SEMINAR ON CHARACTERISTIC CLASSES

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Characteristic classes are cohomological invariants of vector bundles, or more general fibre bundles. More specifically, one associates with each vector bundle $V \to X$ a class $c(V) \in H^*(X)$, such that for each $f : Y \to X$, one has $c(f^*V) = f^*c(V)$. These cohomology classes play an important role in all topological problems that involve bundles. The first part of the seminar discusses the basic theory: we will construct the characteristic classes for complex, real and oriented real vector bundles. In the second part of the seminar, we will see the connection with the *Steenrod algebra*. The elements of the Steenrod algebra are certain natural transformations of $\pmod 2$-cohomology.

The last part of the seminar discusses the connection with the other part of algebraic topology that is related to vector bundles: K-Theory. Then the theory is used to discuss tThe J-homomorphism, a geometrically defined homomorphism

$$J : \pi_k(O) \to \pi_k^{st}$$

from the homotopy groups of the infinite orthogonal group to the stable homotopy groups of spheres, which is important for example in the classification of homotopy spheres. The homotopy of O is known (Bott periodicity), and Adams managed to compute the image of J, up to a slight ambiguity which depends on the "Adams conjecture", proven later by Quillen, Sullivan and others. The homotopy of O is

$$\pi_k(O) = \begin{cases} \mathbb{Z} & k \equiv 3 \pmod 4 \\ \mathbb{Z}/2 & k \equiv 0, 1, \pmod 8 \\ 0 & \text{otherwise.} \end{cases}$$

Adams proved that J is injective if the source group is $\mathbb{Z}/2$, and if the source group is $\mathbb{Z}$, then $J(\pi_{4k-1}O)$ is cyclic of order $\mathrm{den}(B_{2k}/2k)$, a Bernoulli denominator. That $\mathrm{Im}(J)$ is at least as big is done with characteristic classes, and the upper bound follows from the Adams conjecture. The last three talks are a big challenge!!

Sins of omission: we cannot discuss all aspects of characteristic classes. We leave out:

- Euler class and Euler numbers.
- The infinitesimal approach (Chern-Weil theory).
- Characteristic class and cobordism theory.
- Index formulas (Hirzebruch signature formula, Riemann-Roch)

Remark 1. Prerequisites: There has been a number of seminars at this institute in the last semesters which were about topics closely related:

(1) Lecture course on K-theory (Pennig)
(2) Seminar on classical homotopy theory (Bubenzer, Ebert, Knopf)
(3) Seminar on bordism theory (Joachim, Pennig)
(4) Lecture course on the Atiyah-Singer index theorem (Ebert)

and we won't pretend that these courses did not take place. Therefore, this is a fairly advanced seminar, and each talk assumes knowledge of some of the above courses. For each talk, the reference [21] is an essential reference.

Remark on the difficulty level of the talks.

(1) You have already learnt something and want to present these things in a condensed form: opt for 2, 10.

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(2) You have solid background in cohomology theory and want to understand things in detail: opt for 4, 5, 6, 7 or 8.
(3) You are willing to follow large parts of the seminar: 9, 11, 12.
(4) You seek a thrilling challenge: go for 13, 14, 15.

General theory of characteristic classes of vector bundles.

Talk 2. (Vector bundles and principal bundles) Vector bundles. Definition. Fundamental example: the tautological bundle over the Grassmann manifold. Frame bundle of a vector bundle. G-principal bundles in general. Pullback. Change of fibre. Linear Algebra for vector bundles. Literature: [11, §14.1, §14.2, §14.5]. The algebraic constructions are also explained in [20] Prerequisites: This talk is meant to be a survey. Therefore, some previous familiarity with bundle theory is necessary for efficient preparation of the talk.

Talk 3. (Classification of principal bundles) General philosophy of classifying spaces. Homotopy invariance of principal bundles. Milnor's construction of the universal bundle. Classification property. Give concrete models for the classifying spaces of $O(n)$ and $U(n)$ (Grassmann manifolds). Literature: [11, §14.3, §14.4]. Prerequisites: A good understanding of talk 2.

