

Pontrjagin–Thom maps and the homology of the moduli stack of stable curves

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Abstract We study the singular homology (with field coefficients) of the moduli stack $\overline{\mathfrak{M}}_{g,n}$ of stable n -pointed complex curves of genus g . Each irreducible boundary component of $\overline{\mathfrak{M}}_{g,n}$ determines via the Pontrjagin–Thom construction a map from $\overline{\mathfrak{M}}_{g,n}$ to a certain infinite loop space whose homology is well understood. We show that these maps are surjective on homology in a range of degrees proportional to the genus. This detects many new torsion classes in the homology of $\overline{\mathfrak{M}}_{g,n}$.

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1 Introduction

Let $\overline{\mathfrak{M}}_{g,n}$ denote the moduli stack of stable nodal complex curves of genus g with n labelled marked points; this is the Deligne–Mumford–Knudsen compactification of the moduli stack $\mathfrak{M}_{g,n}$ of smooth curves. This object plays a central role in Gromov–Witten theory, conformal field theory, and conjecturally in string topology. The rational cohomology of $\overline{\mathfrak{M}}_{g,n}$ and its tautological subalgebra have been extensively studied in the literature, and the structure of the tautological algebra is at least conjecturally known. However, the mod p (co)homology has received relatively little attention. Here the distinction between the *moduli stack* and the associated *coarse moduli space*

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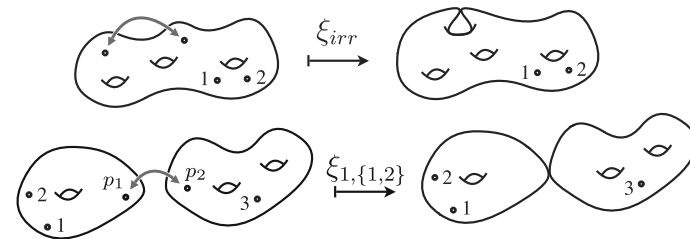
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becomes important because they are only *rationally* homology isomorphic. We take the point of view that the moduli stack is the more fundamental object.

Using the proof of the integrally refined Mumford conjecture by Madsen and Weiss [27], Galatius [9] completely computed the mod p homology of $\mathfrak{M}_{g,n}$ in the Harer–Ivanov stable range; there are large families of torsion classes. Here we address the question of torsion in the homology of the compactified moduli stack.

The boundary $\partial\overline{\mathfrak{M}}_{g,n} := \overline{\mathfrak{M}}_{g,n} \setminus \mathfrak{M}_{g,n}$ of $\overline{\mathfrak{M}}_{g,n}$ is a union of substacks of complex codimension 1. These irreducible boundary components are the images of the ‘gluing’ morphisms between moduli stacks defined by identifying two marked points together to form a node as shown below.



The gluing morphisms are:

$$\begin{aligned} \xi_{irr} : \overline{\mathfrak{M}}_{g-1,n+(2)} &\rightarrow \overline{\mathfrak{M}}_{g,n}, \\ \xi_{h,P} : \overline{\mathfrak{M}}_{h,P \sqcup \{p_1\}} \times \overline{\mathfrak{M}}_{g-h,Q \sqcup \{p_2\}} &\rightarrow \overline{\mathfrak{M}}_{g,P \sqcup Q}, \end{aligned} \tag{1.1}$$

where P, Q are finite sets, $\overline{\mathfrak{M}}_{g-1,n+(2)}$ is the moduli stack of stable curves with $n + 2$ marked points, the first n of which are labelled. These morphisms are representable proper immersions of complex codimension 1 and when P and Q are both nonempty $\xi_{h,P}$ is actually an embedding. These morphisms have transversal (self)-intersections and their images are precisely the various irreducible components of the Deligne–Mumford boundary $\partial\overline{\mathfrak{M}}_{g,n}$.

We study the effect on homology of the Pontrjagin–Thom collapse maps for these morphisms. We show that the self-intersections produce large families of torsion homology classes which are unrelated to the known torsion classes on $\mathfrak{M}_{g,n}$.

Recall that if $f : M^{n-k} \rightarrow N^n$ is a smooth proper embedding of codimension k then the classical Pontrjagin–Thom collapse map

$$N \rightarrow M^{\nu(f)},$$

from N to the Thom space of the normal bundle of M , is defined by collapsing everything outside of a tubular neighbourhood of the image of f to ∞ and identifying the tubular neighbourhood with the normal bundle of f . If f is proper but not an embedding then (by replacing f with an embedding into $N \times \mathbb{R}^l$) the same construction

produces a map

$$PT_f : N \rightarrow QM^{v(f)}, \tag{1.2}$$

where $QX = \Omega^\infty \Sigma^\infty X$ is the free infinite loop space generated by X . We refer to the map PT_f as the *Pontrjagin–Thom map for f* . In the appendix, we extend the classical construction of Pontrjagin–Thom maps to the category of differentiable local quotient stacks. A stack \mathfrak{X} admitting an atlas has an associated *homotopy type* $\text{Ho}(\mathfrak{X})$ (see Sect. 2) which is a space that has the same homological invariants as the stack, and the Pontrjagin–Thom construction produces a map out of the homotopy type.

A lift of the structure group of $v(f)$ along $G \xrightarrow{j} GL_k(\mathbb{R})$ induces a map

$$QM^{v(f)} \rightarrow QBG^{j^* \gamma_k}, \tag{1.3}$$

where γ_k is the universal k -plane bundle over $BGL_k(\mathbb{R})$. Thus we obtain a map

$$\Phi : N \rightarrow QBG^{j^* \gamma_k}$$

as the composition of (1.2) and (1.3).

Let $T(2) = U(1) \times U(1)$ denote the maximal torus in $U(2)$, and let $N(2) \cong \mathbb{Z}/2 \times U(1)^2$ denote the normalizer of the maximal torus. There are homomorphisms

$$T(2) \hookrightarrow N(2) \rightarrow U(1),$$

where the first arrow is the inclusion and the second is defined by multiplying the $U(1)$ components together; we write V for the universal line bundle over $BU(1)$ or its pullback to $BN(2)$ or $BT(2)$. The normal bundles of ξ_{irr} and $\xi_{h,P}$ come with preferred lifts of structure group to $N(2)$ and $T(2)$ respectively. Thus we have Pontrjagin–Thom maps

$$\begin{aligned} \Phi_{irr} &: \text{Ho}(\overline{\mathcal{M}}_{g,n}) \rightarrow QBN(2)^V, \\ \Phi_{h,P} &: \text{Ho}(\overline{\mathcal{M}}_{g,n}) \rightarrow QBT(2)^V, \\ \Phi'_{h,P} &: \text{Ho}(\overline{\mathcal{M}}_{g,n}) \rightarrow QBT(2)^V \rightarrow QBU(1)^V, \end{aligned}$$

where $\Phi'_{h,P}$ is the composition of $\Phi_{h,P}$ with the map induced by the multiplication $T(2) \rightarrow U(1)$. Our main theorem is the following.

Theorem 1.1 *Let g and n be fixed. Let \mathbb{F} be a field.*

- (i) *If the characteristic of \mathbb{F} is different from 2, then the map Φ_{irr} is surjective on $H_i(-, \mathbb{F})$ for $i \leq (2g/7 - 1)$.*
- (ii) *The map $\Phi_{h,\emptyset}$ is surjective on $H_i(-, \mathbb{F})$ for $i \leq (2h/3 - 1)$, $i \leq (2g/(4h+3) - 1)$ (the characteristic of \mathbb{F} is arbitrary).*
- (iii) *The map $\Phi'_{h,\emptyset}$ is surjective on $H_i(-, \mathbb{F})$ for $i \leq (2g/(4h + 3) - 1)$ (the characteristic of \mathbb{F} is arbitrary).*

This theorem detects large families of new torsion classes in the (co)homology of $\overline{\mathfrak{M}}_{g,n}$ as follows. Let Φ be one of the above maps. On cohomology with field coefficients the induced map Φ^* is injective in one of the above ranges of degrees.

Rationally The cohomology of $QBN(2)^V$ with coefficients in \mathbb{Q} is the free commutative algebra on generators $a_{i,j}$ ($i, j \geq 0$) of degree $2 + 2i + 4j$. In this case the image of Φ^* is contained in the tautological algebra; see Sect. 6. The nontriviality of these tautological classes in rational cohomology is probably well-known, though we could not find an explicit statement.

Mod p The mod p Betti numbers of $QBN(2)^V$ are much larger than the rational Betti numbers. If $\text{char}(\mathbb{F}) > 0$, then $H_*(QBN(2)^V; \mathbb{F})$ has a large and rich structure—it is the free graded-commutative algebra over the free Dyer–Lashof module generated by $\tilde{H}_*(BN(2)^V; \mathbb{F})$; see Sect. 5.4 and Appendix B for details. Hence this detects large families of new mod p cohomology classes of $\overline{\mathfrak{M}}_{g,n}$ which are not reductions of rationally nontrivial classes.

Remark 1.2 1. The precise range of degrees in which Theorem 1.1 applies depends on the range of homological stability for the mapping class group of surfaces and symmetric groups. The precise range for the latter ones has been known since Nakaoka. For the mapping class groups, we rely on the improved stability theorem recently shown by S. Boldsen in his thesis. The older results due to Harer and Ivanov give weaker bounds in Theorem 1.1.

2. The cohomology classes produced by the various maps of Theorem 1.1 are distinct in certain stable ranges. With a small modification of the proof, one can show for instance that

$$\Phi_{irr} \times \Phi_{h,\emptyset} : \overline{\mathfrak{M}}_{g,n} \rightarrow QBN(2)^V \times QBT(2)^V$$

is surjective on homology in a range of degrees, although this range will be approximately half the size of the surjectivity ranges for Φ_{irr} and $\Phi_{h,\emptyset}$ individually. See Sect. 5.6 for further details.

3. Note that the range of surjectivity is proportional to g in (i) and (iii) but not in (ii). On the other hand, the homology groups of the target in (ii) are somewhat larger than those of the target in (iii), so $\Phi_{h,\emptyset}$ detects more classes than $\Phi'_{h,\emptyset}$ but in a reduced range of degrees.
4. When $\emptyset \neq P \subsetneq \{1, \dots, n\}$ the gluing morphism $\xi_{h,P}$ is an embedding. Therefore its Pontrjagin–Thom map factors through $BT(2)^V \rightarrow QBT(2)^V$. The cohomology classes pulled back from $BT(2)^V$ all lie in the tautological ring of $\overline{\mathfrak{M}}_{g,n}$. However, one can also consider the quotient $\overline{\mathfrak{M}}_{g,n} // \Sigma_n$, where the symmetric group acts by permuting the labels of the marked points. Now the gluing morphism

$$\xi_{h,P} : \overline{\mathfrak{M}}_{h,P \sqcup \{p_1\}} // \Sigma_P \times \overline{\mathfrak{M}}_{g-h,Q \sqcup \{p_2\}} // \Sigma_Q \rightarrow \overline{\mathfrak{M}}_{g,P \sqcup Q} // \Sigma_{P \sqcup Q}$$

is an immersion with nontrivial self-intersections whenever $h < g/2$. In this case, one can easily adapt the proof of Theorem 1.1 to show for instance that the associated Pontrjagin–Thom map $\Phi'_{h,P}$ is surjective on homology in degrees $i \leq 2g/(4h + 3) - 1$, provided that $n \geq |P|2g/(4h + 3) - 1$.

5. Finally we mention that the restrictions of the Pontrjagin–Thom maps to the open moduli stack $\mathfrak{M}_{g,n}$ of smooth curves are nullhomotopic because the images of the natural morphisms (1.1) lie in $\partial\overline{\mathfrak{M}}_{g,n}$. Thus the torsion classes we detect are *independent* of the torsion classes on $\mathfrak{M}_{g,n}$ which were computed by Galatius [9].

1.1 The idea of the proof

Consider the gluing morphism ξ_{irr} and its associated Pontrjagin–Thom map $\Phi_{irr} : \overline{\mathfrak{M}}_{g,n} \rightarrow QBN(2)^V$. We show that there exists an open stratum $Z \subset \overline{\mathfrak{M}}_{g,n}$ with certain nice properties (an open stratum is a substack consisting of stable curves of a fixed homeomorphism type; only the top stratum is open as a substack of $\overline{\mathfrak{M}}_{g,n}$). Firstly, it lies in the image of ξ , so Φ_{irr} sends it into the zero section subspace $QBN(2)_+ \subset QBN(2)^V$ (this inclusion is surjective on homology). Secondly, by the Bödigheimer–Tillmann stripping-and-splitting theorem [4] (derived from Harer stability) and the homology stability of symmetric groups, $QBN(2)_+$ homologically splits off of Z in a range of degrees. Our theorem is proven by showing that the projection onto this factor in the partial homological splitting of Z agrees (up to homotopy) with the restriction of Φ_{irr} . The other flavours of the theorem are proven by making slightly different choices for Z .

