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A vanishing theorem for characteristic classes of odd-dimensional manifold bundles

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Abstract. We show how the Atiyah–Singer family index theorem for both usual and self-adjoint elliptic operators fits naturally into the framework of the Madsen–Tillmann–Weiss spectra. Our main theorem concerns bundles of odd-dimensional manifolds. Using completely functional-analytic methods, we show that for any smooth proper oriented fibre bundle $E \to X$ with odd-dimensional fibres, the family index $\mathrm{ind}(B) \in K^{-1}(X)$ of the odd signature operator is trivial. The Atiyah–Singer theorem allows us to draw a topological conclusion: the generalized Madsen–Tillmann–Weiss map $\alpha : B\,\mathrm{Diff}^{+}(M^{2m-1}) \to \Omega^{\infty}\,\mathrm{MTSO}(2m-1)$ kills the Hirzebruch $\mathcal{L}$-class in rational cohomology. If $m = 2$, this means that α induces the zero map in rational cohomology. In particular, the three-dimensional analogue of the Madsen–Weiss theorem is wrong. For 3-manifolds M, we also prove the triviality of α in mod p cohomology in many cases. We show an appropriate version of these results for manifold bundles with boundary.

1. Introduction and statement of results

Among the greatest achievements of algebraic topology in the last decade are the two proofs of Mumford's conjecture on the homology of the stable mapping class group by Madsen and Weiss [31] and by Galatius, Madsen, Tillmann and Weiss [22]. The Pontrjagin–Thom construction is crucial for both proofs; it provides a map from the classifying space of the diffeomorphism group of a compact surface to the infinite loop space $\Omega^{\infty}\,\mathrm{MTSO}(2)$ of the Madsen–Tillmann–Weiss spectrum, in other words the Thom spectrum of the inverse of the universal complex line bundle.

The proof in [22] consists of two parts. One part (essentially due to Tillmann [41]) exclusively applies to two-dimensional manifolds, because it relies on two deep results of surface theory (the Harer–Ivanov homological stability theorem and the Earle–Eells theorem on the contractibility of the components of the diffeomorphism group of surfaces of negative Euler number). The other part of the proof, however, is valid for manifolds of arbitrary dimension and with general "tangential structures" and provides a vast generalization of the classical Pontrjagin–Thom theorem relating bordism theory of smooth manifolds and stable homotopy.

Given an oriented (we will ignore more general tangential structures throughout the present paper) closed manifold M of dimension n and a fibre bundle $f : E \to B$ with fibre M

and structural group $\operatorname{Diff}^+(M)$, there exists a map

$$\alpha_E : B \to \Omega^\infty \operatorname{MTSO}(n), \tag{1.1}$$

where $\operatorname{MTSO}(n)$ denotes the Thom spectrum of the inverse of the universal n-dimensional oriented vector bundle. We denote the universal bundle by $E_M \to B\operatorname{Diff}^+(M)$ and its Madsen–Tillmann–Weiss map by α_{E_M}. Let Cob_n^+ be the oriented n-dimensional cobordism category: objects are closed $(n-1)$-dimensional manifolds, morphisms are oriented cobordisms and composition is given by gluing cobordisms. With a suitable topology on object and morphism spaces, Cob_n^+ becomes a topological category. The maps α from (1.1) assemble to a map

$$\alpha^{\mathrm{GMTW}} : \Omega B \operatorname{Cob}_n^+ \to \Omega^\infty \operatorname{MTSO}(n),$$

and the main result of [22] states that α^{GMTW} is a homotopy equivalence. Moreover, for any closed n-manifold M, there is a tautological map $\Phi_M : B\operatorname{Diff}^+(M) \to \Omega B \operatorname{Cob}_n^+$ and $\alpha^{\mathrm{GMTW}} \circ \Phi_M = \alpha_{E_M}$.

The exclusive result for two-dimensional manifolds is that when M is a closed connected oriented surface of genus g, then Φ_M induces an isomorphism on integral homology groups of degrees $* \leq g/2 - 1$. Both theorems together provide an isomorphism of the homology of $B\operatorname{Diff}^+(M)$ and $\Omega^\infty \operatorname{MTSO}(2)$ (in that range of degrees).

In this paper, we study the map α_{E_M} (or, equivalently, Φ_M) when M is an oriented closed manifold of *odd* dimension. It turns out that α_{E_M} fails to be an isomorphism in homology in any range and that no clue about the homology of $B\operatorname{Diff}^+(M)$ can be derived from the study of α_{E_M}. This seems to be an unsatisfactory state of affairs and therefore we attempt to arouse the reader's curiosity by the following remark:

Even if the map α_{E_M} fails to be an "equivalence" of some kind, it still contains interesting information about $B\operatorname{Diff}^+(M)$. Any cohomology class of $\Omega^\infty \operatorname{MTSO}(n)$ (in an arbitrary generalized cohomology theory) yields, via α_{E_M}, a cohomology class of $B\operatorname{Diff}^+(M)$, also known as a characteristic class of smooth oriented M-bundles. One should think of these characteristic classes as "universal" classes in the sense that they are defined for *all* oriented n-manifolds and are defined using only the *local* structure of the manifold.

Examples are the *generalized MMM-classes* (an abbreviation of the names Mumford, Miller and Morita)

$$f_!(c(T_v E)) \in H^{*-n}(B),$$

where $f : E \to B$ is a smooth oriented fibre bundle with vertical tangent bundle $T_v E$ and $c \in H^*(\mathrm{BSO}(n))$ is a characteristic class of oriented vector bundles. The generalized MMM-classes come from spectrum cohomology classes of $\operatorname{MTSO}(n)$.

Other examples come from index theory of elliptic operators. Any sufficiently natural elliptic differential operator D on oriented n-manifolds defines a characteristic class in K^0 (namely, the family index). An application of the Atiyah–Singer index theorem shows that these index-theoretic classes also come from $\operatorname{MTSO}(n)$. More precisely, for any natural differential operator D, there exists a universal symbol class $\operatorname{th}(\sigma_D) \in K^0(\operatorname{MTSO}(n))$ such that

$$\operatorname{ind}(D_E) = \alpha_{E_M}^*(\operatorname{th}(\sigma_D)) \in K^0(B),$$

where D_E is the associated differential operator on the bundle $E \to B$ of closed oriented n-manifolds.

Likewise, a natural self-adjoint elliptic operator has a family index in K^{-1} and so it defines a characteristic class in K^{-1}. Similarly, by the index theorem, there is a universal symbol class and the above formula holds.

On any closed oriented Riemannian manifold of odd dimension, there is the *odd signature operator* $D : \mathcal{A}^{\mathrm{ev}}(M) \to \mathcal{A}^{\mathrm{ev}}(M)$ on forms of even degree. It is self-adjoint, elliptic and its kernel is the space of harmonic form of even degree, which is isomorphic to $H^{\mathrm{ev}}(M;\mathbb{C})$. Given any smooth oriented M-bundle $f : E \to B$ we can choose a Riemannian metric on the fibres and study the induced family of elliptic self-adjoint operators.

Theorem 1.1. *The following assertions hold.*

(i) *The family index of the odd signature operator on an oriented bundle $E \to B$ with odd-dimensional fibres is trivial,* $\mathrm{ind}(D) = 0 \in K^{-1}(B)$.

(ii) *Equivalently, if* $\mathrm{th}(\sigma_{2m-1}) \in K^{-1}(\mathrm{MTSO}(2m-1))$ *is the universal symbol class, then* $\alpha_E^*(\sigma_{2m-1}) = 0$.

The equivalence of the two statements of Theorem 1.1 is given by the index theorem. This result is interesting because σ_{2m-1} is *not* trivial ($\mathrm{ch}(\sigma_{2m-1}) \in H^*(\mathrm{MTSO}(2m-1);\mathbb{Q})$ is essentially the Hirzebruch $\mathcal{L}$-class, see Proposition 4.2). Thus one cannot show Theorem 1.1 (ii) only using algebraic topology. Instead, we give an analytic argument for Theorem 1.1 (i). There are two ingredients for this. The first (well known, but profound) is that the kernel of D has a cohomological meaning by the Hodge–de Rham theorem. The second (not well known, but rather elementary) is a spectral-theoretic argument showing that the family index of any operator with constant kernel dimension is zero. Because $\sigma_{2m-1} \neq 0$, we can then use Theorem 1.1 (ii) to show vanishing theorems for characteristic classes of manifold bundles, by cohomological computations.

Theorem 1.2. *Let M be a closed oriented $(2m-1)$-dimensional manifold. The following assertions hold.*

(i) *The Hirzebruch $\mathcal{L}$-class* $\mathrm{th}_{-L_{2m-1}} \mathcal{L} \in H^{4*-2m+1}(\mathrm{MTSO}(2m-1);\mathbb{Q})$ *is killed by the Madsen–Tillmann–Weiss map* $\Sigma^\infty(B\,\mathrm{Diff}^+(M))_+ \to \mathrm{MTSO}(2m-1)$.

(ii) *For any oriented smooth fibre bundle $f : E \to B$ with fibre M, the generalized MMM-class $f_!(\mathcal{L}(T_v E)) \in H^*(B;\mathbb{Q})$ is trivial.*

The precise meaning of this theorem will be clarified in the main text. Here is an immediate consequence.

Corollary 1.3. *The Madsen–Tillmann–Weiss map* $\alpha : B\,\mathrm{Diff}^+(M) \to \Omega^\infty \mathrm{MTSO}(3)$ *is trivial in rational cohomology (in positive degrees) for any closed oriented 3-dimensional manifold M.*

This is an amusing result. Recently, Hatcher and Wahl [24] showed an analogue of the Harer–Ivanov homological stability for mapping class groups of 3-manifolds. Moreover, for large classes three-dimensional manifolds, it is known that the components of the diffeomorphism group are contractible (but that tends to become wrong after stabilization). One might be tempted to think that these results help make the proof of the analogue of the Mumford conjecture valid, leading to a description of the stable homology of mapping class groups of 3-manifolds in terms of the homology of $\Omega^\infty \mathrm{MTSO}(3)$. Corollary 1.3 shows that this is not the case.

Here is another consequence of Theorem 1.2:

Corollary 1.4. *Let $E \to B$ be an oriented fibre bundle over a closed oriented manifold with odd-dimensional closed fibres. Then* $\operatorname{sign}(E) = 0$.

This is an old theorem, known to Atiyah 40 years ago [5], but apparently he did not publish a proof (perhaps the proof Atiyah had in mind is along the lines of the argument of the present paper). Proofs of Corollary 1.4 were given by Meyer [32] and Lück–Ranicki [28]. In fact, Theorem 1.2 and Corollary 1.4 are equivalent, as we will see in Section 4.4.

Since the odd signature operator is real, it has an index in real K-theory as well. It turns out that in dimensions of the form $4k-1$, this real index is trivial as well. In dimensions of the form $4r+1$, however, the real index is usually not zero. This is discussed in Section 5.

