde{H}_*(X)$, $\tilde{H}_*(A)$ and $H_*(X, A)$.