1.2 Relation with the work of Eliashberg–Galatius

There is a certain overlap between Theorem 1.1 and unpublished work by Eliashberg and Galatius announced in [10], although our results were obtained independently. They have announced a determination of the homotopy type of the moduli stack of stable irreducible curves as the genus tends to infinity—in a stable range and with coefficients in a field of characteristic $\neq 2$ it splits as a tensor product of the cohomology of $QBN(2)^V$ and $\Omega^\infty MT SO(2)$. Their result should imply part (i) of Theorem 1.1. However, their methods do not distinguish between the different irreducible components of the boundary of the full Deligne–Mumford compactification, and therefore parts (ii) and (iii) of our theorem cannot be derived from their work. The reason is that [10] only makes use of the local structure of the singularities of stable curves, while ignoring the global combinatorial structure. We exploit the global combinatorics, giving rise to finer cohomological information.

In spirit, both their work and ours are based on mapping the moduli stack to an appropriate infinite loop space by Pontrjagin–Thom maps. Their map is a combination of the map $\mathfrak{M}_g \rightarrow \Omega^\infty MT SO(2)$ appearing in the proofs of the Mumford Conjecture (see [11, 26, 27]) and a Pontrjagin–Thom map for the whole Deligne–Mumford boundary. However, the techniques used to produce the Pontrjagin–Thom maps in our paper and theirs are quite different: in [10] they construct their map out of the local

differential topology of a family of stable curves, whereas we develop Pontrjagin–Thom maps for stacks in a general setting and then apply this global construction to the moduli stack.

1.3 Outline

In Sect. 2 we recall some material on stacks and explain the notion of the *homotopy type* of a topological stack (this section may be skipped if the reader is willing to pretend that the moduli stacks are manifolds). In Sect. 3 we discuss the Pontrjagin–Thom construction and what it means to extend it to local quotient stacks. Section 4 recalls some needed facts about the moduli stack $\overline{\mathcal{M}}_{g,n}$. In Sect. 5 we give the proof of our main theorem. Section 6 describes how the classes we detect rationally are related to the tautological algebra. Appendix A gives the details of the extension of Pontrjagin–Thom maps from manifolds to local quotient stacks. Appendix B recalls the computation of the homology of a free infinite loop space QX , which is needed in Sect. 5.

2 Some homotopy theory for topological stacks

In this section, we set up the homotopical framework in which the Pontrjagin–Thom maps for stacks will reside.

The reader who is not comfortable with stacks may wish to skip to Sect. 3.2 for our perspective on Pontrjagin–Thom maps and then proceed directly to Sect. 4 while pretending that the moduli stacks are simply manifolds.

2.1 Generalities on stacks

In this section, we will assume that the reader is comfortable with the language of stacks and therefore we will not repeat the basic definitions in detail. A stack over a site \mathbf{S} is a lax sheaf of groupoids over \mathbf{S} . We will consider the sites **diff** and **top** for smooth manifolds and topological spaces. The reader is referred to [15] and [28] for readable introductions to the theory of stacks over the sites **diff** and **top**.

On the site **diff** there is a subtlety in the definition of representable morphisms since one needs transversality for the pullback of two smooth maps to be a smooth manifold. We propose a definition which differs slightly from that given in [15].

- Definition 2.1** 1. A morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of stacks on the site **diff** is a *representable submersion* if for any manifold M and any morphism $M \rightarrow \mathcal{X}$, the fibre product $M \times_{\mathcal{Y}} \mathcal{X}$ is a smooth manifold and the induced map $M \times_{\mathcal{Y}} \mathcal{X} \rightarrow M$ is a submersion.
2. A morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of stacks over **diff** is *representable* if for any representable submersion $f : M \rightarrow \mathcal{Y}$, the pullback $M \times_{\mathcal{Y}} \mathcal{X}$ is a smooth manifold and the induced map $M \times_{\mathcal{Y}} \mathcal{X} \rightarrow M$ is a smooth map.

With this definition any smooth map between manifolds is representable when considered as a morphism of stacks and any morphism from a smooth manifold to a stack

over **diff** is representable. Let \mathfrak{X} be a stack over **diff**. An *atlas* is a smooth manifold X together with a representable surjective submersion $p : X \rightarrow \mathfrak{X}$ (i.e., for any map $Y \rightarrow \mathfrak{X}$, there exists an open covering $(U_i)_{i \in I}$ of Y such that the composition $U = \coprod_{i \in I} U_i \rightarrow Y \rightarrow \mathfrak{X}$ admits a lift through p). A stack which admits an atlas is called a *differentiable stack*.

Similarly, one can define topological stacks. We say that a representable morphism $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ of stacks over **top** has *local sections* if for any space Y and any map $Y \rightarrow \mathfrak{Y}$, the pullback $Y \times_{\mathfrak{Y}} \mathfrak{X} \rightarrow Y$ admits local sections (observe that maps which have local sections are surjective and having local sections is a property which is invariant under base-change). An atlas for a stack \mathfrak{X} over **top** is a space X together with a representable morphism $p : X \rightarrow \mathfrak{X}$ having local sections. A *topological stack* is a stack \mathfrak{X} over **top** which admits an atlas. Our terminology differs from that used by Noohi [28]: the topological stacks defined above are called “pretopological stacks” in [28] and his “topological stacks” satisfy a stronger condition.

We write $\text{STACKS}^{\mathbf{S}}$ for the category of stacks on a site **S** which admit an atlas. Note that $\text{STACKS}^{\mathbf{top}}$ contains the category of spaces as a full subcategory. A topological (or differentiable, respectively) stack is said to be a *Deligne–Mumford stack* if it has a proper étale atlas, i.e., there is an atlas $p : X \rightarrow \mathfrak{X}$ which is a local homeomorphism (local diffeomorphism, respectively) and the map $X \times_{\mathfrak{X}} X \rightarrow X \times X$ is proper (throughout the paper, we use Bourbaki’s definition of properness: a map $f : A \rightarrow B$ is proper if for any topological space C , the product $\text{id}_C \times f : C \times A \rightarrow C \times B$ is a closed map). A differentiable Deligne–Mumford stack is the same as an *orbifold*.

There is also the category $\text{STACKS}^{\mathbf{sch}}$ of algebraic stacks, studied in the book [20]. Moduli stacks of (stable) curves, which constitute the example of interest to us, are most conveniently described (and constructed) as algebraic stacks. There is a functor $\text{STACKS}^{\mathbf{sch}} \rightarrow \text{STACKS}^{\mathbf{top}}$ which extends the “complex points functor”, compare [28], p. 78 f. An atlas $X \rightarrow \mathfrak{X}$ gives rise to a groupoid object in schemes $X \times_{\mathfrak{X}} X \rightrightarrows X$, and the moduli stack of torsors for this groupoid object is canonically equivalent to the original stack. Taking complex points with the analytic topology gives a groupoid in topological spaces which determines a topological stack. The restriction of this functor to smooth stacks in schemes takes values in differentiable stacks, and its restriction to smooth Deligne–Mumford algebraic stacks takes values in differentiable Deligne–Mumford stacks.

2.2 The homotopy type of a topological stack

We now introduce the *homotopy type* of a topological stack. There is a folklore definition of the homotopy type as the classifying space of the groupoid associated to an atlas. We present an axiomatic approach which is equivalent by Proposition 2.3 below. The content here is an ideological reemphasis of ideas which have been present in the literature for some time.

Definition 2.2 Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a representable morphism of topological stacks. Then f is said to be a *universal weak equivalence* if for any test map $Y \rightarrow \mathfrak{Y}$ from a

space Y , the left vertical map in the diagram

$$\begin{array}{ccc}
 Y \times_{\mathfrak{Y}} \mathfrak{X} & \longrightarrow & \mathfrak{X} \\
 \downarrow & & \downarrow f \\
 Y & \longrightarrow & \mathfrak{Y}
 \end{array}$$

is a weak homotopy equivalence of topological spaces.

A *homotopy type* for a topological stack \mathfrak{X} is a pair $(\text{Ho}(\mathfrak{X}), \eta)$, where $\text{Ho}(\mathfrak{X})$ is a paracompact topological space and $\eta : \text{Ho}(\mathfrak{X}) \rightarrow \mathfrak{X}$ is a universal weak equivalence (which is automatically representable, by [28], Corollary 7.3).

Let \mathfrak{X} be a topological stack with an atlas $X_0 \rightarrow \mathfrak{X}$. This determines a simplicial space $X_n = X_0 \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} X_0$ ($n + 1$ copies) which is in fact the nerve of the topological groupoid $X_1 = X_0 \times_{\mathfrak{X}} X_0 \rightrightarrows X_0$. Let $\|X_{\bullet}\|$ be the *thick* realization of the simplicial space X_{\bullet} . The thick realization of a simplicial space is obtained by forgetting the degeneracies and using only the boundary maps. In most cases of interest, the thick geometric realization and the usual geometric realization are homotopy equivalent, see [33, p. 308].

Proposition 2.3 ([29], Theorem 3.11) *If $X_0 \rightarrow \mathfrak{X}$ is an atlas of a topological stack with associated simplicial space X_{\bullet} , then there is a universal weak equivalence $\|X_{\bullet}\| \rightarrow \mathfrak{X}$.*

The space $\|X_{\bullet}\|$ is in general not paracompact, but it is if we assume that X_n is paracompact and Hausdorff for all $n \geq 0$ (see [6]). The class of stacks which admit atlases with this property is quite large and clearly includes all differentiable stacks.

The following statement about the uniqueness and functoriality of homotopy types is stated more precisely and proven in [6].

- Theorem 2.4**
1. *Any two homotopy types of a topological stack are canonically homotopy equivalent. In fact, there is a contractible space of preferred homotopy equivalences between them.*
 2. *Moreover, choosing homotopy types defines a functor from the category of stacks over \mathbf{top} admitting a homotopy type to the homotopy category of spaces (stacks form a 2-category, but 2-isomorphic morphisms are sent to identical homotopy classes). This functor can be refined so as to send each morphism of stacks to a contractible space of maps between the homotopy types.*

2.3 Homotopy types and group actions

There is a pleasant interaction between the notion of the homotopy type of a stack and more familiar topological constructions.

Firstly, if X is a (paracompact) space then we can consider X as a topological stack. Clearly, the identity map $X \rightarrow X$ is a universal weak equivalence and thus $\text{Ho}(X) \simeq X$.

An important class of examples of stacks are the (global) quotient stacks. Let G be a topological group acting on a space X . The quotient stack $X//G$ is defined as follows. If Y is space, then $X//G(Y)$ is the groupoid of triples (P, p, f) ; $p : P \rightarrow Y$ a principal G -bundle and $f : P \rightarrow X$ a G -equivariant map. The isomorphisms are defined in the obvious way. There is a natural morphism $q : X \rightarrow X//G$ defined as follows: consider the trivial principal G -bundle $pr_X : G \times X \rightarrow X$. Note that G acts on $G \times X$ only by group multiplication (and not on $X!$) and that the action map $\mu : G \times X \rightarrow X$ is G -equivariant. Thus $(G \times X, pr_X, \mu)$ is an element of $X//G(X)$, defining a morphism $q : X \rightarrow X//G$. Note that q is a principal G -bundle.

Proposition 2.5 *The homotopy type of $X//G$ is homotopy equivalent to the Borel construction $EG \times_G X$.*

Proof The projection map $EG \times X \rightarrow X$ is G -equivariant while the quotient map $EG \times X \rightarrow EG \times_G X$ is a principal G -bundle, so both maps together define a morphism

$$\eta : EG \times_G X \rightarrow X//G.$$

Clearly, η is a fibre bundle with structure group G and fibre EG : it is associated to the principal bundle $X \rightarrow X//G$. Therefore, if Y is a space and $Y \rightarrow X//G$ a map, then the pullback $Y \times_{X//G} (EG \times_G X) \rightarrow Y$ is a fibre bundle with contractible fibres, hence a weak homotopy equivalence. Hence η is a universal weak equivalence.

An important quotient stack is the moduli stack $\mathfrak{M}_{g,n}$ of smooth complex curves. It is the stack quotient of the Teichmüller space $\mathfrak{T}_{g,n}$ by the action of the mapping class group Γ_g^n of isotopy classes of orientation preserving diffeomorphism of a genus g surface with n marked points. Hence

$$\text{Ho}(\mathfrak{M}_{g,n}) \simeq E\Gamma_g^n \times_{\Gamma_g^n} \mathfrak{T}_{g,n} \simeq B\Gamma_g^n,$$

because the Teichmüller space is contractible.

We will have occasion to deal with group actions on stacks. Suppose \mathfrak{X} is a topological stack with a strict action of a group G (i.e., the action is not just up to coherent 2-morphisms). We will not have to care about group actions which are not strict. Given a strict G -action on \mathfrak{X} and a G -space Y , the notion of an equivariant morphism $Y \rightarrow \mathfrak{X}$ is well-defined.