Talk 4. (The Thom isomorphism in cohomology, Markus Schmetkamp) Orientations of vector bundles. Cohomological orientation. The Thom class and Thom isomorphism theorem. Leray-Hirsch Theorem. Proof. [16]. Literature: This is nicely covered in [16, 4.D]. The relevant results are Theorems 4.D.1, 4.D.8, 4.D.9, 4.D.10. Prerequisites: Cohomology theory.

Talk 5. (Construction of characteristic classes, Georg Frenck) In this talk, the Euler classes, the Stiefel-Whitney and Chern classes are constructed. We follow an approach that is attributed to Dold and worked out in [19]. Begin with the definition of the Euler class [19, 2.4.10] and prove the main properties [19, 2.4.11], which follows quickly from the Thom isomorphism theorem. Do not forget that there are two cases: the oriented and the unoriented one. Then give the definition of the Chern and Stiefel-Whitney classes, as in [19, 3.1]. Theorem 3.1.8 loc.cit. summarizes what is usually known as "axioms" for the characteristic classes and Theorem 3.1.11 gives further properties. Give some computations, for example the characteristic classes of $T\mathbb{CP}^n$ and $T\mathbb{RP}^n$.

Prerequisites: Cohomology theory. Knowledge of the content of talk 4.

Talk 6. (The splitting principle and uniqueness of characteristic classes, Danial Sanusi) In this talk, it is shown that the characteristic classes are uniquely characterized by the axioms presented in the previous talk. This is important later on, when other constructions are introduced. The uniqueness theorem will be invoked to prove that both constructions agree. Literature: [19, Theorem 3.1.16]. Alternatively, you can use the relevant passages of [18, §17]. Prerequisites: Cohomology theory and familiarity with talks 3, 4, 5

Talk 7. (Cohomology of classifying spaces, Lukas Buggisch) The characteristic classes constructed so far can be viewed as cohomology classes of $BO(n)$ or $BU(n)$. In this talk, the cohomology of the classifying spaces $BU(n)$, $BO(n)$ and $BSO(n)$ with various coefficients is proven. Literature: the result is stated as Theorem 3.3.4 in [19]. Do not forget to introduce the Gysin sequence (if it did not happen before), the transfer for finite converings and the Pontrjagin classes.

Prerequisites: a good understanding of what happened in the seminar so far.

Relation to the Steenrod algebra.

Talk 8. (The Steenrod operations, Paul Bubenzer) This is a bit of an aside. The Steenrod operations are certain natural homomorphisms $Sq^i : H^n(X; \mathbb{Z}/2) \to H^{n+i}(X; \mathbb{Z}/2)$. Present the main properties, stated at the beginning [10, VI.15]. They are constructed using homological algebra [10], [9] or homotopy theory [16]. Also mention that there is an analog in $\mathbb{Z}/p$-coefficients for odd primes.

Prerequisites: Cohomology and the methods of acyclic models.

Talk 9. (Stiefel-Whitney classes versus Steenrod operations, Raphael Reinauer) Here, one proves that the Stiefel-Whitney classes can be defined using the Steenrod operations on the Thom space. Literature: [13, §8], [18, 17.9] [10, VI.17]. Keep in mind that some of these sources use the formula as a *definition* of Stiefel-Whitney classes and make sure not to cause confusion. A consequence is the Wu formula, which proves that the Stiefel-Whitney classes of a smooth manifold only depend on the homotopy type. Literature: [10, VI.17], [18, p. 275 f] [13, p. 130 ff]. An interesting consequence is the fact that orientable 3-manifolds are parallelizable. [13, Problem 12.B]

K-Theory and the J-homomorphism.

Talk 10. Recapitulation K-Theory. Bott periodicity theorem. Thom isomorphism in K-theory. Generalized cohomology theories. Prerequisites: K-Theory.