Theorem 1.1 is stronger than Theorem 1.2, because it also has consequences in mod p cohomology. We prove two things for oriented 3-manifolds in that direction, see Section 6. Fix an oriented 3-manifold M and consider

$$\alpha_{E_M}^* : H^{4k-1}(\mathrm{MTSO}(3); \mathbb{F}_p) \to H^{4k-1}(B\,\mathrm{Diff}^+(M); \mathbb{F}_p).$$

We will show that this map is zero in two cases:

- For fixed $k \geq 1$ and almost all odd primes p (Theorem 6.1), and
- for a fixed odd prime p and an infinite number of values of k (Theorem 6.2).

In both cases, the set of pairs (p,k) to which the theorem applies does not depend on M.

In Section 7, we generalize Theorem 1.1 to manifold bundles with boundary. We do not attempt to give a meaning to its first part (which is an analytic result), but only to its second part. Though the map α_E does not exist when $E \to B$ is a manifold bundle with boundary, the map α_E can be defined when the boundary bundle is trivialized. It turns out that the appropriate formulation of Theorem 1.1 is true in this case and so are all conclusions that are derived from it (Theorems 1.2, 6.1 and 6.2).

In a companion paper [18] we show that all of $H^{*>0}(\mathrm{MTSO}(2m); \mathbb{Q})$ is detected on bundles of $2m$-manifolds and that, with the exception of multiples of the Hirzebruch $\mathcal{L}$-class, all of $H^{*>0}(\mathrm{MTSO}(2m+1); \mathbb{Q})$ is detected on bundles of $(2m+1)$-manifolds. This means that Theorem 1.2 is the only vanishing theorem of this type.

1.1. Outline of the paper. Section 2 is a survey on the stable homotopy theory which is needed in this paper. We briefly discuss general Thom spectra, the Madsen–Tillmann–Weiss spectra, the Pontrjagin–Thom construction, the Madsen–Tillmann–Weiss map and Thom isomorphisms. Section 3 provides the necessary constructions from index theory. In Section 4, we discuss the odd signature operator and prove Theorem 1.1. Also, we show Theorem 1.2 and Corollary 1.3. Section 5 discusses the real index of the odd signature operator. Finally, in Section 6, we discuss the vanishing theorem in finite characteristic. Section 7 discusses the extension of the results to the bounded case. In Appendix A, we compute the component group $\pi_0(\mathrm{MTSO}(n))$, refining a computation in [22]. This is needed in Section 5 and might be of independent interest.

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2. Background material on Madsen–Tillmann-Weiss spectra

In this section, we review some material on the Madsen–Tillmann–Weiss spectra. Most of the content of this section is either standard (the reader is referred to the textbook [40] for proofs and much more details on Thom spectra) or we learned it from [22].

2.1. Stable vector bundles and their Thom spectra. For our purposes, a *stable vector bundle* V on a space X is a map $\xi_V : X \to \mathbb{Z} \times \mathrm{BO}$. The *rank* of V is the locally constant function

$$X \xrightarrow{\xi_V} \mathbb{Z} \times \mathrm{BO} \to \mathbb{Z}.$$

By means of the inclusion map $\{d\} \times \mathrm{BO}(d) \to \mathbb{Z} \times \mathrm{BO}$, any ordinary vector bundle is a stable vector bundle. Furthermore, using the Whitney sum map $\mu : \mathbb{Z} \times \mathrm{BO} \times \mathbb{Z} \times \mathrm{BO} \to \mathbb{Z} \times \mathrm{BO}$ and the inversion map $\iota : \mathbb{Z} \times \mathrm{BO} \to \mathbb{Z} \times \mathrm{BO}$, we can add and subtract stable vector bundles.

The *Thom space* of a vector bundle $V \to X$ is the space $\mathrm{Th}(V) = X^V = \mathbb{D}(V)/\mathbb{S}(V)$, the quotient of the unit disc bundle by the unit sphere bundle. A stable vector bundle does not have a Thom space, but there is the *Thom spectrum* $\mathbf{Th}(V)$.

The homotopy type of the spectrum $\mathbf{Th}(W)$ depends only on the homotopy class of ξ_W. We note some standard properties of the Thom spectrum construction. First of all, it is natural with respect to pullbacks of stable vector bundles. If $W \to X$ is an ordinary vector bundle, then the Thom spectrum is homotopy equivalent to the suspension spectrum $\Sigma^\infty X^W$ of the Thom space of W. In particular, the Thom spectrum of the trivial 0-dimensional bundle $\underline{0}$ on X is $\Sigma^\infty X_+$. Let W be a stable vector bundle and V an ordinary vector bundle. Then there is an inclusion map

$$\mathbf{Th}(W) \to \mathbf{Th}(W \oplus V).$$

Let $V \to X$ and $W \to Y$ be two stable vector bundles. There is a canonical homotopy equivalence $\mathbf{Th}(V) \wedge \mathbf{Th}(W) \simeq \mathbf{Th}(V \times W)$. If $X = Y$, we get a diagonal map

$$\mathrm{diag} : \mathbf{Th}(V \oplus W) \to \mathbf{Th}(V) \wedge \mathbf{Th}(W).$$

A special case is the diagonal $\mathbf{Th}(V) \to \Sigma^\infty X_+ \wedge \mathbf{Th}(V)$.

2.2. Orientations and Thom isomorphisms. Assume that A is an associative and commutative ring spectrum with unit (the rather old-fashioned notion of [3] is sufficient for our purposes). Let $V \to X$ be a stable vector bundle of rank $d \in \mathbb{Z}$. The cohomology $A^*(\mathbf{Th}(V))$ is a graded left $A^*(X)$-module; a pair $(x, y) \in A^n(X) \times A^m(\mathbf{Th}(V))$ is sent to the composition

$$x \cdot y : \mathbf{Th}(V) \xrightarrow{\mathrm{diag}} \Sigma^\infty X_+ \wedge \mathbf{Th}(V) \xrightarrow{x \wedge y} \Sigma^n A \wedge \Sigma^m A \to \Sigma^{n+m} A.$$

A *Thom class* or *A-orientation* of V with A-coefficients is a cohomology class $v \in A^d(\mathbf{Th}(V))$ such that for any $x \in X$, the image of v under the restriction map

$$A^d(\mathbf{Th}(V)) \to A^d(\mathbf{Th}(V_x)) \cong A^d(\mathbb{S}^d) \cong A^0(*)$$

is a unit. This is equivalent to saying that $A^*(\mathbf{Th}(V))$ is a free $A^*(X)$-module on the generator v or that the map $\mathrm{th}_V^A : A^*(X) \to A^{*+d}(\mathbf{Th}(V))$; $x \mapsto x \cdot v$ is an isomorphism. If this is the case, then th_V^A is called the *Thom isomorphism*. If A is understood, then the superscript is often omitted.

More generally, we can define a *relative* Thom isomorphism. Let V be a stable vector bundle of rank d and let W be another stable vector bundle of rank e. Assume that V has a Thom class v. Let

$$\mathrm{th}_{W,W\oplus V}^A : A^*(\mathbf{Th}(W)) \to A^{*+d}(\mathbf{Th}(W \oplus V))$$

be the homomorphism which maps $x \in A^n(\mathbf{Th}(W))$ to the composition

$$\mathbf{Th}(W \oplus V) \xrightarrow{\mathrm{diag}} \mathbf{Th}(W) \wedge \mathbf{Th}(V) \xrightarrow{x \wedge v} \Sigma^n A \wedge \Sigma^d A \to \Sigma^{n+d} A;$$

this is an isomorphism of $A^*(X)$-modules. If $v \in A^d(\mathbf{Th}(V))$ and $w \in A^e(\mathbf{Th}(W))$ are Thom classes, then $\mathrm{th}_{W,W\oplus V}^A(v)$ is a Thom class for $V \oplus W$. If the Thom classes of different stable vector bundles are chosen compatibly in this way, then the Thom isomorphisms are compatible in the sense that $\mathrm{th}_{U\oplus V,U\oplus V\oplus W} \circ \mathrm{th}_{U,U\oplus V} = \mathrm{th}_{U,U\oplus V\oplus W}$ whenever U is an arbitrary stable vector bundle. We shall use the short notation $\mathrm{th}_V : A^*(\mathbf{Th}(W)) \to A^*(\mathbf{Th}(V \oplus W))$ if W is understood.

Examples. The examples of ring spectra which play a role in this paper are Eilenberg–MacLane spectra HR for commutative rings R as well as the complex K-theory spectrum K. It is well known that a vector bundle which is oriented in the ordinary sense has a preferred $H\mathbb{Z}$-Thom class (depending on the choice of a generator of $H_1(\mathbb{R}; \mathbb{R} \setminus 0; \mathbb{Z})$). A stable vector bundle has an $H\mathbb{Z}$-orientation if and only if $w_1(V) = 0$. Any complex vector bundle has a K-orientation and so does every complex stable vector bundle, i.e., a formal difference of complex vector bundles. While the universal vector bundle $L_n \to \mathrm{BSO}(n)$ has only two different $H\mathbb{Z}$-orientations, the situation is more complicated for K-theory. If v is a K-orientation of the universal complex vector bundle on $BU(n)$ and $u \in K^0(BU(n))$ is an invertible element, then $u \cdot v$ is another orientation. But there are many units in the ring $K^0(BU(n))$. We follow the convention that the Thom class of a complex vector bundle $\pi : V \to X$ of rank n is represented by the complex

$$0 \to \pi^*\Lambda^0 V \xrightarrow{v\wedge} \pi^*\Lambda^1 V \xrightarrow{v\wedge} \pi^*\Lambda^2 V \to \cdots \to \pi^*\Lambda^n V \xrightarrow{v\wedge} 0.$$

The following observation is important for index theory. Let $V \to X$ be a real vector bundle. Then $V \otimes \mathbb{C}$ has a natural K-orientation. Therefore there is a relative Thom isomorphism

$$K^*(\mathbf{Th}(-V)) \cong K^*(\mathbf{Th}(-V \oplus V \otimes \mathbb{C})) \cong K^*(\mathbf{Th}(V)). \tag{2.1}$$

In this equation we used the Bott periodicity isomorphism $\beta : K^* \to K^{*+2}$ to identify K^* with K^{*+2}. We will do this throughout the whole paper.

2.3. The Pontrjagin–Thom construction. Let M be a closed smooth oriented manifold of dimension n and let $\mathrm{Diff}^+(M)$ be the group of diffeomorphisms of M endowed with the Whitney C^∞-topology. We will study *smooth oriented M-bundles*, i.e., fibre bundles $f: E \to B$ with structural group $\mathrm{Diff}^+(M)$ and fibre M. Let $Q \to B$ be the associated $\mathrm{Diff}(M)$-principal bundle. The *vertical tangent bundle* is the oriented vector bundle

$$T_v E := Q \times_{\mathrm{Diff}^+(M)} TM \to Q \times_{\mathrm{Diff}^+(M)} M = E.$$

The *normal bundle* of f is the stable vector bundle $\nu(f) := -T_v E$.