There are two equivalent descriptions of principal G -bundles over a stack \mathfrak{X} : as a morphism $\mathfrak{X} \rightarrow *//G$, or as a stack \mathfrak{P} with a strict G -action and a G -invariant representable morphism $\mathfrak{P} \rightarrow \mathfrak{X}$ such that the pullback $\mathfrak{P} \times_{\mathfrak{X}} X \rightarrow X$ along any morphism $X \rightarrow \mathfrak{X}$ is a principal G -bundle in the usual sense. An analogous remark applies to arbitrary fibre bundles.

The quotient stack $\mathfrak{X}//G$ is defined in the same way as $X//G$ for spaces X : for a space Y , an object of $(\mathfrak{X}//G)(Y)$ consists of a principal G -bundle $P \rightarrow Y$ and a G -equivariant morphism $P \rightarrow \mathfrak{X}$. Again, it is clear that $\mathfrak{X} \rightarrow \mathfrak{X}//G$ is a principal G -bundle.

Proposition 2.6 *Let \mathfrak{X} be a topological stack with a strict G -action. Then the following hold.*

1. $\mathfrak{X} // G$ is also a topological stack.
2. There exists a homotopy type $\text{Ho}(\mathfrak{X})$ which is a principal bundle on $\text{Ho}(\mathfrak{X} // G)$ such that the universal morphism $\text{Ho}(\mathfrak{X}) \rightarrow \mathfrak{X}$ is G -equivariant.
3. $\text{Ho}(\mathfrak{X} // G) \simeq EG \times_G \text{Ho}(\mathfrak{X})$.

Proof Let $X \rightarrow \mathfrak{X}$ be an atlas, i.e., a representable morphism which admits local sections. Because $\mathfrak{X} \rightarrow \mathfrak{X} // G$ is a bundle, the composite $X \rightarrow \mathfrak{X} // G$ is clearly a representable morphism with local sections. This shows (1).

For (2), choose a homotopy type $\text{Ho}(\mathfrak{X} // G) \rightarrow \mathfrak{X} // G$ and consider the fibre-square

$$\begin{array}{ccc}
 \text{Ho}(\mathfrak{X} // G) \times_{\mathfrak{X} // G} \mathfrak{X} & \longrightarrow & \mathfrak{X} \\
 \downarrow & & \downarrow \\
 \text{Ho}(\mathfrak{X} // G) & \longrightarrow & \mathfrak{X} // G.
 \end{array}$$

Because the right vertical map is a principal G -bundle, so is the left vertical map. Because the bottom horizontal map is a universal weak equivalence, the top horizontal is also a universal weak equivalence. Thus the space $\text{Ho}(\mathfrak{X} // G) \times_{\mathfrak{X} // G} \mathfrak{X} \rightarrow \mathfrak{X}$ is a homotopy type for \mathfrak{X} and is also G -equivariant, which shows (2).

For (3), observe that the natural map $EG \times_G \text{Ho}(\mathfrak{X}) \rightarrow \text{Ho}(\mathfrak{X}) / G = \text{Ho}(\mathfrak{X} // G)$ is a fibre bundle with fibre EG , hence a weak homotopy equivalence.

2.4 Homology of a topological stack

Let \mathfrak{X} be a topological stack and let $\text{Ho}(\mathfrak{X}) \rightarrow \mathfrak{X}$ be a homotopy type. We define

$$H_*(\mathfrak{X}) := H_*(\text{Ho}(\mathfrak{X})).$$

By Theorem 2.4, different choices of atlases give canonical isomorphisms of homology groups and therefore $\mathfrak{X} \mapsto H_*(\mathfrak{X})$ is a well-defined functor. Clearly the homology of a global quotient stack $X // G$ is isomorphic to the Borel-equivariant homology $H_*^G(X) = H_*(EG \times_G X)$. In the same way, we can define cohomology groups of a stack.

For a topological stack \mathfrak{X} , let $\mathfrak{X}^{\text{coarse}}$ be the coarse moduli space (this is the orbit space of a groupoid presenting \mathfrak{X}). There is a natural map $\mathfrak{X} \rightarrow \mathfrak{X}^{\text{coarse}}$ (which is almost never representable) and the composition

$$\mu_{\mathfrak{X}} : \text{Ho}(\mathfrak{X}) \rightarrow \mathfrak{X} \rightarrow \mathfrak{X}^{\text{coarse}} \tag{2.1}$$

is a rational homology equivalence when \mathfrak{X} is an orbifold (see e.g., [12]).

3 The Pontrjagin–Thom construction for differentiable stacks

In this section we describe an extension of the classical Pontrjagin–Thom construction of homotopy-theoretic wrong-way maps to the setting of differentiable stacks. The reader who is not expert with stacks may read Sect. 3.2 for our perspective on the classical construction and then simply take it on faith that an extension to differentiable stacks exists.

3.1 Preliminaries on stable vector bundles and Thom spectra

If $W \rightarrow X$ is a real vector bundle then the *Thom space* of W , denoted X^W is the space obtained by taking the fibrewise one-point compactification of W and then collapsing the section at infinity to the base-point. (If X is compact then this is simply the one-point compactification of W .)

A *virtual vector bundle* on a space X is a pair (E_0, E_1) of real vector bundles on X ; one should think of it as the formal difference $E_0 - E_1$, and we will sometimes use this more suggestive notation. The *rank* of (E_0, E_1) is the difference $\dim E_0 - \dim E_1$. An isomorphism $(E_0, E_1) \rightarrow (F_0, F_1)$ is represented by a pair (V, θ) where V is a vector bundle and

$$\theta : E_0 \oplus F_1 \oplus V \rightarrow E_1 \oplus F_0 \oplus V$$

is a bundle isomorphism. Two pairs $(\theta, V), (\theta', V')$ represent the same morphism if there exists a vector bundle U and an isomorphism $V' \cong V \oplus U$ such that $\theta' \cong \theta \oplus \text{id}_U$ (and then take the equivalence relation that this generates). The composition of $\theta : E_0 \oplus F_1 \oplus V \rightarrow E_1 \oplus F_0 \oplus V$ and $\phi : F_0 \oplus G_1 \oplus W \rightarrow F_1 \oplus G_0 \oplus W$ is defined to be $F_1 \oplus V \oplus W$ together with the composition

$$\begin{array}{ccc} E_0 \oplus F_1 \oplus G_1 \oplus V \oplus W & \xrightarrow{\theta \oplus \text{id}_{G_1 \oplus W}} & E_1 \oplus F_0 \oplus G_1 \oplus V \oplus W \\ & \xrightarrow{\phi \oplus \text{id}_{E_1 \oplus V}} & E_1 \oplus F_1 \oplus G_0 \oplus V \oplus W. \end{array}$$

The category of virtual vector bundles over a fixed space is a groupoid; these form a presheaf of groupoids on the site **top**. Let \mathfrak{K} denote the stackification of the above presheaf. The objects of this stack are slightly more general than virtual bundles; they can locally be presented as formal differences of vector bundles, but globally this might be impossible. Objects of \mathfrak{K} are called *stable vector bundles*.

Let \mathfrak{K}_d denote the full substack consisting of virtual bundles of rank d . For $n \geq d$, let $* \rightarrow \mathfrak{K}_d$ be the arrow representing the stable vector bundle $(\mathbb{R}^n; \mathbb{R}^{n-d})$. It is easy to see that this is an atlas for \mathfrak{K}_d (as a topological stack) and in fact \mathfrak{K}_d is equivalent to the stack $*//O$. Thus

$$\text{Ho}(\mathfrak{K}) = \coprod_{d \in \mathbb{Z}} \text{Ho}(\mathfrak{K}_d) \simeq \mathbb{Z} \times BO,$$

as expected, and 2-isomorphism classes of morphisms $X \rightarrow \mathfrak{K}$ correspond to homotopy classes $X \rightarrow \mathbb{Z} \times BO$.

Stable vector bundles sometimes arise in the following manner. Let $X^{(0)} \subset X^{(1)} \subset \dots \subset X$ be an exhausting filtration, let $V_n \rightarrow X^{(n)}$ be a real vector bundle of rank $n + d$ and assume that isomorphisms $V_{n+1}|_{X^{(n)}} \cong V_n \oplus \mathbb{R}$ are given. On $X^{(n)}$, $W_n := V_n - \mathbb{R}^n$ is a virtual vector bundle and there is a given isomorphism $W_{n+1}|_{X^{(n)}} \cong W_n$. So one obtains from a filtration and a sequence of vector bundles as above a stable vector bundle on X of rank d .

Vice versa, let W be a stable vector bundle on X , given by a map $X \rightarrow \mathfrak{K}_d$. Let $c_W : X \rightarrow \{d\} \times BO$ be a classifying map (aka lift to the homotopy type). Put $X_n := c_W^{-1}(\{d\} \times BO_{d+n})$; this defines an exhausting filtration of X . Let $V_n := c_W^* \gamma_{d+n}$ be the pullback of the $d + n$ -dimensional universal vector bundle. There is an obvious isomorphism $V_{n+1}|_{X_n} \cong \mathbb{R} \oplus V_n$.

For any stable vector bundle W on X , there is an associated *Thom spectrum* $\mathbb{T}h(W)$, produced as follows. Let $X^{(n)}$, V_n be as above. The n th space of $\mathbb{T}h(W)$ is $X_n^{W_n}$ and the structure maps are

$$\Sigma X_n^{W_n} \cong X_n^{\mathbb{R} \oplus W_n} \cong X_n^{W_{n+1}|_{X_n}} \hookrightarrow X_{n+1}^{W_{n+1}}.$$

The homotopy type of the spectrum $\mathbb{T}h(W)$ depends only on the homotopy class of c_W , which can be viewed as an element in the real K -theory group $KO^0(X)$. Furthermore, when W is representable by an actual vector bundle W_0 then the Thom spectrum is homotopy equivalent to the suspension spectrum $\Sigma^\infty X^{W_0}$ of the Thom space. The reader who wants to know more details about Thom spectra is advised to consult [31], chapter IV, §5.

3.2 The classical Pontrjagin–Thom construction

We briefly recall the classical construction. Let $f : M \rightarrow N$ be a proper smooth map of codimension d (i.e., $\dim N - \dim M = d$) between smooth manifolds. The normal bundle

$$\nu(f) := f^*TN - TM$$

is a virtual vector bundle of dimension d on M . For n large enough there exists an embedding $j : M \hookrightarrow \mathbb{R}^n \times N$ such that $pr_N \circ j = f$. The virtual bundle $\nu(j) - \mathbb{R}^n$ is canonically isomorphic to $\nu(f)$.

Choose a tubular neighbourhood U of $j(M)$, identify $U \cong \nu(j)$, and define a map $\mathbb{R}^n \times N \rightarrow M^{\nu(j)}$ as follows: if a point lies in U then it is mapped to the corresponding point in $\nu(j) \subset M^{\nu(j)}$; all points outside U are mapped to the base-point of $M^{\nu(j)}$. Because f is proper, this construction extends to a map $\Sigma^n N_+ \rightarrow M^{\nu(j)}$. The space $M^{\nu(j)}$ is the n^{th} space of the spectrum $\mathbb{T}h(\nu(f))$ and letting n tend to infinity defines a map of spectra

$$PT_f : \Sigma^\infty N_+ \rightarrow \mathbb{T}h(\nu(f))$$

which is the classical Pontrjagin–Thom map. Recall that the functor Σ^∞ from spaces to spectra is left adjoint to the functor Ω^∞ . The adjoint map of PT_f is a map of spaces $N \rightarrow \Omega^\infty \mathbf{Th}(v(f))$, which we also denote by PT_f , because there is no risk of confusion. The homotopy class of PT_f does not depend on the choices involved. The Pontrjagin–Thom map can be used to define umkehr maps in cohomology, see Sect. 6.

These Pontrjagin–Thom maps satisfy a certain naturality condition.

