

# DIVISIBILITY OF MILLER–MORITA–MUMFORD CLASSES OF SPIN SURFACE BUNDLES

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## Abstract

In this paper, we study the divisibility properties of characteristic classes of surface bundles with spin structures. It turns out that the divisibility of the characteristic classes in the non-spin case improves by a power of 2.

## 1. Introduction and statement of results

A surface bundle  $\pi : E \rightarrow B$  is an oriented fiber bundle, whose fiber is a connected oriented closed surface  $F$  and whose structural group is the group  $\text{Diff}^+(F)$  of all orientation-preserving diffeomorphisms of  $F$ , endowed with the Whitney- $C^\infty$ -topology. Let  $Q$  be the associated  $\text{Diff}^+(F)$ -principal bundle. The vertical tangent bundle,  $T_v E$ , is defined to be the oriented 2-dimensional real vector bundle  $Q \times_{\text{Diff}^+(F)} TF$  over  $E$ . The choice of a smoothly varying Riemannian metric on the fibers of  $\pi$  turns  $T_v E$  into a Riemannian vector bundle, alias complex line bundle, and the isomorphism class of this line bundle does not depend on the choice of the metric. The Miller–Morita–Mumford classes [14] of  $E$  are defined to be

$$\kappa_n(E) := \pi_!((c_1(T_v E))^{n+1}) \in H^{2n}(B; \mathbb{Z}), \quad (1)$$

where  $\pi_! : H^*(E) \rightarrow H^{*-2}(B)$  is the Gysin homomorphism. A spin structure on the surface bundle is a spin structure on  $T_v E$ , that is, a square-root  $S$  of  $T_v E$  as a complex line bundle. A spin surface bundle is a surface bundle with a choice of a spin structure. Now we are able to state our theorems.

PROPOSITION 1.1 *If  $E \rightarrow B$  is a spin surface bundle, then  $\kappa_{2n}(E)$  is divisible by  $2^{2n+1}$ .*

Let  $g \geq 2$ . On a closed surface  $F$  of genus  $g$ , there is a numerical invariant for spin structures  $\sigma$ , the Arf invariant  $\text{Arf}(\sigma) \in \{\pm 1\}$ . Two spin structures are conjugate under the action of the mapping class group  $\Gamma_g$  if and only if their Arf invariants agree (for this, see [11]). Let  $\Gamma_g^\epsilon$  be the subgroup fixing a chosen spin structure  $\sigma$  of Arf invariant  $\epsilon$  (this depends on the choice of such a  $\sigma$  only up to conjugation). There is a central  $\mathbb{Z}/2$ -extension  $\hat{\Gamma}_g^\epsilon \rightarrow \Gamma_g^\epsilon$ , such that the space  $B\hat{\Gamma}_g^\epsilon$  classifies isomorphism classes of spin surface bundles whose fiber has genus  $g$  and Arf invariant  $\epsilon$ . The  $n$ th universal Miller–Morita–Mumford class defines an element  $\kappa_n \in H^{2n}(B\hat{\Gamma}_g^\epsilon; \mathbb{Z})$ . We also need the universal surface bundle of genus 0 with spin structure. By a theorem of Smale [15], oriented  $\mathbb{S}^2$ -bundles are classified by  $B\text{SO}(3)$ , and it turns out that  $\mathbb{S}^2$ -bundles with spin structure are classified by the  $\mathbb{R}\mathbb{P}^\infty$ -fibration  $B\text{SU}(2)$  over  $B\text{SO}(3)$  [5]. We allow ourselves to write  $\Gamma_0 = \text{SO}(3)$  and  $\hat{\Gamma}_0^{+1} = \text{SU}(2)$  (the unique spin structure on  $\mathbb{S}^2$  has Arf invariant  $+1$ ).

Based on Harer's work on the homology of  $\Gamma_g^\epsilon$  [9], Bauer [3] showed that the homology group  $H_k(B\hat{\Gamma}_g^\epsilon; \mathbb{Z})$  does not depend on  $g$  or  $\epsilon$ , as long as  $g \geq 2k^2 + 6k + 3$ . We refer to these values of  $k$  as the *stable range*. Our next theorem concerns the optimality of the divisibility given by Proposition 1.1.

**THEOREM 1.2** *In the stable range, the class  $\kappa_{2n} \in H^{2n}(B\hat{\Gamma}_g^\epsilon; \mathbb{Z})$  is not divisible by any non-trivial multiple of  $2^{2n+1}$ .*

In the non-stable range, this theorem is obviously false. For example, if  $4n$  is larger than the cohomological dimension of the moduli space of genus  $g$  spin curves, then  $\kappa_{2n}$  is a torsion class, and thus divisible by an infinite number of integers.