If B is paracompact, then there is an embedding $j: E \to B \times \mathbb{R}^\infty$, i.e., $\mathrm{proj} \circ j = f$ and the image of j has a tubular neighborhood U. Collapsing everything outside U to the base-point defines a map of spectra

$$\mathrm{PT}_f : \Sigma^\infty B_+ \to \mathbf{Th}(\nu(f)),$$

the *Pontrjagin–Thom map* (or PT-map, for short). The space of tubular neighborhoods of a given embedding is contractible and the space of embeddings into $\mathbb{R}^\infty$ is contractible. Thus the map PT_f depends on a contractible space of choices. In particular, its homotopy class only depends on f. For more details on the PT-construction in this parameterized setting, see [21, Section 3].

The Pontrjagin–Thom map can be used to define the *umkehr map* in generalized cohomology. Let $f: E \to B$ be a smooth fibre bundle of dimension n, A a ring spectrum and we assume that $\nu(f)$ has an A-orientation. The umkehr homomorphism $f_! : A^*(E) \to A^{*-n}(B)$ is defined as the composition

$$A^*(E) \xrightarrow{\mathrm{th}_{\nu(f)}} A^{*-n}(\mathbf{Th}(\nu(f))) \xrightarrow{\mathrm{PT}_f^*} A^{*-n}(B).$$

2.4. Madsen–Tillmann–Weiss spectra and Madsen–Weiss maps. Let $n \geq 0$, let $\mathrm{BSO}(n)$ be the classifying space for oriented Riemannian n-dimensional vector bundles and let $L_n \to \mathrm{BSO}(n)$ be the universal oriented vector bundle. The reader should note that the space BSO(0) is homotopy equivalent to the two-point space $\mathbb{S}^0$ and therefore it is *not* the classifying space for the group SO(0). The most natural explanation for this phenomenon occurs in the framework of stacks. Let $\mathrm{Or}(\mathbb{R}^n)$ be the set of orientations of the vector space $\mathbb{R}^n$; the group $O(n)$ acts on $\mathrm{Or}(\mathbb{R}^n)$. The stack of oriented n-dimensional vector bundles is the quotient stack $\mathrm{Or}(\mathbb{R}^n)/\!\!/O(n)$. For $n \geq 1$, the $O(n)$-action on $\mathrm{Or}(\mathbb{R}^n)$ is transitive and hence $\mathrm{Or}(\mathbb{R}^n)/\!\!/O(n) \cong */\!\!/\mathrm{SO}(n)$, while for $n = 0$, we have $\mathrm{Or}(\mathbb{R}^0)/\!\!/O(0) \cong \mathbb{S}^0$.

Definition 2.1. The Thom spectrum of the stable vector bundle $-L_n$ on $BSO(n)$ is called the *Madsen–Tillmann–Weiss spectrum* (or MTW-spectrum) and is denoted by $\mathrm{MTSO}(n)$. Moreover, we denote by $\mathrm{MSO}(n)$ the Thom spectrum of L_n.

Let $f: E \to B$ be a smooth oriented M-bundle. Recall that the space of orientation-preserving bundle maps $\lambda: T_v E \to L_n$ is contractible. Therefore the orientation defines a contractible space of maps

$$\kappa_E = \mathbf{Th}(\lambda) : \mathbf{Th}(-T_v E) \to \mathrm{MTSO}(n).$$

The *Madsen–Tillmann–Weiss map* (or MTW-map) of the bundle $f : E \to B$ is the composition

$$\alpha_E := \kappa_E \circ \mathrm{PT}_f : \Sigma^\infty B_+ \to \mathrm{MTSO}(n), \tag{2.2}$$

which is defined uniquely up to a contractible space of choices.

For the universal oriented M-bundle $E_M \to B\,\mathrm{Diff}^+(M)$, we obtain a universal MTW-map

$$\alpha_{E_M} : \Sigma^\infty(B\,\mathrm{Diff}^+(M))_+ \to \mathrm{MTSO}(n).$$

On the other extreme, the constant map $M \to *$ is a smooth oriented M-bundle and its MTW-map is a map $\alpha_M : \Sigma^\infty \mathbb{S}^0 \to \mathrm{MTSO}(n)$.

By the Thom isomorphism, we have

$$\mathrm{th}_{-L_n} : H^*(\mathrm{BSO}(n); R) = H^*(\Sigma^\infty \mathrm{BSO}(n)_+; R) \xrightarrow{\cong} H^{*-n}(\mathrm{MTSO}(n); R). \tag{2.3}$$

Proposition 2.2. *Let $f : E \to B$ be an oriented n-dimensional manifold bundle, let $\alpha_E : \Sigma^\infty B_+ \to \mathrm{MTSO}(n)$ be its MTW-map and $c \in H^*(\mathrm{BSO}(n))$. Then*

$$\alpha_E^* \,\mathrm{th}_{-L_n}(c) = f_!(c(T_v(E))) \in H^{*-n}(B),$$

where $c(T_v E) \in H^*(E)$ is the value of the characteristic class c on $T_v E$.

Proof. By definition

$$\alpha_E^* \,\mathrm{th}_{-L_n}(c) \stackrel{(2.2)}{=} \mathrm{PT}_f^* \,\mathbf{Th}(\lambda)^* \,\mathrm{th}_{-L_n}(c) = \mathrm{PT}_f^* \,\mathrm{th}_{-T_v E}(c(T_v E)).$$

The second equality expresses the compatibility of Thom isomorphisms and pullbacks. □

Therefore any $c \in H^*(\mathrm{BSO}(n))$ defines a characteristic class of oriented n-manifold bundles. We call these classes "generalized MMM-classes", because the case $n = 2$, $c = \chi^{i+1}$ gives the classes κ_i defined by Mumford [37], Miller [33] and Morita [36].

Recall the adjunction between the two functors Σ^∞ and Ω^∞: given a spectrum $\mathbf{E}$ and a space X, there is a natural bijection

$$[X, \Omega^\infty \mathbf{E}] \cong [\Sigma^\infty X_+, \mathbf{E}].$$

Under this adjunction, α_E corresponds to a map $B \to \Omega^\infty \mathrm{MTSO}(n)$, which is the original MTW-map studied in [30], [31], [22]. We will call this adjoint by the same name and denote it by the same symbol. There is no danger of confusion, because we keep our notation for spaces and spectra entirely disjoint. For the more computational purposes of the present paper, the spectra point of view is more transparent and convenient.

The adjoint $\Sigma^\infty(\Omega^\infty \mathbf{E})_+ \to \mathbf{E}$ of the identity on $\Omega^\infty \mathbf{E}$ induces a map

$$s : A^*(\mathbf{E}) \to A^*(\Omega^\infty \mathbf{E}),$$

the *cohomology suspension*, whenever A is a spectrum. If $A = H\mathbb{Q}$, then the right-hand side is a graded-commutative $\mathbb{Q}$-algebra, but the left-hand side is only a graded $\mathbb{Q}$-vector space.

Let Λ denote the functor which associates to a graded module the free, graded-commutative algebra it generates; s extends to an algebra homomorphism

$$s : \Lambda(H^{*>0}(\mathbf{E};\mathbb{Q})) \to H^*(\Omega_0^\infty \mathbf{E};\mathbb{Q}). \tag{2.4}$$

This is an isomorphism by a classical result of algebraic topology, see [34, p. 262f.].

If X is a space and $\Sigma^\infty X_+ \to \mathbf{E}$ a map with adjoint $X \to \Omega^\infty \mathbf{E}$, then the following diagram commutes:

$$\begin{array}{ccc} A^*(\mathbf{E}) & \longrightarrow & A^*(\Sigma^\infty X_+) \\ \downarrow s & & \| \\ A^*(\Omega^\infty \mathbf{E}) & \longrightarrow & A^*(X). \end{array} \tag{2.5}$$

The rational cohomology of $\Omega^\infty \mathrm{MTSO}(n)$ can be easily computed using (2.5), (2.4), (2.3) and the well-known isomorphisms

$$H^*(\mathrm{BSO}(2m+1);\mathbb{Q}) \cong \mathbb{Q}[p_1,\dots,p_m],$$
$$H^*(\mathrm{BSO}(2m);\mathbb{Q}) \cong \mathbb{Q}[p_1,\dots,p_m,\chi]/(\chi^2 - p_m).$$

3. Background material on index theory

In this section, we will present background material on index theory for bundles of compact manifolds. For details, the reader is referred to either the original source [13] or to the textbook [29].

There are two types of the K-theoretic index theorem: One for usual elliptic operators and another one for self-adjoint elliptic operators on a fibre bundle $f : E \to B$. In the former case, the index is an element in $K^0(B)$ while in the latter one we get an index in $K^{-1}(B)$.

Assume that $f : E \to B$ is a smooth fibre bundle on a paracompact space B with compact closed fibres. Assume that a fibre-wise smooth Riemannian metric on the vertical tangent bundle $T_v E$ is chosen. All vector bundles on E will be fibre-wise smooth (i.e., the transition functions are smooth in the fibre-direction) and all hermitian metrics on vector bundles are understood to be smooth. All differential operations, like exterior derivatives and connections, will be fibre-wise. In particular, when we talk about an elliptic operator on E, we mean what is usually called a family of elliptic operators on E.

For an hermitian vector bundle $V \to E$, we denote $\Gamma_B(V) = \bigcup_{x\in B} \Gamma(E_x; V_x)$, where $E_x = f^{-1}(x)$ and $V_x = V|_{E_x}$. This family of vector spaces over B can be made into a vector bundle (of Fréchet spaces) by requiring that a section $s : B \to \Gamma_B(V)$ is continuous if the associated section of $V \to E$ is continuous in the C^∞-topology. Using the metrics on $T_v E$ and V and a connection on V, we can define the L^2-Sobolev norms $\|\dots\|_r$ on $\Gamma_B(V)$, for all $r \geq 0$. The completion with respect to this norm is a Hilbert bundle which we denote by $W_B^{2,r}(V)$.

There is a technical issue about the structural group of $W_B^{2,r}(V)$; it is discussed and solved in [10, pp. 5, 13f., 38–43]. The structural group is not the general linear group $\mathrm{GL}(W^{2,s})$ with the norm topology, but $\mathrm{GL}(W^{2,s})_{\mathrm{co}}$, the same group with the compactly generated compact-open topology. The group $\mathrm{GL}(W^{2,s})_{\mathrm{co}}$ is contractible (this is much easier than

Kuiper's theorem which asserts that $\mathrm{GL}(W^{2,s})$ with the norm topology is contractible). Moreover, $\mathrm{GL}(W^{2,s})_{\mathrm{co}}$ acts continuously by conjugation on the space of Fredholm operators with a suitably redefined topology. This new space of Fredholm operators is homotopy equivalent to the original one.