Proposition 3.1 *Suppose*

$$\begin{array}{ccc} L \times_N M & \xrightarrow{\tilde{g}} & M \\ \tilde{f} \downarrow & & \downarrow f \\ L & \xrightarrow{g} & N \end{array}$$

is a pullback square in the category of smooth manifolds (f and g need not be transverse; we merely require that the pullback exists as a smooth manifold) with f and \tilde{f} both proper. Suppose in addition that the natural map $T(L \times_N M) \rightarrow TL \times_{TN} TM$ is an isomorphism. Then there is an induced map of Thom spectra $\mathbf{Th}(v(\tilde{f})) \rightarrow \mathbf{Th}(v(f))$ which restricts to \tilde{g} on the zero sections and gives a homotopy commutative (strictly commutative after appropriate choices) diagram

$$\begin{array}{ccc} L & \xrightarrow{g} & N \\ \downarrow \text{PT}_{\tilde{f}} & & \downarrow \text{PT}_f \\ \Omega^\infty \mathbf{Th}(v(\tilde{f})) & \rightarrow & \Omega^\infty \mathbf{Th}(v(f)). \end{array}$$

Proof Choose a proper embedding $j : M \hookrightarrow \mathbb{R}^n$ for some large n . One then has proper embeddings

$$(f, j) : M \hookrightarrow N \times \mathbb{R}^n, \quad (\tilde{f}, j) : L \times_N M \hookrightarrow L \times \mathbb{R}^n$$

and the diagram

$$\begin{array}{ccc} L \times_N M & \xrightarrow{\tilde{g}} & M \\ (\tilde{f}, j) \downarrow & & \downarrow (f, j) \\ L \times \mathbb{R}^n & \xrightarrow{g \times \text{id}_{\mathbb{R}^n}} & N \times \mathbb{R}^n \end{array}$$

is still a pullback diagram. For any $(x, y) \in L \times_N M$, $f(y) = g(x) = z$, we get a pullback diagram of tangent spaces

$$\begin{array}{ccccc} T_{(x,y)}(L \times_N M) & \xrightarrow{\cong} & T_x L \times_{T_z N} T_y M & \xrightarrow{T\tilde{g}} & T_y M \\ & & \downarrow T(\tilde{f}, j) & & \downarrow T(f, j) \\ & & T_x L \oplus \mathbb{R}^n & \xrightarrow{Tg \times \text{id}_{\mathbb{R}^n}} & T_z N \oplus \mathbb{R}^n, \end{array}$$

from which one easily derives that the induced vector bundle map of normal bundles $v(\tilde{f}, j) \rightarrow \tilde{g}^* v(f, j)$ is a monomorphism. Thus there is an induced map of

Thom spaces

$$(L \times_N M)^{v(\tilde{f},j)} \rightarrow M^{v(f,j)}.$$

The suspension spectra of these Thom spaces are identified with the Thom spaces appearing in the statement of the proposition via the canonical isomorphisms of virtual vector bundles

$$v(\tilde{f}, j) \cong v(\tilde{f}) \oplus \mathbb{R}^n, \quad v(f, j) \cong v(f) \oplus \mathbb{R}^n.$$

This defines the desired map $\mathbb{T}h(v(\tilde{f})) \rightarrow \mathbb{T}h(v(f))$ of Thom spectra.

Choose a tubular neighbourhood of the image of (f, j) and identify it with the normal bundle $v(f, j)$. The inverse image of this tubular neighbourhood is a tubular neighbourhood of the image of (\tilde{f}, j) and there is an induced identification of the inverse image with the normal bundle $v(\tilde{f}, j)$. This fact, together with an easy diagram chase shows the compatibility of the two Pontrjagin–Thom maps.

Remark 3.2 The hypothesis that $T(L \times_N M) \rightarrow TL \times_{TN} TM$ is an isomorphism is satisfied, for example, if both f and g are immersions and \tilde{f} is a covering. This is the case that we need later in the proof of the main theorem.

Note that when \tilde{f} is a finite covering map then its normal bundle is trivial of rank 0, its Pontrjagin–Thom map is the transfer, and the map $\mathbb{T}h(v(\tilde{f})) = \Sigma^\infty(L \times_N M)_+ \rightarrow \mathbb{T}h(v(f))$ is induced by the projection onto M followed by the inclusion of the zero section of $v(f)$.

3.3 Normal bundles for stacks and statement of the theorem

To extend the Pontrjagin–Thom construction to stacks one must be able to define the normal bundle of a morphism.

Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a proper representable morphism of differentiable stacks. The *codimension* d of f is by definition $d = \dim(Y) - \dim(Y \times_{\mathfrak{Y}} \mathfrak{X})$, where $Y \rightarrow \mathfrak{Y}$ is an atlas. Let $Y \rightarrow \mathfrak{Y}$ be an atlas and let $X := \mathfrak{X} \times_{\mathfrak{Y}} Y \rightarrow \mathfrak{X}$ be the induced atlas for \mathfrak{X} . The map f pulls back to a map $f_Y : X \rightarrow Y$ which is a proper smooth map. The normal bundle $f^*TY - TX$ is a virtual vector bundle on X , and so it is classified by a morphism $X \rightarrow \mathfrak{K}_d$. Since normal bundles are natural with respect to pullback along submersions, this morphism descends to a morphism

$$v(f) : \mathfrak{X} \rightarrow \mathfrak{K}_d$$

Taking homotopy types produces a map $\text{Ho}(\mathfrak{X}) \rightarrow BO$ which then yields a Thom spectrum $\mathbb{T}h(v(f))$.

We say that two maps are *weakly homotopic* if their restrictions to any compact subset of the domain are homotopic. Weakly homotopic maps induce identical homomorphisms on homotopy groups and in any generalized homology theory.

The following is the main result of this section.

Theorem 3.3 1. *The construction of Pontrjagin–Thom maps extends to the category of proper representable morphisms between differentiable local quotient stacks (see Definition A.1 in Appendix A). More precisely, if $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ is a proper representable morphism of differentiable stacks with \mathfrak{Y} a local quotient stack (this implies that \mathfrak{X} is a local quotient stack as well) then there exists a Pontrjagin–Thom map*

$$PT_f : \text{Ho}(\mathfrak{Y}) \rightarrow \Omega^\infty \mathbf{Th}(v(f)).$$

2. *Given a pullback square of differentiable local quotient stacks*

$$\begin{array}{ccc} \mathfrak{Z} \times_{\mathfrak{Y}} \mathfrak{X} & \rightarrow & \mathfrak{X} \\ \downarrow \tilde{f} & & \downarrow f \\ \mathfrak{Z} & \xrightarrow{g} & \mathfrak{Y} \end{array}$$

such that f and \tilde{f} are both proper and f and g are transverse, the Pontrjagin–Thom maps fit into a weakly homotopy commutative diagram.

$$\begin{array}{ccc} \text{Ho}(\mathfrak{Z}) & \longrightarrow & \text{Ho}(\mathfrak{Y}) \\ \downarrow PT_{\tilde{f}} & & \downarrow PT_f \\ \Omega^\infty \mathbf{Th}(v(\tilde{f})) & \longrightarrow & \Omega^\infty \mathbf{Th}(v(f)) \end{array}$$

Taking \mathfrak{Z} in part (2) to be a manifold explains in which sense this PT-construction extends the classical one. The classical PT-construction is unique in the sense that it depends on a contractible space of choices. It is likely that our extended PT-construction also depends on a contractible space of choices. However, we do not need this fact here and a proof of it would make the already quite technical proof of Theorem 3.3 even more technical. The proof of Theorem 3.3 is given in Appendix A. For the proof of our main result, we need part (2) of Theorem 3.3 under a somewhat weaker condition.

Proposition 3.4 *Part (2) of Theorem 3.3 holds with the transversality condition replaced by the requirement that*

1. *The pullback exists in the category of differentiable local quotient stacks,*
2. *f, g are immersions, \tilde{f} is a covering,*
3. *all stacks in the diagram are Deligne–Mumford (whence the tangent bundles exist).*

The proof of Proposition 3.4 follows immediately from Proposition 3.1 and the remark after it.

4 The moduli stack of stable curves

The stack $\overline{\mathfrak{M}}_{g,n}$ was first constructed in the algebraic category by Deligne, Mumford and Knudsen (in [5] when $n = 0$ and [18] for general n). We will need only the associated orbifold in the category of differentiable stacks. For more information about $\overline{\mathfrak{M}}_{g,n}$, we refer to the textbook [16] or the article [7].

4.1 Definitions and background

A nodal curve is a complete complex algebraic curve C all of whose singularities are nodal, i.e., ordinary double points. The open subset of smooth points of C will be denoted by C^{sm} . The *arithmetic genus* of a connected nodal curve is the dimension of the vector space $H^1(C, \mathcal{O}_C)$; it coincides with the usual genus for smooth curves, and for a nodal curves it is intuitively given by the genus of a smoothing.

All nodal curves in this paper are understood to be connected. Given a finite set P , a P -pointed nodal curve is a nodal curve C with an embedding of P into C^{sm} . Such a curve is *stable* if its automorphism group is finite. This means that the Euler characteristic of each component of $C^{sm} \setminus P$ is negative, or equivalently, C does not contain an irreducible genus 0 component with fewer than 3 marked points and nodes or a genus 1 component with no marked points or nodes.

The stack $\overline{\mathfrak{M}}_{g,P}$ is the lax sheaf of groupoids on the site of schemes over \mathbb{C} in the étale topology which is given by:

1. The objects of $\overline{\mathfrak{M}}_{g,P}(X)$ are pairs $(E \xrightarrow{\pi} X, j : X \times P \hookrightarrow E)$, where π a proper morphism all of whose geometric fibres are reduced connected nodal curves of arithmetic genus g , and j is an embedding over X , and each fibre is a P -pointed stable nodal curve. Such a pair is a *family of pointed curves over X* .
2. An isomorphism of families of pointed stable curves is an isomorphism of schemes over X which respects the embedding j .

Deligne–Mumford–Knudsen [5, 18] constructed a smooth étale atlas for $\overline{\mathfrak{M}}_{g,P}$ in the category of schemes over $\text{spec } \mathbb{C}$. In the complex analytic category an orbifold atlas is given by the degeneration spaces of Bers [1], and another was constructed in [30]. The complex dimension of $\overline{\mathfrak{M}}_{g,P}$ is $3g - 3 + |P|$. An important property of this stack is that its coarse moduli space is compact.

The symmetric group Σ_P acts on $\overline{\mathfrak{M}}_{g,P}$ by permuting the marked points; thus Σ_P acts on $\text{Ho}(\overline{\mathfrak{M}}_{g,P})$.

4.2 Vector bundles on $\overline{\mathfrak{M}}_{g,P}$ and stripping and splitting

On $\overline{\mathfrak{M}}_{g,P}$ there is a complex line bundle L_p for each element $p \in P$; the fibre of L_p over a given curve is the tangent space at the marked point labelled by p in the curve. There is a map

$$\overline{\mathfrak{M}}_{g,P \sqcup Q} \rightarrow \overline{\mathfrak{M}}_{g,P} \times BU(1)^Q$$

which maps to the first factor by forgetting the Q marked points and maps to the second factor by classifying the Q line bundles. This map is known as a *stripping and splitting* map in [4].

Proposition 4.1 *Restricting to the interior of the moduli stack, the map*

$$\mathfrak{M}_{g,P \sqcup Q} \rightarrow \mathfrak{M}_{g,P} \times BU(1)^Q$$

is a homology isomorphism in degrees $* \leq 2g/3 - 1$. Similarly, the map

$$\mathfrak{M}_{g,n+k} \times_{\Sigma_k} (\mathfrak{M}_{h,1})^k \rightarrow \mathfrak{M}_{g,n} \times E\Sigma_k \times_{\Sigma_k} (\mathfrak{M}_h \times BT(2))^k$$

is a homology isomorphism in degrees $* \leq \min\{2g/3 - 1, 2h/3 - 1\}$, and the map

$$\mathfrak{M}_{g,n+2k} // (\Sigma_k \times \Sigma_2^k) \rightarrow \mathfrak{M}_{g,n} \times E\Sigma_k \times_{\Sigma_k} BN(2)^k$$

is a homology isomorphism in degrees $* \leq 2g/3 - 1$.

Proof The first statement is essentially [4, Theorem 1.1], with an improved range due to the recent paper of Boldsen [3]. The second and third statements follow from the first by an easy argument with the Serre spectral sequence.

Remark 4.2 Here is a quick proof of the theorem in [4], easier than the original one. The stripping and splitting map is the middle vertical arrow in the following diagram (whose rows are homotopy-fibrations)

$$\begin{array}{ccccc} \mathfrak{M}_{g,P,Q} & \longrightarrow & \mathfrak{M}_{g,P \sqcup Q} & \longrightarrow & BU(1)^Q \\ \downarrow & & \downarrow & & \parallel \\ \mathfrak{M}_{g,P} & \longrightarrow & \mathfrak{M}_{g,P} \times BU(1)^Q & \longrightarrow & BU(1)^Q, \end{array}$$

where $\mathfrak{M}_{g,P,Q}$ is the moduli stack of smooth curves of genus g with $|P|$ marked points and $|Q|$ additional marked points equipped with a nonzero tangent vector. The left vertical arrow is a homology equivalence in the stable range of $* \leq 2g/3 - 1$ by Boldsen’s improved version [3] of Harer–Ivanov stability. The base space is simply-connected, so both fibrations are simple. Thus a straightforward application of the Leray–Serre spectral sequence finishes the proof.

In order to prove the Φ' flavour of Theorem 1.1, we shall need the following observation.

Lemma 4.3 *The composition*

$$\mathfrak{M}_{g,n+1} \times \mathfrak{M}_{h,m+1} \rightarrow \mathfrak{M}_{g,n} \times \mathfrak{M}_{h,m} \times BT(2) \xrightarrow{\text{proj}} BT(2) \rightarrow BU(1),$$

where the first arrow is stripping-and-splitting and the third arrow is induced by multiplying the two $U(1)$ factors, is surjective on homology in the degrees $\min\{2g/3 - 1, 2h/3 - 1\}$ (whereas the first arrow is a homology isomorphism in $\min\{2g/3 - 1, 2h/3 - 1\}$).