Talk 11. (The Chern-character and the Todd class, William Gollinger) The Chern-character of a complex vector bundle $V \to X$ is a certain cohomology class $\mathrm{ch}(V) \in H^*(X;\mathbb{Q})$, which is a power series in the Chern classes of V. The important properties are that $\mathrm{ch}(V \oplus W) = \mathrm{ch}(V) + \mathrm{ch}(W)$ and $\mathrm{ch}(V \otimes W) = \mathrm{ch}(V)\mathrm{ch}(W)$, so that the Chern character defines a ring homomorphism $\mathrm{ch} : K^*(X) \to H^*(X;\mathbb{Q})$ and it is proven that after tensoring with $\mathbb{Q}$, ch becomes an isomorphism. Literature: [17], p. 97 ff.

One could guess that the Chern character, applied to the K-Theory Thom class, gives the cohomology Thom class, but this is *false*. Instead, there is a correction term which is the *Todd class*, a power series in the Chern classes. This leads to some interesting divisibility theorems for characteristic classes of complex manifolds. [23]

Talk 12. (The J-homomorphism and the e-invariant, William Gollinger) Recapitulation: stable homotopy groups of spheres. Definition of the J-homomorphism. Homomorphism and stablity property. [17, p. 96f]. The rest of the talk is devoted to *lower bounds* for $\mathrm{Im}(J)$, i.e. proving that $Jx \neq 0$ for suitable $x \in \pi_k(O)$. This is done with the e-invariant, which is defined using the Chern character. Literature: [17], p. 99 ff.

Talk 13. (Bundle theoretic reinterpretation of the J-homomorphism, christoph Winges) To prove an upper bound for the J-homomorphism, one needs to show that for a certain number $m(2s)$ and each $x \in \pi_{4s-1}(O)$, we have $J(m(2s)x) = 0$ in the stable homotopy group. The first step to the solution is to "deloop the problem". Using the isomorphism $\pi_{4s-1}(O) \cong \tilde{KO}^0(S^{4s})$, one can consider x as a vector bundle on a sphere. The main result of this talk is to interprete the J-homomorphism as the process that associates with a vector bundle the unit sphere bundle, viewed as a spherical fibration. In precise analogy to the definition of K-theory, one takes the Grothendieck group of fibre homotopy classes of spherical fibrations. *There is no uniform notation for this group in the literature.* The key ingredient is Dold's theory of fibre homotopy equivalences [3], in particular Satz 1, p. 120. Also see [2, p. 398], [18, 16.4, 16.5, 16.6]

Talk 14. (The Adams conjecture I) Formulation of the Adams conjecture. Adams operations in K-theory. Effect of the Adams operations on $KO^0(S^n)$. Then explain why the Adams conjecture implies the upper bound for the J-homomorphism. This follows from the number-theoretic result [6, Theorem 2.7]. Dold's theorem $\pmod k$, [5].

Talk 15. (The Adams conjecture II) The proof follows the paper [4]. It depends crucially on the Becker-Gottlieb transfer [1]. Cover the material in [1] that is necessary for [4] (Note: [1] gives another, more involved proof of the Adams conjecture).

References

[1] Becker, Gottlieb: *The transfer and fibre bundles*
[2] K. Knapp: *Vektorbündel*
[3] A. Dold: *Über faserweise Homotopieäquivalenz von Faserräumen.*
[4] I. Dibag: *On the Adams conjecture*
[5] Adams: *On the groups* $\mathrm{Im}(J)$ *I*
[6] Adams: *On the groups* $\mathrm{Im}(J)$ *II*
[7] Adams: *On the groups* $\mathrm{Im}(J)$ *III*

[8] Adams: *On the groups* Im(J) *IV*
[9] Spanier: *Algebraic topology*
[10] Bredon: *Geometry and topology*
[11] Tom Dieck: *Algebraic topology*
[12] Stong: *Notes on cobordism theory*
[13] Milnor, Stasheff: *Characteristic classes*
[14] Bott, Tu: *Differential forms in algebraic topology*
[15] Steenrod, Epstein: *Cohomology operations*
[16] Hatcher: *Algebraic topology*
[17] Hatcher: *Vector bundles and K-theory*
[18] Husemoller: *Fibre bundles*
[19] Ebert: *Lectures on cobordism theory*
[20] Ebert: *Lectures on the Atiyah-Singer index theorem*
[21] Ebert: *private communication*
[22] Ebert: *Brown's proof of the Adams conjecture*
[23] Karoubi: *K-Theory. An introduction.*

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