Therefore the Hilbert bundles $W_B^{2,r}(V)$ are trivial and the trivialization is unique up to homotopy (in fact, the space of trivializations is contractible).

Let $V_0, V_1 \to E$ be two hermitian vector bundles and let $D : V_0 \to V_1$ be an elliptic operator of order m. Then D has an extension to the bundle of Sobolev spaces

$$D : W_B^{2,s+m}(V) \to W_B^{2,s}(V),$$

which consist of Fredholm operators. We choose, for any vector bundle V, an elliptic pseudo-differential operator A_V of order $-m/2$ which is invertible, for example $A_V = (1+\nabla^*\nabla)^{-m/4}$ for a connection ∇ on V. The operator $A_{V_1} D A_{V_0}$ has order 0 and so it induces a family of Fredholm operator $W_B^{2,0}(V_0) \to W_B^{2,0}(V_1)$. After an application of the trivializations above, we get a continuous map

$$\mathrm{ind}(D) : B \to \mathrm{Fred}(H),$$

where H is a fixed separable, infinite-dimensional Hilbert space. The Atiyah–Jänich theorem states that $\mathrm{Fred}(H)$ is a classifying space for complex K-theory and therefore we get an element $\mathrm{ind}(D) \in K^0(B)$ that does not depend on the choices involved in the construction.

If $D : \Gamma_B(V) \to \Gamma_B(V)$ is formally self-adjoint, we get a map

$$\mathrm{ind}(D) : B \to \mathrm{Fred}_{\mathrm{s.a.}}(H),$$

into the space of self-adjoint operators on H. By work of Atiyah and Singer [12], $\mathrm{Fred}_{\mathrm{s.a.}}(H)$ has three connected components:

$$\mathrm{Fred}_{\mathrm{s.a.}}(H) = \mathrm{Fred}^{+}_{\mathrm{s.a.}}(H) \coprod \mathrm{Fred}^{-}_{\mathrm{s.a.}}(H) \coprod \mathrm{Fred}^{0}_{\mathrm{s.a.}}(H).$$

Here $\mathrm{Fred}^{\pm}_{\mathrm{s.a.}}(H)$ is the subspace of operators such that $\pm A$ is essentially positive (an operator is *essentially positive* if there exists an A-invariant subspace $U \subset H$ of finite codimension, such that $A|_U$ is positive definite). The two spaces $\mathrm{Fred}^{\pm}_{\mathrm{s.a.}}(H) \subset \mathrm{Fred}_{\mathrm{s.a.}}(H)$ are contractible (in fact, $\mathrm{Fred}^{\pm}_{\mathrm{s.a.}}(H)$ is star-shaped with center $\pm\,\mathrm{id}$). The complement $\mathrm{Fred}^{0}_{\mathrm{s.a.}}(H)$ has the homotopy type of the infinite unitary group $U(\infty)$; see [12]. Thus a self-adjoint elliptic operator D on E has an index $\mathrm{ind}(D) \in K^{-1}(B)$ (if D is essentially definite, then $\mathrm{ind}(D) = 0$).

3.1. The topological index. Let $f : E \to B$ be a smooth manifold bundle with vertical cotangent bundle $\pi : T \to E$. Let $D : \Gamma_B(V_0) \to \Gamma_B(V_1)$ be an elliptic differential operator. Recall that the *symbol* of D is a bundle map $\mathrm{smb}_D : \pi^* V_0 \to \pi^* V_1$ which is an isomorphism outside the zero section. If D has order 1, then the symbol is given by

$$\mathrm{smb}_D(\xi)v = i(D(fs) - fDs),$$

where ξ is a vertical cotangent vector at $x \in E$, f is a smooth function such that $df_x = \xi$ and s is a section of V_0 such that $s(x) = v$.

We will constantly identify the vertical cotangent and the vertical tangent bundle. The symbol smb_D defines the *symbol class* $[\mathrm{smb}_D]_0 \in K^0(T; T \setminus 0) = K^0(E^T)$ of D.

Following [8], we can associate a symbol class $[\mathrm{smb}_D]_1 \in K^{-1}(E^T)$ to a self-adjoint elliptic operator D. The symbol map $\mathrm{smb}_D : \pi^* V \to \pi^* V$ is a self-adjoint endomorphism of $\pi^* V$ and it is an isomorphism away from the zero section. Let $\tilde{\pi} : T \oplus \mathbb{R} \to E$. We define $[\mathrm{smb}_D]_1$ to be the class in $K^{-1}(E^T) = K^0((T, T \setminus 0) \times (\mathbb{R}, \mathbb{R} \setminus 0))$ represented by the complex

$$0 \to \tilde{\pi}^* V \xrightarrow{\tilde{\mathrm{smb}}_D} \tilde{\pi}^* V \to 0,$$

where $\tilde{\mathrm{smb}}_D$ is given at the point $(x,t) \in T \oplus \mathbb{R}$ by $\tilde{\mathrm{smb}}_{D(x,t)} := (\mathrm{smb}_D)_x - it\mathbf{1}$. Actually, [8] gives a different formula, but the passage between the two formulations is by an elementary deformation. We leave it to the reader to figure that out.

Recall the relative Thom isomorphism (2.1) $\mathrm{th}_{-T_v E \otimes \mathbb{C}} : K^*(E^{T_v E}) \to K^*(E^{-T_v E})$. The Atiyah–Singer family index theorem ([13] for the usual case, [8] for the self-adjoint case) states that in both cases ($i = 0, 1$)

$$\mathrm{ind}(D) = \beta^{-d} \, \mathrm{PT}_f^* \, \mathrm{th}_{-T_v E \otimes \mathbb{C}}([\mathrm{smb}_D]_i) \in K^i(B).$$

3.2. Universal operators. Now we assume that the first order elliptic operator D on the n-dimensional oriented bundle $E \to B$ family is *symbolically universal*. By that expression, we mean that there exist $\mathrm{SO}(n)$-representations W_0 and W_1 and an $\mathrm{SO}(n-1)$-equivariant map $\gamma : \mathbb{R}^n \times W_0 \to \mathbb{R}^n \times W_1$ such that

(i) $\gamma(v, w) = (v, \gamma_v(w))$, where γ_v is linear and an isomorphism for $v \neq 0$;

(ii) V_0 and V_1 are, as Hermitian vector bundles, isomorphic to the associated bundle

$$\mathrm{Fr}_v(E) \times_{\mathrm{SO}(n)} W_i \to E;$$

(iii) the symbol smb_D is equal to the bundle map

$$\mathrm{Fr}_v(E) \times_{\mathrm{SO}(n)} (\mathbb{R}^n \times W_0) \to \mathrm{Fr}_v(E) \times_{\mathrm{SO}(n)} (\mathbb{R}^n \times W_1)$$

induced by γ.

The map γ defines a class $\sigma_D \in K^0_{\mathrm{SO}(n)}(\mathbb{R}^n, \mathbb{R}^n_0)$. Here we abbreviate $\mathbb{R}^n_0 := \mathbb{R}^n \setminus \{0\}$ and subscribe to an analogous convention for arbitrary vector bundles. The image of σ_D under the standard homomorphism $K^0_{\mathrm{SO}(n)}(\mathbb{R}^n, \mathbb{R}^n_0) \to K^0(L_n, L_n \setminus 0) \cong K^0(\mathrm{MSO}(n))$ is denoted by the same symbol. Clearly, the class σ_D pulls back to the symbol class $[\mathrm{smb}_D]$ under the map $\mathbf{Th}(T_v E) \to \mathrm{MSO}(n)$. By the Thom isomorphism $K^0(\mathrm{MSO}(n)) \cong K^0(\mathrm{MTSO}(n))$, we get a class $\mathrm{th}\,\sigma_D \in K^0(\mathrm{MTSO}(n))$.

The index theorem for the symbolically universal operator D on the fibre bundle $f : E \to B$ reads

$$\mathrm{ind}(D) = \alpha_E^* \, \mathrm{th}\,\sigma_D. \tag{3.1}$$

Similarly, for self-adjoint symbolically universal operators, we get a universal symbol class $\sigma_D \in K^{-1}(\mathrm{MSO}(n))$ and $\mathrm{th}\,\sigma \in K^{-1}(\mathrm{MTSO}(n))$ and the index theorem is expressed by the same formula as in (3.1).

4. The index theorem for the odd signature operator

4.1. The signature operators. Let M be a closed oriented Riemannian manifold of dimension n. Recall that there is the Hodge star operator $* : \mathcal{A}^k(M) \to \mathcal{A}^{n-k}(M)$. There might exist different sign conventions about $*$; we are constantly using the definition given in [11]. The star operator is an complex-linear isometry and satisfies

$$** = (-1)^{k(n-k)} : \mathcal{A}^k(M) \to \mathcal{A}^k(M).$$

The adjoint $d^{\mathrm{ad}} : \mathcal{A}^k(M) \to \mathcal{A}^{k-1}(M)$ of the exterior derivative can be written as

$$d^{\mathrm{ad}} = (-1)^{n(k+1)+1} * d * .$$

If $n = 2m$, then one introduces the involution $\tau := i^{k(k-1)+m} *$ on k-forms [11, p. 574]. Then $D_{2m} = d + d^{\mathrm{ad}}$ satisfies $D_{2m}\tau = -\tau D_{2m}$. If $\mathcal{A}^*_{\pm}(M)$ denote the ± 1-eigenbundles of τ, then the operator $D : \mathcal{A}^*_+(M) \to \mathcal{A}^*_-(M)$ is the (even) *signature operator*. This is an elliptic differential operator of order 1 whose index is the same as the signature of M.

Now let $n = 2m-1$ and note the formulae

$$** = 1 \quad \text{and} \quad d^{\mathrm{ad}} = (-1)^k * d * : \mathcal{A}^k(M) \to \mathcal{A}^{k-1}(M).$$

The *odd signature operator* $D = D_{2m-1} : \bigoplus_{p \geq 0} \mathcal{A}^{2p}(M) \to \bigoplus_{p \geq 0} \mathcal{A}^{2p}(M)$ is defined to be

$$D_{2m-1}\phi = i^m (-1)^{p+1} (*d - d*)\phi$$

whenever $\phi \in \mathcal{A}^{2p}(M)$, see [8].

A straightforward, but tedious, calculation shows that

(i) D is formally self-adjoint and elliptic;

(ii) $D^2 = \Delta = (d + d^{\mathrm{ad}})^2$, the Laplace–Beltrami operator.

Moreover, one observes that

$$\ker(D) = \ker(\Delta) = \bigoplus_{p \geq 0} H^{2p}(M;\mathbb{C})$$

and that consequently

$$\dim \ker D = \sum_{p \geq 0} \dim H^{2p}(M;\mathbb{C}), \tag{4.1}$$

which is the main property needed for the proof of Theorem 1.1.