5 The Pontrjagin–Thom maps for $\overline{\mathfrak{M}}_{g,n}$ in homology

5.1 Outline of the proof

First consider the gluing morphism $\xi_{h,\emptyset} : \overline{\mathfrak{M}}_{h,1} \times \overline{\mathfrak{M}}_{g-h,n+1} \rightarrow \overline{\mathfrak{M}}_{g,n}$. The image of this morphism is one of the irreducible components of the boundary of $\overline{\mathfrak{M}}_{g,n}$. We will show that there is an open stratum $Z \subset \overline{\mathfrak{M}}_{g,n}$ sitting at a k -fold self-intersection of the image of $\xi_{h,\emptyset}$ for some fairly large k , such that the composition

$$Z \hookrightarrow \overline{\mathfrak{M}}_{g,n} \xrightarrow{\Phi} QBT(2)^V$$

is surjective on homology in a range of degrees. The reason for the surjectivity is the following. The stratum Z is chosen so that, using the Bödigeheimer–Tillmann striping and splitting theorem, (Proposition 4.1), Z homologically splits off a factor of $E\Sigma_k \times_{\Sigma_k} BT(2)^k$ in a range of degrees. By the Barratt–Priddy–Quillen–Segal Theorem and homological stability of symmetric groups, this factor has the homology of $QBT(2)_+$ in a range of degrees. The Bödigeheimer–Tillmann range in this case is proportional to the minimum of $g - kh$ and h , while the second range is proportional to k , so one must choose k to maximise the overlap of these two ranges. Since Z lies in the image of $\xi_{h,\emptyset}$, Φ maps it to the subspace $QBT(2)_+ \subset QBT(2)^V$, and the key idea of the proof is that this coincides with the projection onto the $QBT(2)_+$ factor in the approximate homological splitting of Z . It is an easy calculation that the inclusion $QBT(2)_+ \hookrightarrow QBT(2)^V$ is surjective on homology with field coefficients.

The other flavours of the theorem are proven by appropriately modifying the choice of the stratum Z .

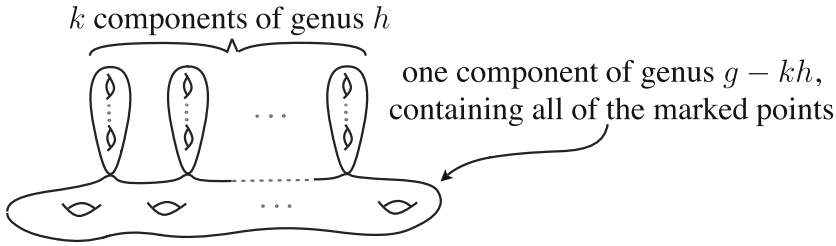
Remark 5.1 As we will see in the proof of this theorem, the homology surjectivity comes from boundary components that have high numbers of self-intersections. Thus Pontrjagin–Thom maps for the boundary components which are embedded (rather than immersed) factor as

$$\overline{\mathfrak{M}}_{g,n} \rightarrow BT(2)^V \rightarrow Q(BT(2)^V).$$

Such maps cannot be surjective in homology in a range because the second map is not. This is why the theorem refers only to self-intersecting boundary components.

5.2 The choice of detecting stratum Z

First consider the gluing morphism $\xi_{h,\emptyset} : \overline{\mathfrak{M}}_{h,1} \times \overline{\mathfrak{M}}_{g-h,n+1} \rightarrow \overline{\mathfrak{M}}_{g,n}$ and the resulting Pontrjagin–Thom map $\Phi_{h,\emptyset} : \overline{\mathfrak{M}}_{g,n} \rightarrow QBT(2)^V$. We choose Z to be the open stratum consisting of curves with precisely k separating nodes, each of which pinches off a component of genus h with no marked points on it; the component to which each of these is attached via a node has genus $g - kh$ and contains all of the marked points. This is pictured below.

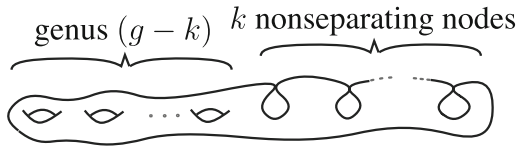


Thus $Z \cong \mathfrak{M}_{g-kh,n+k} \times_{\Sigma_k} (\mathfrak{M}_{h,1})^k$. There is a stripping-and-splitting map for this stratum; its target is

$$\left[\mathfrak{M}_{g-kh,n} \times E\Sigma_k \times_{\Sigma_k} (\mathfrak{M}_h)^k \right] \times \left[E\Sigma_k \times_{\Sigma_k} BT(2)^k \right]. \tag{5.1}$$

and it is a homology isomorphism in degrees $* \leq \min\{2(g - kh)/3 - 1, 2h/3 - 1\}$. In Sect. 5.3 we shall relate the second factor in the above partial homological splitting to $QBT(2)_+$.

For the gluing map $\xi_{irr} : \overline{\mathfrak{M}}_{g-1,n+2} // \Sigma_2 \rightarrow \overline{\mathfrak{M}}_{g,n}$, we take Z to be the stratum consisting of those curves which have precisely k nonseparating nodes and are irreducible (i.e., no subset of the nodes separates the curve), as shown below.



Thus, $Z \cong \mathfrak{M}_{g-k,n+2k} // (\Sigma_k \times \Sigma_2^k)$, and the target of the relevant stripping-and-splitting map is

$$\mathfrak{M}_{g-k,n} \times E \left(\Sigma_k \times \Sigma_2^k \right) \times_{\Sigma_k \times \Sigma_2^k} BU(1)^{2k} = \mathfrak{M}_{g-k,n} \times E\Sigma_k \times_{\Sigma_k} BN(2)^k. \tag{5.2}$$

It is a homology isomorphism in degrees $* \leq 2(g - k)/3 - 1$.

In either of these cases, the important point is that the projection onto the second factor in the partial homological splitting is surjective on homology in the stated range.

5.3 Symmetric groups and QX_+

Here we recall the relation between $E\Sigma_k \times_{\Sigma_k} X^k$ and QX_+ . Let X be a connected space. There is an k -fold covering

$$E(\Sigma_{k-1} \times 1) \times_{\Sigma_{k-1} \times 1} X^k \rightarrow E\Sigma_k \times_{\Sigma_k} X^k$$

induced by the index k inclusion $\Sigma_{k-1} \times 1 \hookrightarrow \Sigma_k$. The Becker–Gottlieb transfer is a map

$$\text{trf} : E\Sigma_k \times_{\Sigma_k} X^k \rightarrow Q_{(k)}(E(\Sigma_{k-1} \times 1) \times_{\Sigma_{k-1} \times 1} X^k)_+,$$

where $Q_{(k)}$ denotes the k^{th} component. When X is a manifold or local quotient stack the transfer can be described as the Pontrjagin–Thom map for the covering projection. The *group completion map* is then the composition

$$\text{gc}_k : E\Sigma_k \times_{\Sigma_k} X^k \xrightarrow{\text{trf}} Q_{(k)}(E(\Sigma_{k-1} \times 1) \times_{\Sigma_{k-1} \times 1} X^k)_+ \rightarrow Q_{(k)}X_+,$$

where the second map is induced by projecting onto the k th component of X^k .

The name stems from the following. One can put a monoid structure on the union

$$\coprod_k E\Sigma_k \times_{\Sigma_k} X^k,$$

and the maps $\{\text{gc}_k\}$ assemble to a monoid map $\text{gc} : \coprod_k E\Sigma_k \times_{\Sigma_k} X^k \rightarrow QX_+$. The Barratt–Priddy–Quillen–Segal Theorem (see e.g. [33]) asserts that this map is the homotopy-theoretic group completion of the above monoid.

Lemma 5.2 *For any connected space X , the map $\text{gc}_k : E\Sigma_k \times_{\Sigma_k} X^k \rightarrow Q_{(k)}X_+$ induces an isomorphism in homology with field coefficients in degrees $* \leq k/2 - 1$.*

Proof This is a well-known consequence of homology stability for symmetric groups (with twisted coefficients). After choosing a base-point in X one has stabilization maps

$$j_k : E\Sigma_k \times_{\Sigma_k} X^k \rightarrow E\Sigma_{k+1} \times_{\Sigma_{k+1}} X^{k+1}$$

whose colimit is denoted by $E\Sigma_\infty \times_{\Sigma_\infty} X^\infty$. The induced map

$$\text{gc}_\infty : E\Sigma_\infty \times_{\Sigma_\infty} X^\infty \rightarrow QX_+$$

is a homology isomorphism onto the base-point component by the group completion theorem (see e.g. [25, 32]). Up to a shift of component the stabilization map from $E\Sigma_k \times_{\Sigma_k} X^k$ to the colimit followed by gc_∞ agrees with gc_k . The stabilization map j_k induces isomorphism in homology in degrees $* \leq k/2 - 1$, and epimorphism when $* \leq k/2$. This has probably been known for a long time (homological stability for symmetric groups with constant coefficients was first proven by Nakaoka). One proof can be found in [13], based on a result of [2]. Another proof is a combination of Proposition 1.6 in [17] with the main result of [19]. The authors are not aware of a previously published proof.

5.4 Homology of the Thom spectra and their infinite loop spaces

Here we show that the maps $QBG_+ \rightarrow QBG^V$ (for $G = T(2), N(2)$, or $U(1)$) induced by inclusion of the zero section into the Thom space are surjective of homology with field coefficients. This is the only part of the proof of Theorem 1.1 where field coefficients are used seriously. The homology surjectivity is immediate from the following two lemmata.

Lemma 5.3 *The inclusion of the zero section*

$$BN(2) \hookrightarrow BN(2)^V$$

induces a surjection on homology with coefficients in a field of characteristic $\neq 2$. The inclusions of the zero sections

$$BT(2) \hookrightarrow BT(2)^V, \quad BU(1) \hookrightarrow BU(1)^V$$

induce surjections in homology with coefficients in an arbitrary field.

Proof Let ι denote one of the three above inclusions. The statement is equivalent to the statement that ι^* is injective on cohomology. The composition of the Thom isomorphism followed by ι^* is equal to multiplication by the Euler class $e(V)$ of V , so it suffices to show that $e(V)$ is not a zero-divisor in each of the three cases. This is well known. For $BN(2)$, the Euler class is a zero divisor in characteristic 2.

Lemma 5.4 *If $f : X \rightarrow Y$ is a pointed map between pointed spaces which is surjective in homology with coefficients in a field \mathbb{F} , then the induced map $Qf : QX \rightarrow QY$ is surjective on homology with coefficients in \mathbb{F} .*

Lemma 5.4 is a well-known fact which we discuss in Appendix B.

5.5 Proof of Theorem 1.1

We shall now relate the projection onto the second factor of the partial homological splitting of Z (given by (5.1) in the separating case, and (5.2) in the nonseparating case) to the restriction of the Pontrjagin–Thom map Φ to Z . We will describe the argument in the separating case, and then indicate how to modify it for the nonseparating case.

Let \tilde{Z} denote the moduli stack of curves of the type in Z equipped with a marking of one node. Forgetting the marking gives a k -fold covering $\pi : \tilde{Z} \rightarrow Z$. Naturality of Pontrjagin–Thom maps/transfers applied to the pullback squares

$$\begin{array}{ccccc}
 E(\Sigma_{k-1} \times 1) \times_{\Sigma_{k-1} \times 1} BT(2)^k & \longleftarrow & \tilde{Z} & \longrightarrow & \overline{\mathfrak{M}}_{g-h,n+1} \times \overline{\mathfrak{M}}_{h,1} \\
 \downarrow & & \downarrow \pi & & \downarrow \xi_{h,\emptyset} \\
 E\Sigma_k \times_{\Sigma_k} BT(2)^k & \longleftarrow & Z & \longrightarrow & \overline{\mathfrak{M}}_{g,n}
 \end{array}$$

gives a weakly homotopy commutative diagram

$$\begin{array}{ccccc}
 E\Sigma_k \times_{\Sigma_k} BT(2)^k & \longleftarrow & Z & \longrightarrow & \overline{\mathfrak{M}}_{g,n} \\
 \text{trf} \downarrow & & \downarrow \text{PT}_\pi = \text{trf} & & \downarrow \text{PT}_{\xi_h, \emptyset} \\
 Q^{(k)} \left(E(\Sigma_{k-1} \times 1) \times_{\Sigma_{k-1} \times 1} BT(2)^k \right)_+ & \longleftarrow & Q^{(k)} \tilde{Z}_+ & \longrightarrow & \Omega^\infty \mathbf{Th}(v_{\xi_h, \emptyset})
 \end{array}$$

by Theorem 3.3 part (2). The lower right horizontal arrow of the above diagram factors as

$$Q\tilde{Z}_+ \rightarrow Q(\overline{\mathfrak{M}}_{g-h,n+1} \times \overline{\mathfrak{M}}_{h,1})_+ \rightarrow Q(\overline{\mathfrak{M}}_{g,n})_+ \xrightarrow{\text{PT}_{\xi_h, \emptyset}} \Omega^\infty \mathbf{Th}(v_{\xi_h, \emptyset})$$

and one observes that the diagram

$$\begin{array}{ccccc}
 Q^{(k)} \tilde{Z}_+ & \longrightarrow & Q^{(k)}(\overline{\mathfrak{M}}_{g-h,n+1} \times \overline{\mathfrak{M}}_{h,1})_+ & \longrightarrow & \Omega^\infty \mathbf{Th}(v_{\xi_h, \emptyset}) \\
 \downarrow & & \downarrow & & \downarrow \\
 Q^{(k)} \left(E(\Sigma_{k-1} \times 1) \times_{\Sigma_{k-1} \times 1} BT(2)^k \right)_+ & \longrightarrow & Q^{(k)} BT(2)_+ & \xrightarrow{\iota} & QBT(2)^V
 \end{array}$$

is weakly homotopy commutative, where the middle vertical arrow is induced by the classifying maps for the two line bundles associated with the two marked points that $\xi_{h, \emptyset}$ glues together.