For the odd classes  $\kappa_{2n-1}$ , we need to exclude subtle torsion phenomena.

**CONVENTION 1.3** If  $B$  is a space and  $x \in H^*(B, \mathbb{Z})$  and  $0 \neq a \in \mathbb{Z}$ , then the statement  *$x$  is divisible by  $a$*  means that the image of  $x$  in the free abelian group  $H_{\text{free}}^*(B; \mathbb{Z}) := H^*(B, \mathbb{Z})/T$  is divisible by  $a$  ( $T$  denotes the torsion subgroup).

Note that a class which is divisible in the naive sense is also divisible in the sense of Convention 1.3, but not conversely.

**THEOREM 1.4** *If the surface bundle  $\pi : E \rightarrow B$  has a spin structure, then  $\kappa_{2n-1}(\pi)$  is divisible by  $2^{2n} \text{den}(B_n/2n)$ .*

Here,  $B_n$  denotes the  $n$ th *Bernoulli number*, which is defined by the expansion

$$\text{td}(z) := \frac{z}{1 - e^{-z}} = 1 + \frac{z}{2} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{B_n}{(2n)!} z^{2n}; \quad (2)$$

here  $B_n$  is a rational positive number, and von Staudt's theorem gives the prime decomposition of its denominator. For a non-zero integer  $k$  and a prime  $p$ , denote the  $p$ -adic valuation of  $k$  by  $v_p(k)$ . With this notation, we have

$$\text{den}\left(\frac{B_n}{2n}\right) = \prod_{(p-1)|2n} p^{1+v_p(2n)}.$$

Note that this is always an even number.

The first divisibility theorem for the class  $\kappa_{2n-1}$  without the presence of a spin structure was obtained by Morita [14]. He showed that  $\kappa_{2n-1}$  is divisible by  $\text{den}(B_n/2n)$  (in the sense of Convention 1.3). More systematically, the divisibility for oriented surface bundles was studied by Galatius, Madsen and Tillmann in [7]. Their main result is given below.

**THEOREM 1.5** [7] *Let  $D_k$  be the maximal divisor of the class  $\kappa_k$ . Then  $D_{2n} = 2$  and  $D_{2n-1} = \text{den}(B_n/2n)$ .*

A link between the divisibility results of [14] and [7] is given by Akita [1]. He conjectured that the mod- $p$  reduction of  $\kappa_n$  vanishes if and only if  $n \equiv -1 \pmod{p-1}$  and gave some evidence for this. Akita's conjecture was also shown in [7].

The method for our treatment of  $\kappa_{2n}$  is completely different from the method in [7]. In the proof of Theorem 1.4 we use the same method as Morita, namely the Atiyah–Singer family index theorem. However, I was not able to show the optimality of the divisibility in Theorem 1.4.

This paper is a condensed version of parts of my Ph.D. Thesis [5] and I want to thank my advisor, Bödighheimer, for his support and patience during my time as a Ph.D. student.

## 2. Proofs

NOTATION For the proof of Proposition 1.1, Theorems 1.2 and 1.4, we assume the following situation. Let us assume that the surface bundle  $\pi : E \rightarrow B$  admits a spin structure, that is, there is a complex line bundle  $S \rightarrow E$  and an isomorphism  $S^{\otimes 2} \cong T_\nu E$ . We denote  $y := c_1(S)$ ; thus  $2y = c = c_1(T_\nu E)$ . Convention 1.3 only applies in the proof of Theorem 1.4.

*Proof of Proposition 1.1.* This is trivial:

$$\kappa_n = \pi_1(c^{n+1}) = 2^{n+1}\pi_1(y^{n+1}). \tag{3}$$

This argument also shows that  $\kappa_{2n-1}$  is divisible by  $2^{2n}$ , which is contained in Theorem 1.4.

An equivalent formulation of Theorem 1.2 is to say that for any integer  $p > 1$ , the reductions of  $\kappa_{2n}$  in  $H^{4n}(B\hat{\Gamma}_g^\epsilon; \mathbb{Z}/(2^{2n+1}p))$  is non-trivial (in the stable range). Thus, it is sufficient to construct a single example of a surface bundle with a spin structure such that the reductions of  $\kappa_{2n}$  as above is non-zero. In general, it is a difficult matter to construct non-trivial surface bundles of high fiber genus and to do computations with them. But the modern homotopy theory of moduli spaces offers a way out.