Both signature operators are symbolically universal. The even one is associated with the map of $\mathrm{SO}(2m)$-equivariant vector bundles on $\mathbb{R}^n$

$$\gamma_{2m} : \mathbb{R}^{2m} \times \Lambda^*_+(\mathbb{R}^{2m}) \otimes \mathbb{C} \to \mathbb{R}^{2m} \times \Lambda^*_-(\mathbb{R}^{2m}) \otimes \mathbb{C},$$
$$\gamma_{2m}(v, \omega) = i(v \wedge \omega - *(v \wedge (*\omega))).$$

The odd signature operator is associated with the representation $\Lambda^{\mathrm{ev}}(\mathbb{R}^{2m-1})$ and the map

$$\gamma_{2m-1}(v, \omega) = i^{m-1}(-1)^p(*(v \wedge \omega) - v \wedge (*\omega)).$$

Abbreviate $\mathbb{R}^n_0 := \mathbb{R}^n \setminus 0$. Let $\sigma_{2m-1} \in K^{-1}_{\mathrm{SO}(2m-1)}(\mathbb{R}^{2m-1}, \mathbb{R}^{2m-1}_0)$ be the universal symbol class of the odd signature operator and $\sigma_{2m} \in K^0_{\mathrm{SO}(2m)}(\mathbb{R}^{2m}, \mathbb{R}^{2m}_0)$ be the universal symbol class of the even signature operator. We denote their images in $K^*(\mathrm{MSO}(n))$ by the same symbol.

4.2. The vanishing theorem. Let $f : E \to B$ be a smooth oriented M-bundle, M a closed oriented $(2m-1)$-manifold. Assume that we choose a Riemannian metric on the vertical tangent bundle (for any bundle on a paracompact base space such a metric exists; the space of these metrics is contractible). The odd signature operators on the fibres of f fit together to a family of self-adjoint elliptic differential operators. Therefore we have the family index

$$\mathrm{ind}(D) \in K^{-1}(B),$$

which does not depend on the auxiliary Riemannian metric, but which is an invariant of smooth oriented M-bundles. In the universal case, we get an element $\mathrm{ind}(D) \in K^{-1}(B\,\mathrm{Diff}^{+}(M))$.

The proof of the first part of Theorem 1.1 is an immediate consequence of (4.1) and Theorem 4.1 below. The second part follows immediately from the first part and the index formula (3.1).

Theorem 4.1 seems to be well known to experts in operator theory, see, e.g., [17, 5.1.4], and it is certainly implicitly contained in [12]. I have included the following rather elementary proof for the convenience of the reader.

Theorem 4.1. *Let B be a space and let $A : B \to \mathrm{Fred}^0_{\mathrm{s.a.}}(H)$; $x \mapsto A_x$ be a continuous map such that $x \mapsto \dim\ker A_x$ is locally constant. Then A is homotopic to a constant map.*

Proof. Step 1: First we show that we can deform A into a family A' consisting of invertible operators.

To this end, we note that because the dimension of $\ker A_x$ is locally constant, the union $\ker(A) := \bigcup_{x\in B} \ker(A_x)$ is a (finite-dimensional) vector bundle on B. Therefore the projection operator p_x onto the kernel of A_x depends continuously on x and p_x commutes with A_x because A_x is self-adjoint. Therefore $A_x + tp_x$ is Fredholm for all $t \in \mathbb{R}$ and $\mathrm{Spec}(A_x + tp_x) = \mathrm{Spec}\,A_x \setminus \{0\} \cup \{t\} \subset \mathbb{R}_{\neq 0}$. Thus for $t \neq 0$, $A_x + tp_x$ is invertible (and neither essentially negative nor positive).

Step 2: By step 1, we assume that A_x is invertible for all $x \in B$, in other words $\mathrm{Spec}(A_x) \subset \mathbb{R} \setminus 0$ for all $x \in B$. Let $h : \mathbb{R} \setminus 0 \to \mathbb{R}$ be the signum function. For any x and any $t \in [0, 1]$, the operator

$$th(A) + (1-t)A$$

is a self-adjoint invertible operator (the latter statement is easy to see because A and $h(A)$ commute). For $t = 0$, we get A and for $t = 1$, we get $h(A)$ which is a self-adjoint involution which is neither essentially positive nor negative.

Step 3: By step 2, we can assume that A is a map from B into the space $\mathcal{P}(H)$ of all self-adjoint involutions F on H such that $\mathrm{Eig}(F, \pm 1)$ are both infinite-dimensional. Let us show that $\mathcal{P}(H)$ is contractible. The unitary group $U(H)$ acts transitively on $\mathcal{P}(H)$ (by conjugation) and the isotropy group at a given F_0 is $U(\mathrm{Eig}(F_1, 1)) \times U(\mathrm{Eig}(F_0, -1))$. Thus we have a continuous bijection

$$U(H)/U(\mathrm{Eig}(F_0, 1)) \times U(\mathrm{Eig}(F_0, -1)) \to \mathcal{P}(H). \tag{4.2}$$

The map $U(H) \to \mathcal{P}(H)$, $u \mapsto uF_0u^{-1}$ has a local section and thus the bijection above is a homeomorphism. Here is a construction of the local section. Let $H_{\pm} := \mathrm{Eig}(F_0; \pm 1)$. For a given F, let $\tilde{u}_F$ be $\frac{1}{2}(1 \pm F)$ on $H_{\pm}$. The operator $\tilde{u}_F$ depends continuously on

F; $\tilde{u}_{F_0} = 1$. Therefore, for F close to F_0, $\tilde{u}_F$ is an isomorphism. A direct calculation shows that $\tilde{u}_F(H_\pm) = \mathrm{Eig}(F; \pm 1)$. An application of the Gram–Schmidt process defines a continuous family $F \mapsto u_F$ of unitary operators on a neighborhood of F_0 such that $u_F(H_\pm) = \mathrm{Eig}(F; \pm 1)$, in other words, $u_F F u_F^{-1} = F_0$, which is what we want.

The left-hand side space in (4.2) is contractible by Kuiper's theorem [27] and the long exact homotopy sequence, which completes the proof of the theorem. □

In the proof of the theorem we had the choice between two different contractible spaces of nullhomotopies of A; in the first step, we could choose either a positive value or a negative value of the real parameter t (put in another way: the spectral value can be pushed either in the positive or in the negative direction). The concatenation of these two nullhomotopies defines a map $B \to \Omega\, \mathrm{Fred}^0_{\mathrm{s.a.}} \simeq \Omega U \simeq \mathbb{Z} \times BU$, in other words an element in $K^0(B)$. It is not hard to see that this is the same as the class of the bundle $\ker(A) \to B$.

In the case of the odd signature operator on the smooth oriented fibre bundle, this is the K-theory class of the flat bundle $\bigoplus_{p \geq 0} H^{2p}(E/B; \mathbb{C})$ of even cohomology groups. This K-theory class is a characteristic class of smooth oriented fibre bundle, nevertheless, it is *not* induced by an element in $K^0(\mathrm{MTSO}(2m-1))$. This can be seen as follows. There exist odd-dimensional manifold bundles $f : E \to B$ such that

$$\sum_{p \geq 0} [H^{2p}(E/B; \mathbb{C})] \neq 0 \in K^0(B).$$

For example, ones takes orientation reversing involutions on $\mathbb{S}^1$ and $\mathbb{S}^{2m-2}$. The diagonal action $\mathbb{Z}/2 \curvearrowright \mathbb{S}^1 \times \mathbb{S}^{2m-2}$ is then orientation-preserving. The bundle

$$E\mathbb{Z}/2 \times_{\mathbb{Z}/2} (\mathbb{S}^1 \times \mathbb{S}^{2m-2}) \to B\mathbb{Z}/2$$

has the desired property. This shows that this K^0-invariant is nontrivial in all dimensions $2m - 1 > 1$.

On the other hand, $K^0(\mathrm{MTSO}(2m-1)) = 0$ by the following argument. By the Thom isomorphism in twisted K-theory ([19]), one computes

$$K^0_{\mathrm{SO}(2m-1)}(\mathbb{R}^{2m-1}, \mathbb{R}^{2m-1}_0) \cong K^{1+\tau}_{\mathrm{SO}(2m-1)}(*),$$

where τ is the twist induced from the central extension $\mathrm{Spin}^c(2m-1) \to \mathrm{SO}(2m-1)$. But $K^{-1+\tau}_{\mathrm{SO}(2m-1)}(*) = 0$, see [19, p. 11]. Moreover, writing $E(G; (X, Y)) := (EG \times_G X, EG \times_G Y)$ for a topological group G acting on a space pair (X, Y), we have

$$\begin{aligned} K^0(\mathrm{MTSO}(2m-1)) &\cong K^0(\mathrm{MSO}(2m-1)) \\ &\cong K^0(E(\mathrm{SO}(2m-2); (\mathbb{D}^{2m-1}, \mathbb{S}^{2m-2}))) \end{aligned}$$

By the Atiyah–Segal completion theorem [9] (or rather a version for space pairs that be deduced from [9, Corollary 2.2] by a 5-Lemma argument, together with [6, Proposition 10.12]), it follows that

$$K^0(E(\mathrm{SO}(2m-2); (\mathbb{D}^{2m-1}, \mathbb{S}^{2m-2}))) \cong (K^0_{\mathrm{SO}(2m-1)}(\mathbb{D}^{2m-1}, \mathbb{S}^{2m-2}))^\wedge = 0,$$

here $()^\wedge$ denotes completion at the augmentation ideal.

4.3. Cohomology calculation. In this section, we indicate how Theorem 1.2 is derived from Theorem 1.1. The essential computations are at least implicitly done in [11] and [8] and we shall give only a sketch. First note that the second statement of Theorem 1.2 is an immediate consequence of the first one in view of Proposition 2.2. Consider the commutative diagram

$$\begin{array}{ccc} K^0(\mathrm{MSO}(2m)) & \xrightarrow{\rho^*} & K^0(\Sigma^1\,\mathrm{MSO}(2m-1)) \\ \big\downarrow{\scriptstyle \mathrm{th}_{-L_{2m}\otimes\mathbb{C}}} & & \big\downarrow{\scriptstyle \mathrm{th}_{-L_{2m-1}\otimes\mathbb{C}}} \\ K^0(\mathrm{MTSO}(2m)) & \xrightarrow{\eta^*} & K^0(\Sigma^{-1}\,\mathrm{MTSO}(2m-1)) \\ \big\downarrow{\scriptstyle \mathrm{ch}} & & \big\downarrow{\scriptstyle \mathrm{ch}} \\ H^*(\mathrm{MTSO}(2m);\mathbb{Q}) & \xrightarrow{\eta^*} & H^*(\mathrm{MTSO}(2m-1);\mathbb{Q}). \end{array}$$

Let $\tilde{\mathcal{L}} \in H^*(\mathrm{BSO};\mathbb{Q})$ be the multiplicative sequence in the Pontrjagin classes associated with the formal power series $\sqrt{x}\coth(\frac{\sqrt{x}}{2})$. Recall that the Hirzebruch $\mathcal{L}$-class is associated with the formal power series $\sqrt{x}\coth(\sqrt{x})$; see [25]. Note that the degree $4k$ parts in $H^*(\mathrm{BSO}(2m))$ are related by

$$\tilde{\mathcal{L}}_{4k} = 2^{m-k}\mathcal{L}_{4k} \in H^{4k}(\mathrm{BSO}(2m);\mathbb{Q}). \tag{4.3}$$

Proposition 4.2. *The following assertions hold.*

(i) *The image of* $\sigma_{2m} \in K^0(\mathrm{MTSO}(2m))$ *under the restriction homomorphism* ρ^* *is equal to* $2\sigma_{2m-1}$.