Assembling the above diagrams, one sees that the composition

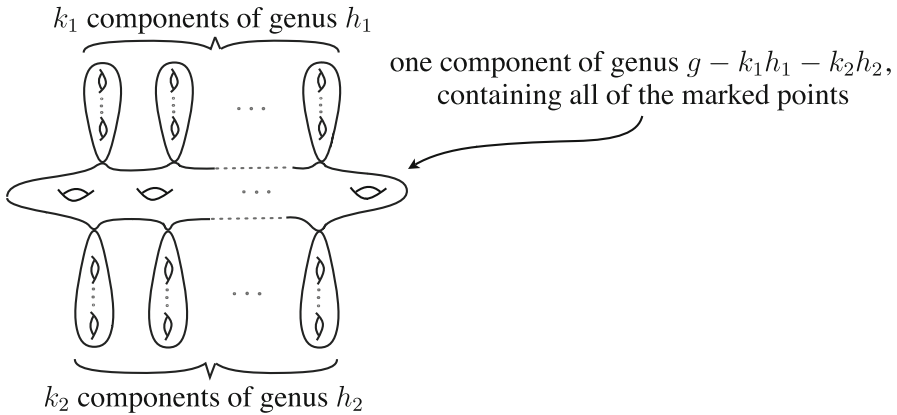
$$Z \rightarrow E\Sigma_k \times_{\Sigma_k} BT(2)^k \xrightarrow{\text{eg}_k} Q^{(k)} BT(2)_+ \xrightarrow{\iota} QBT(2)^V \tag{5.3}$$

coincides (up to weak homotopy) with the restriction of $\Phi_{h, \emptyset}$ to Z . The first arrow is the projection onto the second factor of an approximate homological splitting, so it is homology surjective in degrees $* \leq \min\{2(g - kh)/3 - 1, 2h/3 - 1\}$. The second map is a homology isomorphism in degrees $* \leq k/2 - 1$ by Lemma 5.2, and the third is homology surjective in all degrees. Choosing k (and Z) to maximise the overlap of these ranges yields part (ii) of the main theorem.

Part (i) is proved by essentially the same argument, using the appropriate stratum Z , and replacing $BT(2)$ with $BN(2)$ throughout. Part (iii) is proved as for part (ii), using in addition Lemma 4.3.

5.6 Independence of the detected classes

Using a slight modification of the argument in the proof of Theorem 1.1 one can show that the cohomology classes produced by the various maps Φ are independent in certain stable ranges. For example, the open stratum $Z \subset \overline{\mathfrak{M}}_{g,n}$ of curves with one component of genus $g - k_1h_1 - k_2h_2$ attached to k_1 components of genus h_1 and k_2 components of genus h_2 , as shown below,



detects that the map

$$\overline{\mathfrak{M}}_{g,n} \xrightarrow{\Phi_{h_1,\emptyset} \times \Phi_{h_2,\emptyset}} QBT(2)^V \times QBT(2)^V$$

is surjective on homology in degrees

$$* \leq \min\{2(g - k_1h_2 - k_2h_2)/3 - 1, 2h_1/3 - 1, 2h_2/3 - 1, k_1/2 - 1, k_2/2 - 1\}$$

and one can choose k_1 and k_2 to optimize this range.

6 Comparison to the tautological algebra

Here we explain the relationship between the rational cohomology classes detected via Theorem 1.1 and the tautological algebra of $\overline{\mathfrak{M}}_{g,n}$.

Proposition 6.1 *The image of the homomorphism*

$$\Phi_{irr}^* : H^*(QBN(2)^V; \mathbb{Q}) \rightarrow H^*(\overline{\mathfrak{M}}_{g,n}; \mathbb{Q})$$

is contained in the cohomology tautological algebra $\mathcal{R}^(\overline{\mathfrak{M}}_{g,n})$. The analogous statement is true for the other maps studied in Theorem 1.1.*

Before we can explain the definition of $\mathcal{R}^*(\overline{\mathfrak{M}}_{g,n})$ and the proof of Proposition 6.1, we need to say a few words about umkehr maps (also called “pushforward” or “Gysin map”) in cohomology and their relation to the Pontrjagin–Thom construction.

Let $f : M \rightarrow N$ be a proper smooth map between manifolds (or a proper representable morphism between differentiable local quotient stacks) of codimension d , and let $PT_f : \Sigma^\infty N_+ \rightarrow \mathbb{T}\mathbf{h}(v(f))$ be its Pontrjagin–Thom map. A *cohomological orientation* of f is by definition a Thom class in $H^d(\mathbb{T}\mathbf{h}(v(f)))$. This orientation induces a *Thom isomorphism* $th : H^*(M) \rightarrow H^{*+d}(M^{v(f)})$ (see [31], ch. V for details). The

umkehr map $f_!$ is defined as the composition

$$H^*(M) \cong H^*(\Sigma^\infty M_+) \xrightarrow{\text{th}} H^{*+d}(M^{v(f)}) \xrightarrow{\text{PT}^*} H^{*+d}(\Sigma^\infty N_+) \cong H^{*+d}(N). \tag{6.1}$$

The tautological algebra has been studied by many authors; we refer to the survey paper [34]. Here is the definition. One considers all natural morphisms $\overline{\mathfrak{M}}_{g,n+1} \rightarrow \overline{\mathfrak{M}}_{g,n}$ (forget the last point and collapse an unstable component if it shows up), $\overline{\mathfrak{M}}_{g-1,n+2} \rightarrow \overline{\mathfrak{M}}_{g,n}$, $\overline{\mathfrak{M}}_{h,k+1} \times \overline{\mathfrak{M}}_{g-h,n-k+1} \rightarrow \overline{\mathfrak{M}}_{g,n}$ (the gluing morphisms) and $\overline{\mathfrak{M}}_{g,n} \rightarrow \overline{\mathfrak{M}}_{g,n}$ (given by a permutation of the labelling set $\{1, \dots, n\}$). All these morphisms are representable morphisms of complex-analytic stacks and so they have canonical orientations. Thus there are umkehr maps in integral cohomology for these morphisms. There is another, more traditional way to define the umkehr maps for complex orbifolds, based on rational Poincaré duality for the coarse moduli spaces, but this only works in rational cohomology.

Definition 6.2 The collection of tautological algebras

$$\mathcal{R}^*(\overline{\mathfrak{M}}_{g,n}) \subset H^{2*}(\overline{\mathfrak{M}}_{g,n}; \mathbb{Q})$$

is the smallest system of unital \mathbb{Q} -subalgebras which contain all classes $\psi_i = c_1(L_i) \in H^2(\overline{\mathfrak{M}}_{g,n}; \mathbb{Q})$, for all g, n and $i = 1, \dots, n$ and which is closed under pushforward by the natural morphisms above.

We prove Proposition 6.1 only for the map $\Phi_{irr} : \overline{\mathfrak{M}}_{g,n} \rightarrow QBN(2)^V$, which is sufficient to clarify the pattern.

First recall that $H^*(QBN(2)^V; \mathbb{Q}) = \mathbb{Q}[a_{i,j}]$, where

$$a_{i,j} = \text{th}(y_1^i y_2^j) \in H^{2+2i+4j}(BN(2)^V),$$

and y_i is the i th Chern class of the 2-dimensional complex vector bundle on $BN(2)$ induced by the inclusion $N(2) \rightarrow U(2)$. Thus we need to argue that $\Phi_{irr}^*(\text{th}(y_1^i y_2^j))$ is in the tautological algebra. By the definition of Φ_{irr} , this is precisely $\text{PT}_{\xi_{irr}}^*(\text{th}(c_1(W)^i c_2(W)^j))$, where $W = L_{n+1} \oplus L_{n+2} \rightarrow \overline{\mathfrak{M}}_{g-1,n+(2)}$ is the sum of the natural line bundles (which is well-defined, although the last two points are permuted). This can be rewritten, using the definition of the umkehr map, as

$$(\xi_{irr})!(c_1(W)^i c_2(W)^j) = (\xi_{irr})!(\psi_{n+1} + \psi_{n+2})^i (\psi_{n+1} \psi_{n+2})^j.$$

This obviously lies in the tautological ring. There is a little argument needed, because we used the PT-map starting from $\overline{\mathfrak{M}}_{g-1,n+(2)}$, while the tautological algebra is defined using the map from the twofold cover $\overline{\mathfrak{M}}_{g-1,n+2}$. We leave this to the reader.

Appendix A: Construction of the Pontrjagin–Thom map

A.1 Local quotient stacks

In this appendix, we prove Theorem 3.3. First we have to introduce local quotient stacks, which we view as the natural setting for the Pontrjagin–Thom construction.

Definition A.1 A local quotient stack is a topological stack \mathfrak{X} , such that

1. there exists a paracompact atlas for \mathfrak{X} .
2. there exists a countable cover of open substacks $\mathfrak{X}_k \subset \mathfrak{X}$ such that $\mathfrak{X}_k \cong X_k // G_k$ for some Hausdorff space X_k and some compact Lie group G_k .
3. The diagonal morphism $\mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$ is representable and proper (recall that we use Bourbaki’s definition of a proper map).

A differentiable stack is a local quotient stack if the spaces X_k are smooth manifolds with smooth G_k -actions.

Lemma A.2 ([8], Lemma A.14) *If \mathfrak{Y} is a local quotient stack and $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ is a representable separated (i.e., pullbacks along maps from Hausdorff spaces are Hausdorff spaces) morphism of topological stacks then \mathfrak{X} is a local quotient stack as well. In particular, every open substack of a local quotient stack is a local quotient stack. The analogous statements for differentiable local quotient stacks are also true.*

Proof First suppose that \mathfrak{Y} is a global quotient $Y // G$. Let $X := \mathfrak{X} \times_{\mathfrak{Y}} Y \rightarrow \mathfrak{X}$ be the atlas of \mathfrak{X} obtained by pulling back the atlas $Y \rightarrow \mathfrak{Y}$. One easily checks that $X \times_{\mathfrak{X}} X \cong G \times X$ and that the two arrows $X \times_{\mathfrak{X}} X = G \times X \rightrightarrows X$ are the projection onto X and a group action. Furthermore, one can check that $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ is represented by a G -equivariant map $X \rightarrow Y$. Now suppose \mathfrak{Y} is a local quotient stack with a covering by global quotients $\{\mathfrak{Y}_k \cong Y // G_k\}$. The substacks $\mathfrak{X}_k := \mathfrak{Y}_k \times_{\mathfrak{Y}} \mathfrak{X}$ form an open cover of \mathfrak{X} and by the above, $\mathfrak{X}_k \cong X_k // G_k$.

Lemma A.2 indicates that the class of local quotient stacks is large and robust. Orbifolds are local quotient stacks and so are global quotient stacks of the form $Y // \Gamma$, where Γ is a (possibly noncompact) Lie group which acts properly on Y . A very general result by Zung [35] states that any proper Lie groupoid represents a local quotient stack.

Lemma A.3 *The coarse moduli space \mathfrak{X}^{coarse} of a differentiable local quotient stack \mathfrak{X} is a paracompact Hausdorff space.*

Proof Given an atlas $X \rightarrow \mathfrak{X}$, one sees that the associated groupoid $X \times_{\mathfrak{X}} X \rightrightarrows X$ is proper in the sense of [23]. The coarse moduli space is the orbit space of this groupoid and hence it is Hausdorff and paracompact.

As an application of this lemma, we have existence of locally finite smooth partitions of unity subordinate to any open cover of \mathfrak{Y} as follows. Any open cover of \mathfrak{Y} gives an open cover of \mathfrak{Y}^{coarse} , because the open subsets of \mathfrak{Y}^{coarse} are precisely the coarse moduli spaces of the open substacks of \mathfrak{Y} , compare [28], Sect. 4.3. On \mathfrak{Y}^{coarse} , we have partitions of unity, which can then be pulled back via $\mathfrak{Y} \rightarrow \mathfrak{Y}^{coarse}$.