*Proof of Theorem 1.2.* Assume that we know a universal space  $X$ , maps  $\alpha_{g,\epsilon} : B\hat{\Gamma}_g^\epsilon \rightarrow X$ , which are homology equivalences in the stable range and cohomology classes  $b_n \in H^{2n}(X; \mathbb{Z})$  such that  $\alpha_{g,\epsilon}^*(b_n) = \kappa_n$  for all  $n > 0$ . Assume further that we have a spin surface bundle  $E \rightarrow B$  with fiber genus 0 and  $\kappa_n \neq 0 \in H^{4n}(B; \mathbb{Z}/(2^{2n+1}p))$  for all  $p > 1$ . It follows that  $b_{2n} \neq 0 \in H^{4n}(X; \mathbb{Z}/(2^{2n+1}p))$  and hence Theorem 1.2 follows.

The first candidate is the classifying space  $B\hat{\Gamma}_\infty^\epsilon$  of the infinite spin mapping class group. Unfortunately, there is no map  $B\hat{\Gamma}_g^\epsilon \rightarrow B\hat{\Gamma}_\infty^\epsilon$  because the infinite mapping class group is the colimit of mapping class groups of surfaces *with boundary* and we are considering closed surfaces.

We need a more sophisticated construction to treat closed surfaces. The spin version of the Madsen–Weiss theorem tells us the following. There exists a spectrum  $\mathbb{G}_{-2}^{\text{Spin}}$  with  $\pi_0(\mathbb{G}_{-2}^{\text{Spin}}) \cong \mathbb{Z} \oplus \mathbb{Z}/2$  and maps  $\alpha_{g,\epsilon} : B\hat{\Gamma}_g^\epsilon \rightarrow \Omega_{g-1,\epsilon}^\infty \mathbb{G}_{-2}^{\text{Spin}}$  into the  $(g-1, \epsilon)$ -component of the infinite loop space of the spectrum. These maps induce isomorphisms in integral homology in the stable range. There are two proofs of this result. The first one is a generalization of the arguments of [13] to the spin case and this is done in [6]. The second proof is contained in [8]. The most readable construction of the spectrum and the maps can be found in [8]. Moreover, there are cohomology classes  $b_n \in H^{2n}\Omega_{0,0}^\infty \mathbb{G}_{-2}^{\text{Spin}}$ , which are pulled back to the Miller–Morita–Mumford classes by the composition  $B\hat{\Gamma}_g^\epsilon \rightarrow \Omega_{0,0}^\infty \mathbb{G}_{-2}^{\text{Spin}}$  of the map  $\alpha_{g,\epsilon}$  with the translation map  $\Omega_{g-1,\epsilon}^\infty \mathbb{G}_{-2}^{\text{Spin}} \rightarrow \Omega_{0,0}^\infty \mathbb{G}_{-2}^{\text{Spin}}$ . Details can be found in [7].

Thus, all the conditions on the universal space  $X$  mentioned above are satisfied by the space  $\Omega_{0,0}^\infty \mathbb{G}_{-2}^{\text{Spin}}$ .

Our candidate for the bundle  $E \rightarrow B$  is the universal  $\mathbb{S}^2$ -bundle with spin structure. This is the sphere bundle in the universal 3-dimensional vector bundle on  $B \text{SU}(2) = B \text{Spin}(3)$ . The cohomology of  $B \text{Spin}(3)$  is well known:  $H^*(B \text{Spin}(3); \mathbb{Z}) = \mathbb{Z}[u]$ , where  $u \in H^4(B \text{Spin}(3))$  is the first Pontryagin class of the universal bundle.

The total space is easily described as well:  $E = E \text{Spin}(3) \times_{\text{Spin}(3)} \mathbb{S}^2 = E \text{Spin}(3) \times_{\text{Spin}(3)} \text{Spin}(3)/\text{Spin}(2) \simeq B \text{Spin}(2) \simeq \mathbb{C}\mathbb{P}^\infty$ . Because the Euler class of the universal vector bundle on  $B \text{Spin}(3)$  is zero, the associated Gysin sequence becomes

$$0 \longrightarrow H^p(B \text{Spin}(3)) \xrightarrow{\pi^*} H^p(B \text{Spin}(2)) \xrightarrow{\pi_!} H^{p-2}(B \text{Spin}(3)) \longrightarrow 0. \quad (4)$$