(ii) *The image of* σ_{2m} *under* $\mathrm{ch}\circ\mathrm{th}_{-L_{2m}\otimes\mathbb{C}}$ *in* $H^*(\mathrm{MTSO}(2m);\mathbb{Q})$ *is the class* $\tilde{\mathcal{L}}$.

Proof. The first part is contained in the proof of Lemma 4.2 in [8], the discussion in [7, p. 63] is also relevant. We give a sketch; the computations are rather tedious (they become more transparent when Clifford algebras are used, see [23, p. 261f.]). The point is that on a $2m$-manifold with boundary, the operator D_{2m-1} is the boundary part of D_{2m}. The calculation, which is on the operator level in the quoted references is in fact completely symbolic.

Recall that on $\Lambda^*\mathbb{R}^{2m-1}$, $** = 1$ and $*$ interchanges the even and the odd forms. Let

$$\gamma'_{2m-1} := \gamma_{2m-1} \oplus *\gamma_{2m-1}.$$

This is an endomorphism of the trivial vector bundle $\mathbb{R}^{2m-1}\times\Lambda^*\mathbb{R}^{2m-1}$. We have to show that γ_{2m} is $\mathrm{SO}(2m-1)$-equivariantly isomorphic to the endomorphism

$$\delta : (v,t,\omega) \mapsto (v,t,-it\omega + \gamma'_{2m-1}(v,\omega))$$

of $\mathbb{R}^{2m-1}\times\mathbb{R}\times\Lambda^*\mathbb{R}^{2m-1}\otimes\mathbb{C}$, which represents $2\sigma_{2m-1}$ in K-theory.

Let $\varphi : \Lambda^*\mathbb{R}^{2m}\otimes\mathbb{C} \to \Lambda^*_+\mathbb{R}^{2m}\otimes\mathbb{C}$ be the linear map $1+\tau$ (τ is defined as in Section 4.1). It is an $\mathrm{SO}(2m-1)$-equivariant isomorphism. Let c be the $\mathrm{SO}(2m-1)$-equivariant isomorphism $(v,\omega)\mapsto v\wedge\omega - \iota_v\omega$. The tedious calculation is that $c\circ\varphi\circ\delta\circ\varphi^{-1} = \gamma_{2m}$.

The second part is done in [11, Section 6]. The diagram

$$\begin{array}{ccc} K^0(\mathrm{MSO}(2m)) & \xrightarrow{\theta} & H^*(\mathrm{MSO}(2m);\mathbb{Q}) \\ \big\downarrow{\scriptstyle \mathrm{th}_{-L_{2m}\otimes\mathbb{C}}} & & \big\downarrow{\scriptstyle \mathrm{th}_{-L_{2m}\otimes\mathbb{C}}} \\ K^0(\mathrm{MTSO}(2m)) & \xrightarrow{\mathrm{ch}} & H^*(\mathrm{MTSO}(2m);\mathbb{Q}) \end{array}$$

is commutative, where $\theta(a) := T \cdot \mathrm{ch}(a)$ and $T \in H^*(\mathrm{BSO}(2m);\mathbb{Q})$ is the multiplicative sequence associated with the power series $\sqrt{x}/(4\sinh(\sqrt{x}/2)^2)$ (this is the usual commutation rule between Chern character and Thom isomorphism). Under the injective map

$$H^*(\mathrm{MTSO}(2m);\mathbb{Q}) \to H^*(\mathrm{MTSO}(2);\mathbb{Q}) \otimes \cdots \otimes H^*(\mathrm{MTSO}(2);\mathbb{Q}),$$

the element $\mathrm{ch}(\sigma_{2m})$ maps to $\mathrm{ch}(\sigma_2) \otimes \cdots \otimes \mathrm{ch}(\sigma_2)$ (the multiplicativity of the signature). The class $\mathrm{ch}(\sigma_2)$ is easily evaluated; the result of the computation is that $\mathrm{ch}(\sigma_{2m})$ is the multiplicative sequence associated with $2\sinh(\sqrt{x})$. The result follows. □

Theorem 1.2 follows immediately from Theorem 1.1 (ii), (4.3) and Proposition 4.2.

4.4. Applications of Theorem 1.2.

Proof of Corollary 1.3. Recall the power series expansion

$$\sqrt{x}\coth(\sqrt{x}) = \sum_{k=0}^{\infty} \frac{2^{2k} B_{2k}}{(2k)!} x^{2k}$$

and recall that B_{2k} is a nonzero rational number. On the other hand $H^*(\mathrm{BSO}(3)) = \mathbb{Q}[p_1]$ and therefore

$$\mathcal{L} = \sum_{k=0}^{\infty} \frac{2^{2k} B_{2k}}{(2k)!} p_1^k.$$

Thus the components of $\mathcal{L}$ form an additive basis of $H^*(\mathrm{BSO}(3))$. Therefore,

$$H^*(\mathrm{MTSO}(3);\mathbb{Q}) \to H^*(\Sigma^\infty(B\,\mathrm{Diff}^+(M))_+;\mathbb{Q})$$

is trivial, by Theorem 1.2. By (2.4), this finishes the proof. □

Proof of Corollary 1.4. Because $TE \cong f^*TB \oplus T_vE$, we have

$$\begin{aligned} \text{(4.4)} \qquad \mathrm{sign}(E) &= \langle \mathcal{L}(TE);[E]\rangle \\ &= \langle \mathcal{L}(f^*TB)\mathcal{L}(T_vE);[E]\rangle = \langle \mathcal{L}(TB)f_!(\mathcal{L}(T_vE));[B]\rangle. \end{aligned}$$

By Theorem 1.2, $f_!(\mathcal{L}(T_vE)) = 0$. □

To derive Theorem 1.2 from Corollary 1.4, observe first that

$$H_*(B\,\mathrm{Diff}^+(M);\mathbb{Q}) \cong \Omega_*^{\mathrm{fr}}(B\,\mathrm{Diff}^+(M)) \otimes \mathbb{Q}$$

(the framed bordism group) by Pontrjagin's theorem and Serre's finiteness theorem. Therefore, to show that $\alpha_M^*\,\mathrm{th}\,\mathcal{L} = 0$, it suffices to show that one has $h^*\alpha_M^*\,\mathrm{th}\,\mathcal{L} = 0$ whenever $h : B \to B\,\mathrm{Diff}^+(M)$ is a map with B being a framed manifold, classifying an M-bundle $f : E \to B$. If B is framed, then $\mathcal{L}(TB) = 1$ and therefore

$$0 = \mathrm{sign}(E) = \langle \mathcal{L}(TE);[E]\rangle = \langle f_!(\mathcal{L}(T_vE))\mathcal{L}(TB);[B]\rangle = \langle f_!(\mathcal{L}(T_vE));[B]\rangle.$$

by (4.4) and Corollary 1.4. Hence, Theorem 1.2 follows.

5. A real refinement and the one-dimensional case

Let $f : E \to B$ be a smooth oriented fibre bundle of fibre dimension $2m-1$. Recall the formula for the odd signature operator: $D = i^m(-1)^{p+1}(*d - d*)$ on a $(2m-1)$-dimensional manifold. If m is even ($2m-1 = 3, 7, \dots$), D is real and self-adjoint and so it has an index in $KO^1(B)$, see [12, corollary on p. 307].

If m is odd ($2m-1 = 1, 5, \dots$), then $-iD$ is a *real, skew-adjoint* operator, acting on real-valued differential forms. As such, it has an index in $KO^{-1}(B)$, compare [12, Theorem A].

The question we consider is whether this refined index is also trivial. If m is even, it is obvious that the deformations in steps 1 and 2 in the proof of Theorem 4.1 take place in the space of self-adjoint Fredholm operators. Moreover, Kuiper's theorem is true for the isometry group of a real Hilbert space as well. Therefore step 3 of the proof works in the real case as well and thus the refined index is trivial for even values of m.

The case of odd m is a bit more subtle. We have to check whether the argument in the proof of Theorem 4.1 goes through with $-iD$ instead of D in the space of real, skew-adjoint Fredholm operators. It turns out that step 2 can be changed appropriately (we deform an invertible operator into one with $F^2 = -1$). The argument for step 3 can be applied to the space of skew-adjoint real Fredholm operators F with $F^2 = -1$, again because Kuiper's theorem is true in the real case. The problem is with step 1.

Let $H \cong \bigoplus_{p \geq 0} H^{2p}(E/B; \mathbb{R}) \to B$ be the finite-dimensional real vector bundle formed out of the kernels of the real odd signature operator.

In order to make sense out of the deformation in step 1, it is not enough to know that H is a real vector bundle, but also that H admits a skew-adjoint invertible endomorphism. Such an endomorphism is, up to homotopy, the same as a complex structure on H. Therefore:

Theorem 5.1. *Let $f : E \to B$ be an oriented smooth M-bundle, M of dimension $4r+1$. Then the real family index of the odd signature operator* $\mathrm{ind}_{\mathbb{R}} D \in KO^{-1}(B)$ *is trivial if and only if the K-theory class $[H] \in KO^0(B)$ lies in the image of the realification map $K^0(B) \to KO^0(B)$.*

The first obstruction to find a complex structure on H is of course the parity of its dimension $\dim H \pmod 2$. This agrees with the Kervaire semi-characteristic $\mathrm{Kerv}(M)$, see the discussion in the appendix. More generally, we can interpret this result in terms of the exact sequence

$$K^{-2}(B) \xrightarrow{\gamma} KO^0(B) \xrightarrow{\delta} KO^{-1}(B),$$

compare, e.g., [26, Theorem 5.18]. The map γ is the inverse to the Bott map, composed with the realification map $K^0 \to KO^0$ and δ is the product with the generator of $KO^{-1}(*) \cong \mathbb{Z}/2$. The image $\delta([H]) \in KO^{-1}(B)$ agrees with the real index. So the real index vanishes if and only if there is a complex structure on H.

