Why there cannot be a three-dimensional Madsen-Weiss theorem

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Let M be any oriented closed smooth n-manifold, let $B_M := B\operatorname{Diff}^+(M)$ and let $p : E_M \to B_M$ be the universal oriented M-bundle. Let $\mathrm{MTSO}(n) := \mathbb{T}\mathbf{h}(-L_n)$ be the Madsen-Tillmann spectrum, i.e., the Thom spectrum of the inverse $-L_n$ of the universal n-dimensional oriented vector bundle $L_n \to B\,\mathrm{SO}(n)$. There is a map $\alpha_M : B_M \to \Omega^\infty \mathrm{MTSO}(n)$, the *Madsen-Tillmann map* [4]. The importance of this construction is that all cohomology classes of B_M (alias characteristic classes of smooth M-bundles) which are derived from the vertical tangent bundle $\pi : T_v E_M \to E_M$ are induced from classes on $\Omega^\infty \mathrm{MTSO}(n)$ via the map α_M.

To be more precise, let us compute the rational cohomology of the unit component $\Omega^\infty_0 \mathrm{MTSO}(n)$. There is the Thom isomorphism

$$\tau : H^*\big(B\,\mathrm{SO}(n)\big) \cong H^{*-n}\big(\mathrm{MTSO}(n)\big)$$

and the isomorphism

$$s : \Lambda H^*_{>0}\big(\mathrm{MTSO}(n);\mathbb{Q}\big) \cong H^*\big(\Omega^\infty_0 \mathrm{MTSO}(n);\mathbb{Q}\big),$$

where Λ is the functor which associates to a graded $\mathbb{Q}$-vector space the free-graded commutative algebra generated by it. On the other hand

$$H^*\big(B\,\mathrm{SO}(2m+1);\mathbb{Q}\big) = \mathbb{Q}[p_1,\dots,p_m], \text{ and}$$
$$H^*\big(B\,\mathrm{SO}(2m)\big) = \mathbb{Q}[p_1,\dots,p_m,\chi]/(\chi^2 - p_m).$$

Given any $c \in H^*(B\,\mathrm{SO}(n))$, then

$$p_!\big(c(T_v E_M)\big) = \alpha_M^* s\tau(c).$$

Another source of characteristic classes of smooth fiber bundles, this time with values in the topological K-theory of the base space, is the index of natural differential elliptic operators. In this talk, we consider only self-adjoint operators. The case of general operators is parallel (and better known). Let D be a family of self-adjoint elliptic differential operators on the universal bundle $E_M \to B_M$. By [2], these data have an index $\operatorname{ind}(D) \in K^1(B_M)$. On the other hand, the Atiyah-Singer family index theorem holds and yields

$$\operatorname{ind}(D) = (p\circ\pi)_!(\mathrm{smb}_D)_{\mathrm{sa}},$$

where $(\mathrm{smb}_D)_{\mathrm{sa}} \in K^1(\mathbb{T}\mathbf{h}(T_v E_M))$ is the self-adjoint symbol class of D [1], and where $(p\circ\pi)_! : K^1(\mathbb{T}\mathbf{h}(T_v E_M)) \to K^1(B_M)$ is the umkehr map in K-theory, which is defined as the composition of the Thom isomorphism $K^1(\mathbb{T}\mathbf{h}(T_v E_M)) \cong K^1(\mathbb{T}\mathbf{h}(-T_v E_M))$ and the map $K^1(\mathbb{T}\mathbf{h}(-T_v E_M)) \to K^1(B_M)$ induced by Pontrjagin-Thom collapse.

If the operator D is natural then there exists an element $\sigma_D \in K^1(\mathbb{T}\mathbf{h}(L_n))$ such that σ_D maps to $(\mathrm{smb}_D)_{\mathrm{sa}}$ under the map $\mathbb{T}\mathbf{h}(T_v E_M) \to \mathbb{T}\mathbf{h}(L_n)$ which comes from the classifying map for the vertical tangent bundle. In this case

$$\operatorname{ind}(D) = \alpha_M^* \operatorname{th}^{-1}\sigma_D, \tag{1}$$

where $\operatorname{th} : K^1(\mathrm{MTSO}(n)) \to K^1(\mathbb{T}\mathbf{h}(L_n))$ is the Thom isomorphism.

Now let M be a $2m+1$-dimensional closed oriented manifold. The even signature operator [1] $D : \bigoplus_{p\geq 0} \mathcal{A}^{2p}(M) \to \bigoplus_{p\geq 0} \mathcal{A}^{2p}(M)$ is defined to be

$$D\phi = i^{m+1}(-1)^{p+1}(*d - d*)\phi$$

whenever $\phi \in \mathcal{A}^{2p}(M)$. It is a self-adjoint, elliptic differential operator, and it is natural. Furthermore, it is related to the Laplace-Beltrami operator on forms by $D^2 = \Delta$. Moreover

$$\ker(D) = \ker(\Delta) = \bigoplus_{p\geq 0} H^{2p}(M;\mathbb{C}) \tag{2}$$

by the Hodge theorem. Now choose a fiberwise smooth metric on the vertical tangent bundle of the universal M-bundle. The even signature operators on the fibers define a family of self-adjoint elliptic differential operators and hence we have an index $\operatorname{ind}(D) \in K^1(B_M)$. Here is our main result.

Theorem 1. [3] *For any odd-dimensional closed oriented manifold M, the family index of the even signature operator* $\operatorname{ind}(D) \in K^1(B_M)$ *is trivial.*

The proof is purely analytic (it uses spectral theory, Kuiper's theorem on the contractibility of the unitary group of a Hilbert space and, crucially, the constancy of the dimension of the kernel, which follows from equation 2.

Because the proof of the vanishing of the index is purely analytical, the Atiyah-Singer index theorem 1 allows us to draw topological conclusions. Apply the Chern character to equation 1. A routine calculation of characteristic classes shows that $\operatorname{ch}(\alpha_M^* \operatorname{th}^{-1} \sigma_D) = \alpha_M^* s\tau\mathcal{L}$, where $\sigma_D \in K^1(\mathbb{T}\mathbf{h}(L_n))$ is the universal symbol for the even signature operator and $\mathcal{L} \in H^{4*}(B\,\mathrm{SO}(2m+1);\mathbb{Q})$ is the Hirzebruch L-class. Therefore Theorem 1 implies:

Theorem 2. [3] *For any closed oriented $2m+1$-manifold M, the Madsen-Tillmann map $\alpha_M : B_M \to \Omega_0^\infty \mathrm{MTSO}(2m+1)$ annihilates*

$$s\tau\mathcal{L} \in H^{4*-2m-1}(\Omega_0^\infty \mathrm{MTSO}(2m+1);\mathbb{Q}).$$

If $m = 1$, then the k-th component $\mathcal{L}_k$ generates $H^{4k}(B\,\mathrm{SO}(3);\mathbb{Q})$. Therefore, the map $B_{M^3} \to \Omega_0^\infty \mathrm{MTSO}(3)$ induces the zero map in rational cohomology. This is in sharp contrast to the 2-dimensional case, where Madsen and Weiss [5] showed that $B_M \to \Omega_0^\infty \mathrm{MTSO}(2)$ induces an isomorphism in integral homology in degrees $* < g/2-1$, whenever M is a connected closed oriented surface of genus g.

References

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