In order to identify the vertical tangent bundle, we observe that the unit tangent bundle of  $\mathbb{S}^2$  is diffeomorphic to  $\text{SO}(3)$ , with  $\text{Spin}(3)$  acting on it by the usual 2-fold covering. Thus, the unit vertical tangent bundle  $T$  of the universal sphere bundle is homotopy equivalent to  $E \text{Spin}(3) \times_{\text{Spin}(3)} \text{SO}(3) \simeq \mathbb{R}\mathbb{P}^\infty$ . It follows that  $T$  is isomorphic to either  $L^{\otimes 2}$  or  $L^{\otimes -2}$ , where  $L$  is the universal complex line bundle. This is because the unit sphere bundle of  $L^{\otimes k}$  is a  $K(\mathbb{Z}/k; 1)$ -space for any non-zero integer  $k$  and because any line bundle on  $\mathbb{C}\mathbb{P}^\infty$  is a tensor power of  $L$ . It follows that

$$\kappa_{2n} = \pi_!(c_1(L^{\otimes \pm 2})^{2n+1}) = \pm 2^{2n+1} \pi_!(c_1^{2n+1}) = \pm 2^{2n+1} u^n. \quad (5)$$

Thus the reduction of  $\kappa_{2n}$  in  $H^{4n}(B\hat{\Gamma}_g^\epsilon; \mathbb{Z}/(2^{2n+1}p))$  is non-trivial for all  $p > 1$ . This finishes the proof of Theorem 1.2.

It should be noted that the sign ambiguity in (5) depends on the identification of  $\mathbb{C}\mathbb{P}^\infty$  with the total space of the universal sphere bundle. In [5, p. 50], it is explained which choice leads to which sign.

A similar argument can be used to give a simple proof of the fact that in the oriented case,  $\kappa_{2n}$  is not divisible by any multiple of 2. This was proven in [7] using different methods. Consider the inclusion  $j : \mathbb{S}^1 \rightarrow \text{SO}(3)$  of a maximal torus. The universal  $\mathbb{S}^2$ -bundle is homotopy equivalent to  $Bj : \mathbb{C}\mathbb{P}^\infty \rightarrow B \text{SO}(3)$  [5, p. 49]. The Miller–Morita–Mumford classes of  $Bj$  are ([5] or also [12, p. 370])

$$\kappa_{2n} = 2(p_1)^n.$$

It remains to show that  $(p_1)^n \in H^{4n}(B \text{SO}(3); \mathbb{Z})$  is not divisible. This is because  $Bj^*(p_1) = z^2$ ;  $z \in H^2(\mathbb{C}\mathbb{P}^\infty)$  ([5] or also [12, p. 370]).

*Proof of Theorem 1.4.* Because the number  $\text{den}(B_n/2n)$  is an even number, Theorem 1.4 does not follow from the Morita relation, together with equation (3). Suppose first that the base space  $B$  of the spin surface bundle  $\pi : E \rightarrow B$  is compact. We can choose complex structures on the fibers of  $\pi$ . For any fiberwise holomorphic vector bundle  $V \rightarrow E$ , there is the twisted Cauchy–Riemann operator  $\bar{\partial}_V$ , whose index defines an element  $\text{ind}(\bar{\partial}_V) \in K^0(B)$ . The Atiyah–Singer family index theorem [2], or rather a corollary of it, asserts that

$$\text{ch}(\text{ind}(\bar{\partial}_V)) = \pi_!(\text{td}(c) \text{ch}(V)) \in H^*(B; \mathbb{Q}). \quad (6)$$

For example, it follows that

$$s_{2n-1}(\text{ind}(\bar{\partial}_1)) = (-1)^{n-1} \frac{B_n}{2n} \kappa_{2n-1}. \quad (7)$$

As usual,  $s_k$  denotes the integral Chern character class  $s_k = k! \text{ch}_k \in H^{2k}(BU; \mathbb{Z})$ . For the proof of Theorem 1.4, we study the index bundle of the operator  $\bar{\partial}_{1+S^*}$ . We compute

$$\text{ch}(\text{ind } \bar{\partial}_{1+S^*}) = \pi_!(\text{td}(2y)(1 + e^{-y})) = \pi_!(2 \text{td}(y)). \tag{8}$$

In particular,

$$s_{2n-1}(\text{ind } \bar{\partial}_{1+S^*}) = \frac{(-1)^{n+1} B_n}{2^{2n-1} 2n} \kappa_{2n-1}. \tag{9}$$

Because  $\text{num}(B_n/2n)$  and  $2^{2n-1} \text{den}(B_n/2n)$  are coprime by von Staudt’s theorem, this implies that  $2^{2n-1} \text{den}(B_n/2n)$  divides  $\kappa_{2n-1}$ . If we show that  $s_{2n-1}(\text{ind } \bar{\partial}_{1+S^*})$  is divisible by 2, then Theorem 1.4 follows for surface bundles on a compact base space.