It is worth studying the one-dimensional case explicitly. The MTW-spectrum is

$$\mathrm{MTSO}(1) \cong \Sigma^{-1}\Sigma^\infty \mathbb{S}^0.$$

It is well known that $\mathrm{Diff}^+(\mathbb{S}^1) \simeq \mathbb{S}^1$; therefore $B\,\mathrm{Diff}^+(\mathbb{S}^1) \simeq \mathbb{CP}^\infty$. The MTW-map $\alpha : \Sigma^\infty \mathbb{CP}^\infty_+ \to \Sigma^{-1}\Sigma^\infty \mathbb{S}^0$ can be identified with the *circle transfer*. The restriction $\Sigma^\infty \mathbb{S}^0 \to \Sigma^{-1}\Sigma^\infty \mathbb{S}^0$ of α to the base-point is simply the generator $\eta \in \pi_1(\Sigma^\infty \mathbb{S}^0) \cong \mathbb{Z}/2$.

The odd signature operator on $\mathbb{S}^1$ is simply $D = -i * d$ on $C^\infty(\mathbb{S}^1)$. If $\mathbb{S}^1$ has a Riemannian metric with volume a and x is a coordinate $\mathbb{S}^1 \to \mathbb{R}/a\mathbb{Z}$ preserving orientation and length, then $D = -i\frac{d}{dx}$. The symbol is $\mathrm{smb}_D(dx) = -1$. Using this, it is easy to see that $\sigma_1 \in K^{-1}_{\mathrm{SO}(1)}(\mathbb{R}, \mathbb{R}_0) = K^0(\mathbb{R}^2, \mathbb{R}^2_0)$ is the Bott class. Thus the universal symbol is a generator of $K^{-1}(\mathrm{MTSO}(1)) \cong \mathbb{Z}$.

The vanishing theorem (Theorem 1.1) in this case can be obtained much easier because $K^{-1}(\mathbb{CP}^\infty) = 0$. In fact, the vanishing theorem for the topological index follows immediately from this fact, without any use of elliptic operator theory.

On the other hand, the Kervaire semi-characteristic of $\mathbb{S}^1$ is clearly nonzero and therefore the real index of the signature operator is nonzero; it is a generator of $KO^{-1}(\mathbb{CP}^\infty) \cong \mathbb{Z}/2$. The latter isomorphism follows from the main result of [4], which states in this case that $KO^{-1}(\mathbb{CP}^\infty)$ is the cokernel of the completion of the map $RU(\mathbb{S}^1) \to RO(\mathbb{S}^1)$ from the complex to the real representation ring. The restriction of the real index to the base-point is the generator of $KO^{-1}(*) = \mathbb{Z}/2$.

6. Vanishing theorems in mod p cohomology and an open problem

Let M be a closed oriented 3-manifold and $\alpha_{E_M} : B\,\mathrm{Diff}^+(M) \to \mathrm{MTSO}(3)$ be the MTW-map. We have seen that α_{E_M} is trivial in rational cohomology. In this section, we sketch two methods to derive from Theorem 1.1 that

$$\alpha_{E_M}^* : H^{4k-3}(\mathrm{MTSO}(3); \mathbb{F}_p) \to H^{4k-3}(B\,\mathrm{Diff}^+(M); \mathbb{F}_p)$$

vanishes for certain values of k and primes p.

Theorem 6.1. *Let M be an oriented closed 3-manifold. Fix $k \geq 1$. Then we have that $\alpha_{E_M}^* : H^{4k-3}(\mathrm{MTSO}(3); \mathbb{F}_p) \to H^{4k-3}(B\,\mathrm{Diff}^+(M); \mathbb{F}_p)$ is trivial for all primes p with $p \geq 2k$ and p not dividing the numerator of B_k (these are almost all primes, for a fixed k).*

Theorem 6.2. *Let M be an oriented closed 3-manifold. Fix an odd prime p. Then $\alpha_{E_M}^* : H^{4k+1}(\mathrm{MTSO}(3); \mathbb{F}_p) \to H^{4k+1}(B\,\mathrm{Diff}^+(M); \mathbb{F}_p)$ is trivial for all k that can be written as $k = \frac{1}{2}(p-1)i$ for some $i \in \mathbb{N}$.*

Note that neither of the sets of pairs (k, p) provided by the two theorems contains the other one. Also, they do not exhaust all values of (k, p). Neither theorem makes a statement about the prime 2. The methods of the proof of both theorems can be used to derive vanishing theorems for all odd dimensions (and the method of Theorem 6.2 gives a result about the prime 2 as well), but here we confine ourselves to the case of dimension 3.

Proof of Theorem 6.1. Let us consider the symbol of the odd signature operator $\mathrm{th}(\sigma) \in K^{-1}(\mathrm{MTSO}(3))$ as a map $\sigma_3 : \mathrm{MTSO}(3) \to \Sigma^{-1} K$. Because the homotopy groups of $\mathrm{MTSO}(3)$ vanish below degree -3 and because the odd homotopy groups of K are zero, there is a unique (up to homotopy) lift κ to the -4-connected cover:

$$\begin{array}{ccc} & & \Sigma^{-3}\mathbf{k} = (\Sigma^{-1}K)\langle -4 \rangle \\ & \overset{\kappa}{\nearrow} & \downarrow \\ \kappa : \mathrm{MTSO}(3) & \xrightarrow{\sigma_3} & \Sigma^{-1}K. \end{array}$$

The composition $\kappa \circ \alpha_{E_M} : B\,\mathrm{Diff}^+(M)_+ \to \mathrm{MTSO}(3) \to \Sigma^{-3}\mathbf{k}$ is still nullhomotopic. The theorem follows from a theorem of Adams [1], [2] about the spectrum cohomology of $\mathbf{k}$. In general, the class $s_r := r!\,\mathrm{ch}_r \in H^{2r}(BU;\mathbb{Z})$ is not a spectrum cohomology class, i.e., it does not lie in the image of the cohomology suspension $H^{2r}(\mathbf{k};\mathbb{Z}) \to H^{2r}(BU;\mathbb{Z})$. The result of [1], [2] is that a certain multiple $m(r)\,\mathrm{ch}_r$ actually *is* a spectrum cohomology class. The number $m(r)$ is given by $m(r) := \prod_p p^{[\frac{r}{p-1}]}$. The product goes over all prime numbers and for $x \in \mathbb{R}$, $[x]$ is the largest integer which is less or equal than x (thus, it involves only primes p with $p-1 \leq r$). Moreover, $u_r := m(r)\,\mathrm{ch}_r$ is a generator of $H^{2r}(\mathbf{k};\mathbb{Z}) \cong \mathbb{Z}$, $r \geq 0$ (all other cohomology groups of $\mathbf{k}$ are trivial). If p is an odd prime then $H^*(\mathrm{MTSO}(3);\mathbb{Z})$ has no p-torsion and so

$$H^*(\mathrm{MTSO}(3);\mathbb{Z}) \otimes \mathbb{F}_p \cong H^*(\mathrm{MTSO}(3);\mathbb{F}_p).$$

Thus, if $\kappa^*(\Sigma^{-3}u_r) \in H^{2r-3}(\mathrm{MTSO}(3))$ reduces to a generator of $H^{2r-3}(\mathrm{MTSO}(3);\mathbb{F}_p)$, then $\alpha : H^{2r-3}(\mathrm{MTSO}(3);\mathbb{F}_p) \to H^{2r-3}(B\,\mathrm{Diff}^+(M);\mathbb{F}_p)$ is the zero map. Up to powers of 2, which we can disregard since p is assumed to be odd, κ^* maps $\Sigma^{-3}u_{2r} \in H^{2r-3}(\Sigma^{-3}\mathbf{k})$ to

$$\pm \frac{B_r}{(2r)!} m(2r)(u_{-3}p_1^r).$$

(it is not hard to derive that this class is integral from Von Staudt's theorem and [38, Lemma 2.1]). This reduces to a generator mod p if p does not divide $\frac{B_r}{(2r)!}m(2r)$ which is certainly the case if $p \geq 2r$ and p does not divide the numerator of B_r. □

Proof of Theorem 6.2. Look at the diagram

$$\begin{array}{ccccc} H^1(\mathrm{MTSO}(3);\mathbb{F}_p) & \twoheadleftarrow & H^1(\mathrm{MTSO}(3);\mathbb{Z}) & \longrightarrow & H^1(\mathrm{MTSO}(3);\mathbb{Q}) \\ \downarrow & & \downarrow & & \downarrow 0 \\ H^1(B\,\mathrm{Diff}^+(M);\mathbb{F}_p) & \longleftarrow & H^1(B\,\mathrm{Diff}^+(M);\mathbb{Z}) & \rightarrowtail & H^1(B\,\mathrm{Diff}^+(M);\mathbb{Q}). \end{array}$$

The right-hand vertical arrow is zero by Corollary 1.3 and the lower right horizontal arrow is injective ($H^1(X;\mathbb{Z}) \to H^1(X;\mathbb{Q})$ is injective for any space X). The upper left horizontal arrow is surjective since p is odd. An easy diagram chase shows that

$$H^1(\mathrm{MTSO}(3);\mathbb{F}_p) \to H^1(B\,\mathrm{Diff}^+(M);\mathbb{F}_p)$$

is also trivial.