A.2 The Whitney embedding theorem for local quotient stacks

First we fix some notations. If $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ is a map of stacks and $\mathfrak{U} \subset \mathfrak{Y}$ an open substack, we denote $f^{-1}(\mathfrak{U}) := \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{U}$. Similarly, if \mathfrak{U}_1 and \mathfrak{U}_2 are open substacks of \mathfrak{Y} , then $\mathfrak{U}_1 \cap \mathfrak{U}_2 := \mathfrak{U}_1 \times_{\mathfrak{Y}} \mathfrak{U}_2$; similarly the intersection of a finite number of substacks is defined.

Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a proper representable morphism of differentiable local quotient stacks. Let $Y \rightarrow \mathfrak{Y}$ be an atlas which we will use to present the homotopy type $\text{Ho}(\mathfrak{Y})$. A homotopy type of \mathfrak{X} and f is then obtained by pullback:

$$\begin{CD} \text{Ho}(\mathfrak{X}) @>\eta_{\mathfrak{X}}>> \mathfrak{X} \\ @V\text{Ho}(f)VV @VVfV \\ \text{Ho}(\mathfrak{Y}) @>\eta_{\mathfrak{Y}}>> \mathfrak{Y}. \end{CD}$$

Note that $\text{Ho}(f)$ is still a proper map. If we have an open cover $(\mathfrak{Y}_i)_{i \in I}$ of \mathfrak{Y} , we obtain open substacks $\mathfrak{X}_i := f^{-1}(\mathfrak{Y}_i)$ which cover \mathfrak{X} . If $(\mathfrak{Y}_i)_{i \in I}$ is locally finite, then so is $(\mathfrak{X}_i)_{i \in I}$. The cover $(\mathfrak{Y}_i)_{i \in I}$ induces open covers $(\text{Ho}(\mathfrak{Y}_i))_{i \in I}$ and $(\text{Ho}(\mathfrak{X}_i))_{i \in I}$ of $\text{Ho}(\mathfrak{Y})$ and $\text{Ho}(\mathfrak{X})$, respectively. Likewise, given a vector bundle $\mathfrak{V} \rightarrow \mathfrak{Y}$, we obtain a vector bundle $\text{Ho}(\mathfrak{V}) := \eta_{\mathfrak{Y}}^* \mathfrak{V} \rightarrow \text{Ho}(\mathfrak{Y})$; similarly we treat vector bundles on \mathfrak{X} or vector bundles that are defined on open substacks.

To construct the Pontrjagin–Thom map for f , we of course would like to find an embedding $f : \mathfrak{X} \rightarrow \mathfrak{Y} \times \mathbb{R}^n$ whose first component is just f . Usually, this is not possible. We can ask for less, namely an embedding $f : \mathfrak{X} \rightarrow \mathfrak{V}$ into some finite-dimensional vector bundle on \mathfrak{Y} . This is not possible in general, either. It is, however, always possible to find such an embedding locally on \mathfrak{Y} . We could glue these together by means of a partition of unity, but that would lead to the consideration of infinite-dimensional vector bundles on \mathfrak{Y} that we rather would like to avoid.

In the course of this section, we will meet the following set of data several times.

Definition A.4 Let Y be a space or a stack. A *crude vector bundle* on Y consists of the following data:

1. a countable, locally finite cover $(Y_S)_{S \in J}$ of Y indexed on a partially ordered set J such that if $S \leq T$, then $Y_T \subset Y_S$,
2. finite-dimensional vector bundles $V_S \rightarrow Y_S$, monomorphisms $b_{S,T} : V_S|_{Y_T} \rightarrow V_T$ for each pair $S \leq T$ such that $b_{S,U} = b_{T,U} \circ b_{S,T}$ whenever $S \leq T \leq U$ and $b_{S,S} = \text{id}$.

Here is a typical way to obtain crude vector bundles.

Example A.5 Let $(Y_i)_{i \in I}$ be a countable locally finite open cover, and for each i let $V_i \rightarrow Y_i$ be a finite dimensional vector bundle. Let J be the set of finite nonempty subsets of I , partially ordered by inclusion. Put $Y_S := \bigcap_{i \in S} Y_i$ and $V_S := \bigoplus_{i \in S} V_i|_{Y_S}$. The monomorphisms $b_{S,T}$ are just the obvious inclusions. By construction, this is a crude vector bundle.

Given a crude vector bundle, we define $\bigcup V_S := \coprod_{S \in J} \{S\} \times V_S / \sim$; the equivalence relation is generated by $(S, v) \sim (T, w)$ if $S \leq T$, $v \in V_S|_{Y_T}$ and $w = b_{S,T}(v)$ and endow it with the quotient topology. There is a natural map $\pi : \bigcup_{S \in J} V_S \rightarrow Y$. To simplify notation, we will occasionally pretend that all the monomorphisms $b_{S,T}$ are actually inclusions.

Suppose that $f : X \rightarrow Y$ is a map. An embedding $e : X \rightarrow \bigcup_{S \in J} V_S$ over f is then an embedding such that $\pi \circ e = f$. We will call this set of data a *crude embedding over f* . We can also talk about tubular neighborhoods of crude embeddings. First recall some facts about tubular neighborhoods in the setting of ordinary manifolds. Let $e : M \rightarrow N$ be an embedding. A tubular neighborhood is a pair (E, ι) , where $E \rightarrow M$ is a vector bundle and $\iota : E \rightarrow N$ is an open embedding that restricts to e on the zero section of E . The restriction of the differential to the zero section induces an isomorphism $TE|_M \xrightarrow{T_e} e^*TN$. The composition $E \rightarrow TE|_M \rightarrow e^*TM \rightarrow e^*TN/TM = \nu(e)$ is an isomorphism. Therefore a bundle E as above is canonically isomorphic to the normal bundle. Two tubular neighborhoods (E_i, ι_i) , $i = 0, 1$ are called equivalent if there is an isomorphism $a : E_0 \rightarrow E_1$ with $\iota_1 \circ a = \iota_0$. The set of equivalence classes can thus be identified with a subset of the mapping space $Map(\nu(e), N)$ and we put the subspace topology on it. There is an obvious generalization of these notions to the G -equivariant case, where G is a compact Lie group. It is a standard fact that the space of tubular neighborhoods obtained in this way is contractible.

Now suppose that $e : X \rightarrow \bigcup_{S \in J} V_S$ is a crude embedding. A crude tubular neighborhood of e consists of the following data for each S :

1. a crude vector bundle on X , consisting of vector bundles $E_S \rightarrow X_S := f^{-1}(Y_S)$ and monomorphisms $c_{S,T} : E_S|_{X_T} \rightarrow E_T$,
2. and open embeddings $\iota_S : E_S \rightarrow V_S$ whose restriction to the zero section is e_S ,

and these data are subject to the following compatibility condition:

if $S \leq T$, then $\iota_T \circ c_{S,T} = b_{S,T} \circ (\iota_S|_{(E_S|_{Y_T})})$.

Note that a crude tubular neighborhood defines a map $\iota : \bigcup_S E_S \rightarrow \bigcup_S V_S$ and it is entirely determined by this.

Proposition A.6 *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a proper representable morphism of differentiable local quotient stacks. Then there exists a crude vector bundle $\pi : \mathfrak{V} \rightarrow \mathfrak{Y}$, a crude embedding $e : \mathfrak{X} \rightarrow \mathfrak{V}$ over f with a crude tubular neighborhood.*

Proof First assume that $\mathfrak{Y} = Y//G$, where G is a compact Lie group acting on the manifold Y with finite orbit type, i.e., the number of conjugacy classes of isotropy subgroups is finite. Then $\mathfrak{X} \cong X//G$ and f is represented by a G -equivariant map $X \rightarrow Y$ (compare the proof of Lemma A.2). Because f is proper, the action on X also has finite orbit type. Mostow showed ([24], p. 444 f) that there exists a finite-dimensional G -representation V and a G -equivariant embedding $i : X \hookrightarrow V$. Then $X \rightarrow Y \times V$ is also a G -equivariant embedding. For an arbitrary G -representation V , an embedding $\mathfrak{X} \rightarrow (Y \times V)//G$ over \mathfrak{Y} is the same as a G -equivariant embedding $X \rightarrow Y \times V$ over Y . Therefore we get an embedding of stacks $X//G \hookrightarrow (Y \times V)//G$.

Now let \mathfrak{Y} be a local quotient stack. Let $(U_i//G_i)_{i \in I}$ be a countable locally finite open cover of \mathfrak{Y} by global quotients. Such an open cover exists by Lemma A.3 and

Lemma 1.21 of [14]. For each $i \in I$, we choose an open substack $\mathfrak{Y}_i = Y_i // G_i \subset U_i // G_i$ such that $Y_i \subset U_i$ is relatively compact and the collection of all \mathfrak{Y}_i covers \mathfrak{Y} . Let $\mathfrak{X}_i := f^{-1}(\mathfrak{Y}_i)$, $i \in I$ be the induced open cover of \mathfrak{X} . Then the G_i -action on Y_i is of finite orbit type and therefore we can choose a finite-dimensional vector bundle $\mathfrak{V}_i \rightarrow \mathfrak{Y}_i$ and an embedding $\tilde{e}_i : \mathfrak{X}_i \rightarrow \mathfrak{V}_i$ over $f|_{\mathfrak{X}_i}$.

As in example A.5, let J be the set of all finite nonempty subsets of I . For $S \in J$, we put $\mathfrak{Y}_S = \bigcap_{i \in S} \mathfrak{Y}_i$ and $\mathfrak{V}_S = \bigoplus_{i \in S} \mathfrak{V}_i|_{\mathfrak{Y}_S}$. The \mathfrak{Y}_S cover \mathfrak{Y} and by construction, these data form a crude vector bundle on \mathfrak{Y} . Next, let $(\lambda_i)_{i \in I}$ be a locally finite partition of unity on \mathfrak{Y} subordinate to the cover $(\mathfrak{Y}_i)_{i \in I}$ and $(\mu_i)_{i \in I}$ be a family of bump functions, i.e., $\text{supp}(\mu_i) \subset \mathfrak{Y}_i$ and $\mu_i \lambda_i = \lambda_i$. For $x \in \mathfrak{X}$, set

$$e(x) := (\mu_i(f(x))\tilde{e}_i(x))_{i \in I} \in \bigcup_{S \in J} \mathfrak{V}_S.$$

This is clearly a crude embedding $e : \mathfrak{X} \rightarrow \bigcup_{S \in J} \mathfrak{V}_S$ over f .

It remains to construct a crude tubular neighborhood for e . As usual, let $e_S = e|_{\mathfrak{X}_S} : \mathfrak{X}_S \rightarrow \mathfrak{V}_S$. Let $\mathfrak{E}_S \rightarrow \mathfrak{X}_S$ be the normal bundle of e_S . The collection of all \mathfrak{E}_S is a crude vector bundle on \mathfrak{X} ; let $q : \bigcup_{S \in J} \mathfrak{E}_S \rightarrow \mathfrak{X}$ be the projection. For each $S \in J$, \mathfrak{Y}_S is a global quotient stack $Y_S // G_S$ because it is an open substack of the global quotient stack \mathfrak{Y}_i whenever $i \in S$ (c.f. Lemma A.2). The embedding $e_S : \mathfrak{X}_S \rightarrow \mathfrak{V}_S$ arose from a G_S -equivariant embedding $X_S \rightarrow Y_S \times V_S$. For $S \in J$, let \mathcal{F}_S denote the space of all tubular neighborhoods of e_S in \mathfrak{V}_S ; as we remarked above, \mathcal{F}_S is contractible. We will use this contractibility to find a compatible collection of tubular neighborhoods of the embeddings e_S which can be glued together to form a crude tubular neighborhood.

For $S \subset T$, let $r_{S,T} : \mathcal{F}_S \rightarrow \mathcal{F}_T$ denote the map defined by sending the tubular neighborhood $\iota_S : \mathfrak{E}_S \rightarrow \mathfrak{V}_S$ of e_S to the tubular neighborhood of e_T given by

$$\iota_S \oplus \text{Id} : \mathfrak{E}_S|_{\mathfrak{X}_T} \oplus \mathfrak{V}_{T \setminus S}|_{\mathfrak{Y}_T} \rightarrow \mathfrak{V}_S|_{\mathfrak{Y}_T} \oplus \mathfrak{V}_{T \setminus S}|_{\mathfrak{Y}_T} \cong \mathfrak{V}_T.$$

Let $\Delta_S \subset \mathbb{R}^S$ denote the $(|S| - 1)$ -dimensional simplex spanned by S . Because the spaces \mathcal{F}_S are all contractible, by induction on $|S|$ it is possible to choose maps $h_S : \Delta_S \rightarrow \mathcal{F}_S$ satisfying the compatibility conditions that whenever $S \subset T$ then $h_T|_{\Delta_S} = r_{S,T} \circ h_S$. For $x \in \mathfrak{X}$, let $S(x) := \{i \in I | f(x) \in \mathfrak{Y}_i\}$, and define

$$\iota(e) := h_{S(q(e))} \left(\sum_{i \in S(q(e))} \lambda_i(f(q(e))) \cdot \{i\} \right) (e) \in \bigcup_{S \in J} \mathfrak{V}_S.$$

By construction, this is a crude tubular neighborhood of e .

