http://www.math.uni-bonn.de/people/ebert/ Dr. Johannes Ebert
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ÜBUNGEN ZUR VORLESUNG TOPOLOGIE II

Aufgabenblatt 5
Abgabe: Mittwoch, 20.5.2009 in der Vorlesung.

Exercise 5.1. (The suspension isomorphism) Let X be a topological space. The *suspension* of X is the space $\Sigma X := [0,1] \times X/\sim$, where the equivalence relation $\sim$ is $(0,x) \sim (0,y)$ and $(1,x) \sim (1,y)$ for all $x, y, \in X$. Show that there is a natural isomorphism (for all $n \in \mathbb{N}$) $\tilde{H}_n(X) \cong H_{n+1}(\Sigma X)$.

Exercise 5.2. Let $f : \mathbb{S}^n \to \mathbb{S}^n$ be a continuous map without fixed points. Show that $\deg(f) = (-1)^{n+1}$. Show that any continuous map $\mathbb{RP}^{2n} \to \mathbb{RP}^{2n}$ has a fixed point.

Let n be odd. Construct a tangential vector field on $\mathbb{S}^n$ without zeroes and construct a map $\mathbb{RP}^n \to \mathbb{RP}^n$ without fixed points.

Exercise 5.3. The map $[0,1] \to \mathbb{S}^1 \subset \mathbb{C}$, $t \mapsto \exp(2\pi i t)$ can be viewed as a singular 1-chain $c \in C_1(\mathbb{S}^1)$. Show that c is a cycle and that its homology class $[c] \in H_1(\mathbb{S}^1) \cong \mathbb{Z}$ is a generator (i.e. it corresponds to ± 1 under the isomorphism $H_1(\mathbb{S}^1)$). Hint: follow the inductive proof of the isomorphism $H_1(\mathbb{S}^1) \cong \mathbb{Z}$ given in the lecture.

Conclude that the Hurewicz Homomorphism $\pi_1(\mathbb{S}^1) \to H_1(\mathbb{S}^1)$ is an isomorphism. Use this fact to compute the mapping degree of the map $f_k : \mathbb{S}^1 \to \mathbb{S}^1$, $f_k(z) = z^k$, $k \in \mathbb{Z}$.

Extend the first part of the exercise to higher-dimensional spheres. More precisely, then Δ^n be the standard simplex and let $\partial\Delta^n$ be the boundary of Δ^n, i.e. the set of convex combinations $\sum_{i=0}^n t_i e_i \in \mathbb{R}^{n+1}$ such that at least one of the numbers t_i is 0. Convince yourself that $\partial\Delta^n$ is homeomorphic to $\mathbb{S}^{n-1}$. Let $\iota_n \in C_n(\Delta^n)$ be identity viewed as a simplex. Show that

(1) ι_n represents a generator of $H_n(\Delta^n, \partial\Delta) \cong \mathbb{Z}$,
(2) The boundary of ι_n represents a generator of $H_{n-1}(\partial\Delta^n) \cong \mathbb{Z}$.

Hint: follow the inductive proof of these isomorphisms from the lecture.

Exercise 5.4. Let $\hat{\mathbb{C}}$ be the Riemann sphere, i.e. the one-point compactification of the complex plane $\mathbb{C}$. Recall that that $\hat{\mathbb{C}}$ is homeomorphic to $\mathbb{S}^2$. Let $p(z)$ be any polynomial in one complex variable. Show that $p : \mathbb{C} \to \mathbb{C}$ extends to a continuous map $\hat{p} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and compute the degree of $\hat{p}$. Use this to show the fundamental theorem of algebra.

More generally, let $p(z) = \frac{f(z)}{g(z)} = \frac{f_n z^n + f_{n-1} z^{n-1} + \ldots f_0}{z^m + g_{m-1} z^{m-1} + \ldots g_0}$ be a rational function, where $f_{n-1} \neq 0$ and f, g are coprime. Again, p extends to a map $\hat{p} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Compute the degree of $\hat{p}$. Hint: the answer is $\max\{m, n\}$.

Exercise 5.5. Let $X_0 \subset X_1 \subset X_2 \ldots$ be a sequence of spaces and let $X = \bigcup_{n \geq 0} X_n$. In other words, $n \mapsto X_n$ is a functor from $\mathbb{N}$ (viewed as a directed set) to **Top** and X is its colimit. Suppose that any compact subset of X is contained in one of the spaces X_n. Show that $H_*(X) \cong \operatorname{colim}_{\mathbb{N}} H_*(X_n)$. Show, by an example, that this conclusion does not hold without the assumption on compact subsets of X.