It is clear that  $s_{2n-1}(\text{ind } \bar{\partial}_{1+S^*}) = s_{2n-1}(\text{ind } \bar{\partial}_1) + s_{2n-1}(\text{ind } \bar{\partial}_{S^*})$ . Equations (7) and (9) imply that

$$s_{2n-1}(\text{ind } \bar{\partial}_1) = 2^{2n-1} s_{2n-1}(\text{ind } \bar{\partial}_{1+S^*}). \tag{10}$$

Thus, 2 divides  $s_{2n-1}(\text{ind } \bar{\partial}_1)$ . Furthermore, note that

$$\text{ind}(\bar{\partial}_{S^*}) = \ker(\bar{\partial}_{S^*}) - \text{coker}(\bar{\partial}_{S^*}) = \ker(\bar{\partial}_{S^*}) - \ker(\bar{\partial}_{T^* \otimes S})^* = \ker(\bar{\partial}_{S^*}) - \ker(\bar{\partial}_{S^*})^* \tag{11}$$

by the Serre duality theorem, which is also true in the parametrized setting. Finally,  $s_{2n-1}(V^*) = -s_{2n-1}(V)$  for any (virtual) complex vector bundle, whence  $s_{2n-1}(\text{ind}(\bar{\partial}_{S^*})) = 2s_{2n-1}(\ker(\bar{\partial}_{S^*}))$ , which finishes the proof of Theorem 1.4 if the base space  $B$  is compact.

If  $B$  is a space of finite homological type (that is, all homology groups are finitely generated), then  $H_{\text{free}}^*(B; \mathbb{Z}) \cong \text{Hom}(H_*(B; \mathbb{Z}); \mathbb{Z})$ . Because any homology class has compact support, Theorem 1.4 also follows if  $B$  is of finite type. Finally, we argue that  $B\hat{\Gamma}_g^\epsilon$  is of finite homological type. This is because of the following. By the Leray–Serre spectral sequence for the central extension  $\mathbb{Z}/2 \rightarrow \hat{\Gamma}_g^\epsilon \rightarrow \Gamma_g^\epsilon$ , it suffices to show that  $B\Gamma_g^\epsilon$  is of finite homological type. Consider a torsion free subgroup  $\Delta$  of the full mapping class group  $\Gamma_g$  of finite index. Then there is a finite  $K(\Delta, 1)$ -complex. This is a result of Harvey and Ivanov; the proof can be found in [10]. It follows that  $\Delta$  is a group of type  $FP_\infty$  [4, pp. 199 f]. Thus  $\Gamma_g$  and  $\Gamma_g^\epsilon$  are of type  $FP_\infty$  [4, p. 197]. It follows that  $B\Gamma_g^\epsilon$  is of finite homological type [4, p. 198].

### 3. Concluding remarks

It would be nice to have an analogue of Theorem 1.2 for the odd classes, that is the statement that the divisibility bound in Theorem 1.4 is the best possible. Unfortunately, the method applied in [7] does not carry over to the spin case. The key point in their proof is the construction of finite cyclic group actions on surfaces  $G \curvearrowright M$ , such that the induced bundles  $EG \times_G M \rightarrow BG$  have non-trivial Miller–Morita–Mumford classes. In the spin case, this does not work. An action of a finite group on a surface generates a spin surface bundle if and only if the 2-Sylow-subgroups act freely [5, p. 45]. The whole problem is certainly a 2-primary problem and if  $G$  is cyclic of order  $2^r$ , then the condition means that  $G$  itself acts freely. But under this circumstance, all Miller–Morita–Mumford classes of the group action are trivial.

Another possibility would be to use Morita’s construction in [14], which was used by him to show the algebraic independence of the Miller–Morita–Mumford classes. It is not difficult to show that

his ‘ $m$ -construction’ on a spin surface bundle yields a spin surface bundle if  $m$  is odd. This can be used to show that  $\kappa_1$  for spin surface bundles is divisible by precisely  $2^2 \operatorname{den}(B_1/2) = 48$ . However, in higher dimensions the Miller–Morita–Mumford classes become much larger than needed for the proof of such an optimality theorem.

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