Now we pass to homotopy types. Choose an atlas $Y \rightarrow \mathfrak{Y}$. Using Y , we get a model $\eta_{\mathfrak{Y}} : \text{Ho}(\mathfrak{Y}) \rightarrow \mathfrak{Y}$ for the homotopy type. We pull back all the data we have

constructed so far by the map $\eta_{\mathfrak{Y}}$. The result is a commuting diagram of spaces

$$\begin{array}{ccc}
 \mathrm{Ho}(\mathfrak{E}) & \xrightarrow{\mathrm{Ho}(t)} & \mathrm{Ho}(\mathfrak{Y}) \\
 \mathrm{Ho}(s) \uparrow & \nearrow \mathrm{Ho}(e) & \downarrow \mathrm{Ho}(\pi) \\
 \mathrm{Ho}(\mathfrak{X}) & \xrightarrow{\mathrm{Ho}(f)} & \mathrm{Ho}(\mathfrak{Y}),
 \end{array}$$

where $\mathrm{Ho}(\mathfrak{E}) \rightarrow \mathrm{Ho}(\mathfrak{X})$ and $\mathrm{Ho}(\mathfrak{Y}) \rightarrow \mathrm{Ho}(\mathfrak{Y})$ are crude vector bundles, $\mathrm{Ho}(f)$ is a proper map, $\mathrm{Ho}(e)$ is an embedding and $\mathrm{Ho}(t)$ is an open embedding. We would like to embed the crude vector bundle $\mathrm{Ho}(\mathfrak{Y})$ into the trivial vector bundle $\mathrm{Ho}(\mathfrak{Y}) \times \mathbb{R}^\infty$, but we have to do this with some care, because we wish to have an embedding $\mathrm{Ho}(\mathfrak{Y}) \rightarrow \mathrm{Ho}(\mathfrak{Y}) \times \mathbb{R}^\infty$ which locally (and not merely pointwise) goes into some finite stage $\mathbb{R}^n \subset \mathbb{R}^\infty$. The vector bundles $\mathrm{Ho}(\mathfrak{Y}_S)$ are of course not necessarily trivial, but the real problem is that there might not exist an embedding $\mathrm{Ho}(\mathfrak{Y}_S) \rightarrow \mathbb{R}^{n_S}$. We must work locally on the homotopy type rather than just locally on the stack.

Lemma A.7 *Let Y be a paracompact space with $(Y_i)_{i \in I}$ a countable locally finite cover, for each i let $\pi_i : V_i \rightarrow Y_i$ be a finite-dimensional vector bundle, and let $(V_S \rightarrow Y_S)_{S \in J}$ be the crude vector bundle constructed as in Example A.5 and $V = \bigcup_S V_S$. Assume that there exist open neighborhoods \tilde{Y}_i of Y_i and vector bundles $\tilde{V}_i \rightarrow \tilde{Y}_i$ with $\tilde{V}_i|_{Y_i} = V_i$, such that $(\tilde{Y}_i)_{i \in I}$ is a locally finite cover of Y . Then there exists a monomorphism $\epsilon : V \rightarrow Y \times \mathbb{R}^\infty$ such that any $y \in Y$ has an open neighborhood $U \subset Y$ such that $\epsilon(V|_U) \subset U \times \mathbb{R}^{n_U}$.*

Proof First suppose that I and hence J consists of a single element, i.e., $\pi : V \rightarrow Y$ is a single vector bundle. According to Lemma 1.21 of [14], we can find a countable cover $(Y_k)_{k \in K}$ of Y and trivialisations $\varphi_k : V|_{Y_k} \cong Y_k \times \mathbb{R}^n$. Let $(\mu_k)_{k \in K}$ be a partition of unity subordinate to $(Y_k)_{k \in K}$. For $v \in V$, put

$$\epsilon(v) := (\mu_k(\pi(v)) \cdot \varphi_k(v))_{k \in K} \in (\mathbb{R}^n)^K \cong \mathbb{R}^\infty,$$

where the last isomorphism is induced by a bijection $\mathbb{N} \times K \cong \mathbb{N}$. Clearly, ϵ has the desired property.

In the general case, let $\epsilon_i : \tilde{V}_i \rightarrow Y \times \mathbb{R}^\infty$, $i \in I$, be embeddings as just constructed. Let $(v_i)_{i \in I}$ be a family of cut-off functions, i.e., $\mathrm{supp}(v_i) \subset \tilde{Y}_i$ and $v_i|_{Y_i} \equiv 1$. Now, for $v \in \bigcup_{S \in J} \tilde{V}_S$, we put

$$\epsilon(v) = (v_i(\pi(v)) \cdot \epsilon_i(v))_{i \in I} \in (\mathbb{R}^\infty)^I \cong \mathbb{R}^\infty,$$

where the last isomorphism is induced by a bijection $\mathbb{N} \times I \cong \mathbb{N}$. The restriction of ϵ to V has the desired properties.

Now we apply Lemma A.7 to the space $\mathrm{Ho}(\mathfrak{Y})$, the open cover $(\mathrm{Ho}(\mathfrak{Y}_i))_{i \in I}$ and the vector bundles $\mathrm{Ho}(\mathfrak{Y}_i) \rightarrow \mathrm{Ho}(\mathfrak{Y}_i)$. The additional property that the bundles $\mathrm{Ho}(\mathfrak{Y}_i)$ can be extended to neighborhoods of $\overline{\mathrm{Ho}(\mathfrak{Y}_i)}$ can easily be satisfied by a minor adjustment of the construction in the proof of Proposition A.6.

Let us summarize our progress so far.

- Proposition A.8** 1. *There is an open exhaustion $\text{Ho}(\mathfrak{Y})_1 \subset \text{Ho}(\mathfrak{Y})_2 \subset \dots$ of $\text{Ho}(\mathfrak{Y})$, with the preimages giving an open exhaustion $\text{Ho}(\mathfrak{X})_1 \subset \text{Ho}(\mathfrak{X})_2 \subset \dots$ of $\text{Ho}(\mathfrak{X})$, and an embedding $g : \text{Ho}(\mathfrak{X}) \rightarrow \text{Ho}(\mathfrak{Y}) \times \mathbb{R}^\infty$ over $\text{Ho}(f)$ that sends $\text{Ho}(\mathfrak{X})_n$ into the subspace $\text{Ho}(\mathfrak{Y})_n \times \mathbb{R}^n$.*
2. *There exist vector bundles $W_n \rightarrow \text{Ho}(\mathfrak{X})_n$ and open embeddings $W_n \rightarrow \text{Ho}(\mathfrak{Y})_n \times \mathbb{R}^n$ extending $g|_{\text{Ho}(\mathfrak{X})_n}$. Moreover, there are isomorphisms $W_{n+1}|_{\text{Ho}(\mathfrak{X})_n} \cong W_n \oplus \mathbb{R}$ which are compatible with these embeddings.*
3. *There is an isomorphism of the stable vector bundle W represented by the bundles W_n with $\eta_{\mathfrak{X}}^* \nu(f)$, the stable normal bundle of f .*

Proof For part (1), let $\text{Ho}(\mathfrak{Y})_n \subset \text{Ho}(\mathfrak{Y})$ be the subspace of all points y such that if $y \in \text{Ho}(\mathfrak{Y}_S)$ for some S then there exists an open neighborhood $U_y \subset \text{Ho}(\mathfrak{Y}_S)$ with $\epsilon(\text{Ho}(\mathfrak{Y}_S)|_{U_y}) \subset U_y \times \mathbb{R}^n$, where ϵ is the bundle monomorphism from A.7. By construction, $\text{Ho}(\mathfrak{Y})_n$ is open and $\text{Ho}(\mathfrak{X})_n = \text{Ho}(f)^{-1}(\text{Ho}(\mathfrak{Y})_n)$ is mapped by $g = \epsilon \circ \text{Ho}(e)$ into the subbundle $\text{Ho}(\mathfrak{Y})_n \times \mathbb{R}^n$.

For part (2), let $W_{n,S}$ denote the vector bundle on $\text{Ho}(\mathfrak{X})_n \cap \text{Ho}(\mathfrak{X}_S)$ given by

$$\text{Ho}(\mathfrak{E}_S) \oplus \text{Ho}(f)^* \epsilon(\text{Ho}(\mathfrak{Y}_S)|_{\text{Ho}(\mathfrak{Y})_n \cap \text{Ho}(\mathfrak{Y}_S)})^\perp,$$

where the orthogonal complement is taken in the ambient trivial \mathbb{R}^n bundle. As S varies these bundles $W_{n,S}$ canonically glue together to define the desired bundle W_n . An open embedding of $W_{n,S}$ into $\text{Ho}(\mathfrak{Y})_n \cap \text{Ho}(\mathfrak{Y}_S) \times \mathbb{R}^n$ is given by $\text{Ho}(\iota_S) \circ \epsilon$ on the first summand and the identity on the second factor. These embeddings also glue together as S varies.

For part (3), it is enough to construct *locally* on $\text{Ho}(\mathfrak{X})$ a *canonical* isomorphism, which is a straightforward matter (the bundles \mathfrak{E}_S in the proof of A.6 were introduced as certain normal bundles of f).

From Proposition A.8, it is now to entirely straightforward to construct a Pontrjagin–Thom map $\Sigma^\infty \text{Ho}(\mathfrak{Y})_+ \rightarrow \mathbf{Th}(W) \simeq \mathbf{Th}(\nu(f))$ by the usual collapse construction.

Appendix B: A quick review of homology of infinite loop spaces

We recall the description of the homology of the free infinite loop space QX with coefficients in a field \mathbb{F} . In characteristic zero the description is easy and classical; in finite characteristic the standard reference is [21].

If \mathbb{F} is a field, V a graded \mathbb{F} -vector space, then we denote by $\Lambda(V)$ the free graded-commutative \mathbb{F} -algebra generated by V .

Let X be a pointed space. There is a natural map $X \rightarrow Q(X)$, adjoint to the identity on $\Sigma^\infty X$ and thus a homomorphism $H_*(X) \rightarrow H_*(QX)$. Because QX is a homotopy-commutative H -space, the Pontrjagin product endows the homology $H_*(QX; \mathbb{F})$ with the structure of a graded-commutative \mathbb{F} -algebra. Thus we obtain a ring homomorphism $\Lambda(\tilde{H}_*(X; \mathbb{F})) \rightarrow H_*(QX; \mathbb{F})$. If $\text{char}(\mathbb{F}) = 0$ then this is an isomorphism. This is a standard result of algebraic topology, see [22, p. 262 f].

If $\text{char } \mathbb{F} = p > 0$ then the homology $H_*(QX; \mathbb{F})$ is much richer. The homology algebra $H_*(QX; \mathbb{F})$ is a module over an algebra of homology operations known as the

Dyer–Lashof operations (they are also known as Araki–Kudo operations if $p = 2$). These operations measure the failure of chain-level commutativity of the Pontrjagin product.

For $p \neq 2$, these operations are:

$$\beta^\epsilon Q^s : H_n(QX; \mathbb{F}) \rightarrow H_{n+2s(p-1)-\epsilon}(QX; \mathbb{F})$$

for $\epsilon \in \{0, 1\}$ and $s \in \mathbb{Z}_{\geq \epsilon}$. Given a sequence $I = (\epsilon_1, s_1, \dots, \epsilon_n, s_n)$, I is *admissible* if $s_{i+1} \leq ps_i - \epsilon_i$ for $i = 1, \dots, n - 1$. One defines the *excess*

$$e(I) = 2s_1 - \epsilon_1 - \sum_{i=2}^n (2s_i(p - 1) - \epsilon_i)$$

and $b(I) = \epsilon_1$. Such a sequence determines an iteration of operations which is written Q^I .

When $p = 2$ the operations are of the form

$$Q^s : H_n(QX; \mathbb{F}) \rightarrow H_{n+s}(QX; \mathbb{F})$$

for $s \in \mathbb{Z}_{\geq 0}$. A sequence $I = (s_1, \dots, s_n)$ is *admissible* if $s_{i+1} \leq 2s_i$ for each $i = 1, \dots, n - 1$. The excess is defined to be $e(I) = s_1 - \sum_{i=2}^n s_i$, and for convenience one puts $b(I) = 0$.

Let V be a graded \mathbb{F} -vector space and let B be a homogeneous basis of V . The *free unstable Dyer–Lashof module generated by V* is the \mathbb{F} -vector space $DL_{\mathbb{F}}(V)$ on the basis

$$\{Q^I x \mid x \in B, I \text{ admissible}, e(I) + b(I) \geq \deg(x)\}.$$

Because $H_*(QX; \mathbb{F})$ has Dyer–Lashof operations, there is a ring homomorphism, compatible with the Dyer–Lashof operations

$$\Lambda(DL_{\mathbb{F}}(\tilde{H}_*(X; \mathbb{F}))) \rightarrow H_*(QX; \mathbb{F}),$$

and it is proven in [21] that this is an isomorphism. This calculation immediately implies Lemma 5.4.

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