Let us denote the mod p Steenrod algebra by $\mathcal{A}_p$. Then we have that the restriction of α to $\mathcal{A}_p \cdot H^1(\mathrm{MTSO}(3))$ is trivial because α^* commutes with the Steenrod algebra. Let $x := p_1 \in H^4(\mathrm{BSO}(3);\mathbb{F}_p)$ be the first Pontrjagin class and let $\mathcal{P}$ be the total Steenrod power operation. The component $\mathcal{P}^i$ has degree $4ri$, $r := \frac{1}{2}(p-1)$. We will compute

$\mathcal{P}(\mathrm{th}_{-L_3}(x)) \in H^*(\mathrm{MTSO}(3); \mathbb{F}_p)$. This is done using a formula by Wu (compare [35, Theorem 19.7]). Wu's formula states, in the present context, that

$$\mathcal{P}(\mathrm{th}_{-L_3} x) = \mathrm{th}_{-L_3}((x + x^p)(1 + x^r)^{-1}).$$

Therefore $\mathcal{P}^i(\mathrm{th}_{-L_3} x) \in H^{1+4ri}(\mathrm{MTSO}(3); \mathbb{F}_p)$ agrees with $\mathrm{th}_{-L_3}(x^{ri+1})$, multiplied by the coefficient of z^{ri+1} in the power series

$$(z + z^p)(1 + z^r)^{-1} = \sum_{l \geq 0} (-1)^l (z^{rl+1} + z^{rl+p}).$$

It is clear that this coefficient is a unit in $\mathbb{F}_p^\times$. Thus $\mathcal{P}^i(\mathrm{th}_{-L_3} x) \in H^{1+4ri}(\mathrm{MTSO}(3); \mathbb{F}_p)$ is a generator and we have argued above that $\alpha^* \mathcal{P}^i(\mathrm{th}_{-L_3} x) = 0$. This concludes the proof. □

It appears to be quite difficult to find an example of a 3-manifold such that

$$\alpha : \Sigma^\infty B\,\mathrm{Diff}^+(M)_+ \to \mathrm{MTSO}(3)$$

is nonzero in cohomology. Many computations which will not be reproduced here suggest that α could very well vanish in cohomology with arbitrary coefficients. However:

Proposition 6.3. *For $M = \mathbb{S}^3$, the MTW-map* $\Sigma^\infty \mathrm{BSO}(4)_+ \to \mathrm{MTSO}(3)$ *of the universal $\mathbb{S}^3$-bundle is not nullhomotopic.*

Proof. (The author owes this argument to O. Randal-Wiliams.) The map α fits into a cofibre sequence (see Proposition A.4):

$$\Sigma^\infty \mathrm{BSO}(4)_+ \xrightarrow{\alpha} \mathrm{MTSO}(3) \xrightarrow{\eta} \Sigma\,\mathrm{MTSO}(4).$$

If α were nullhomotopic, then there exists a splitting $s : \Sigma\,\mathrm{MTSO}(4) \to \mathrm{MTSO}(3)$ of η (i.e., $s \circ \eta = \mathrm{id}$). In the sequel we assume that the map

$$\eta^* : H^*(\mathrm{MTSO}(4); \mathbb{F}_3) \to H^*(\Sigma^{-1}\mathrm{MTSO}(3); \mathbb{F}_3)$$

has a right inverse s^* as a map of $\mathcal{A}_3$-modules and show that this is impossible. Let u_{-3} be the Thom class of $-L_3$ and we write $u_{-3} \cdot x$ for $\mathrm{th}_{-L_3}(x)$ (recall that this is a module structure); similarly for MTSO(4). Since $\eta^* u_{-4} = \Sigma^{-1} u_3$, it follows that $s^* \Sigma^{-1} u_{-3} = u_{-4}$ and thus that

$$Q u_{-4} = \Sigma^{-1} s^* Q u_{-3} \tag{6.1}$$

for any $Q \in \mathcal{A}_3$. Put $Q = \mathcal{P}^3 - \mathcal{P}^2\mathcal{P}^1$. Recall the formulae

$$\begin{aligned} \mathcal{P}^1 p_1 &= p_1^2 + p_2; & \mathcal{P}^2 p_1 &= p_1^3; \\ \mathcal{P}^1 p_1^2 &= -p_1(p_1^2 + p_2); & \mathcal{P}^1 p_2 &= p_1 p_2 \end{aligned} \tag{6.2}$$

for the Steenrod operations on BSO(4) (and hence, by putting $p_2 = 0$, also on BSO(3)). The general formula for the action of $\mathcal{A}_p$ on the cohomology of BSO(n) was given by Borel and Serre [16, Théorème 14.1]. It is, however, easier to derive the low-dimensional special case (6.2) directly from the splitting principle and general properties of the Steenrod operations.

Moreover, the effect of Steenrod operations on the Thom class is expressed by the formula

$$\mathcal{P}(u_4) = u_{-4}(K(p_1, p_2)),$$

where K is the multiplicative sequence associated with $(1+x)^{-1}$. Its lowest terms are $K(p_1, p_2) = 1 - p_1 + p_1^2 - p_1^3 - p_1 p_2 + \cdots$, compare [35, Theorem 19.7]. From this, we get

$$(\mathcal{P}^3 - \mathcal{P}^2\mathcal{P}^1)(u_{-4}) = u_{-4} p_1 p_2$$

on MTSO(4). The analogous computation for u_{-3} yields $\mathcal{P}(u_{-3}) = 0$, which is a contradiction in view of (6.1). □

We conclude this section by asking the question:

Question 6.4. Does there exist an oriented closed 3-manifold M and a prime p, such that $\alpha_{E_M}^* : \tilde{H}^*(\mathrm{MTSO}(3); \mathbb{F}_p) \to \tilde{H}^*(B\,\mathrm{Diff}^+(M); \mathbb{F}_p)$ is nontrivial?

7. What happens for manifold bundles with boundary?

In this section we study manifold bundles with boundary and ask whether the vanishing theorem 1.2 still holds for such bundles. Let A be a spectrum and $x \in A^*(\mathrm{MTSO}(n))$. If $E \to B$ is a manifold bundle with boundary, then there is no map $\alpha_E : \Sigma^\infty B_+ \to \mathrm{MTSO}(n)$ in general. Under certain circumstances, however, we still can define an element in $A^*(B)$ that plays the role of $\alpha_E^* x$.

We have to distinguish two cases. The first case is when we require the boundary bundle to be trivialized. In this case, Theorem 1.2, interpreted appropriately, is still true. The second case is when the boundary bundle is not required to be trivial. In this case the generalized MMM-classes are not defined in general and therefore the analogue of Theorem 1.2 does not make sense, as we will discuss briefly.

7.1. Manifold bundles with boundary. Let M be an n-dimensional (oriented, smooth, compact) manifold with boundary. There are two types of smooth M-bundles that come to mind.

We can study the structural group $\mathrm{Diff}^+(M)$ of all orientation-preserving diffeomorphisms, with no condition on the boundary. We say a bundle with structural group $\mathrm{Diff}^+(M)$ and fibre M has *free boundary*. Or we can consider the group $\mathrm{Diff}^+(M; \partial)$ of diffeomorphisms of M that coincide with id on a small neighborhood of ∂M. Bundles with this structural groups are said to have *fixed boundary*.

7.2. The Pontrjagin–Thom construction for bundles with boundary.
Let $f : E \to B$ be a manifold bundle with boundary $\partial f : \partial E \to B$ and of fibre dimension n. The isomorphism $T_v \partial E \oplus \mathbb{R} \cong T_v E|_{\partial E}$ defines a spectrum map

$$\eta_E : \mathbf{Th}(-T_v \partial E) \to \Sigma \mathbf{Th}(-T_v E)$$

that fits into a commutative diagram (the rest of the diagram is explained below)

$$\begin{array}{ccc}
\Sigma^{\infty} B_+ & & \\
\downarrow{\scriptstyle \mathrm{PT}_{\partial E}} & \searrow{\scriptstyle \mathrm{PT}_E \; \simeq *} & \\
\mathbf{Th}(-T_v \partial E) & \xrightarrow{\eta_E} & \Sigma \mathbf{Th}(-T_v E) \\
\downarrow{\scriptstyle \kappa_{\partial E}} & & \downarrow{\scriptstyle \kappa_E} \\
\mathrm{MTSO}(n-1) & \xrightarrow{\eta} & \Sigma\, \mathrm{MTSO}(n) \\
 & & \downarrow{\scriptstyle x} \\
 & & \Sigma A.
\end{array} \tag{7.1}$$

Choose an embedding $j : E \to B \times [0,\infty) \times \mathbb{R}^{\infty-1}$ such that $\partial E = j^{-1}(B \times \{0\} \times \mathbb{R}^{\infty-1})$ and $j(E) \subset B \times [0,1) \times \mathbb{R}^{k-1}$. The collapse construction defines a spectrum map

$$\mathrm{PT}_E : [0,\infty] \wedge \Sigma^{\infty-1} B_+ \to \mathbf{Th}(-T_v E),$$

here $\infty \in [0,\infty]$ serves as a basepoint. If $t \gg 1$, then the composition

$$\Sigma^{\infty-1} B_+ \cong \{t\} \wedge \Sigma^{\infty-1} B_+ \to [0,\infty] \wedge \Sigma^{\infty-1} B_+ \xrightarrow{\mathrm{PT}_E} \mathbf{Th}(-T_v E)$$

is the constant map. On the other hand, if $t = 0$, then

$$\Sigma^{\infty-1} B_+ \cong \{0\}_+ \wedge \Sigma^{\infty-1} B_+ \to [0,\infty] \wedge \Sigma^{\infty-1} B_+ \xrightarrow{\mathrm{PT}_E} \mathbf{Th}(-T_v E)$$

is the composition

$$\eta_E \circ \mathrm{PT}_{\partial E} : \Sigma^{\infty-1} B_+ \to \Sigma^{-1} \mathbf{Th}(-T_v \partial E) = \mathbf{Th}(-T_v E|_{\partial E}) \to \mathbf{Th}(-T_v E).$$

In other words, the Pontrjagin–Thom map PT_E can be interpreted as a nullhomotopy of the composition $\eta_E \circ \mathrm{PT}_{\partial E}$, as displayed in diagram (7.1).

Let A be a spectrum and $x : \mathrm{MTSO}(n) \to A$ be a map. By composing the nullhomotopy PT_E with $x \circ \kappa_E$, we get a nullhomotopy P of the composition

$$x \circ \kappa_E \circ \eta_E \circ \mathrm{PT}_{\partial E} : \Sigma^{\infty-1} B_+ \to \Sigma A.$$

Suppose that there is a second nullhomotopy Q of the same map, but written as the composition

$$\Sigma^{\infty} B_+ \xrightarrow{\mathrm{PT}_{\partial}} \mathbf{Th}(-T_v \partial E) \xrightarrow{\kappa_{\partial E}} \mathrm{MTSO}(n-1) \to \Sigma\, \mathrm{MTSO}(n-1) \to \Sigma A$$

(this is the same as $x \circ \eta \circ \alpha_{\partial E}$). Such a nullhomotopy typically arises from a vanishing theorem concerning the bundle ∂E and it only involves ∂E and some choices that do not depend on E. We can glue the two nullhomotopies Q and P and obtain a map $\Sigma^{\infty} B_+ \to A$. If the nullhomotopy Q is defined for the universal bundle $E_M \to B\,\mathrm{Diff}^+(M)$, we can use this procedure to define characteristic classes of smooth M-bundles. More precisely, even though the map $\alpha_{E_M} : \Sigma^{\infty} B\,\mathrm{Diff}^+(M)_+ \to \mathrm{MTSO}(n)$ does not exist, we can make sense out of the element $\alpha_{E_M}^*(x) \in A(B\,\mathrm{Diff}^+(M))$.

We list a few examples of situations to which the above philosophy can be applied. In each of these cases, we would need to make the map x as well as the nullhomotopy Q precise on the point-set level. We indicate how this works in the example that is of interest to us: the second example.

(i) If $\partial E = \emptyset$ and $x = \mathrm{id} : \mathrm{MTSO}(n) \to \mathrm{MTSO}(n)$ and Q is the constant nullhomotopy, then we get of course the MTW-map α_E back.

(ii) (generalization of the first example) If $x = \mathrm{id}_{\mathrm{MTSO}(n)}$ and E has fixed boundary, then $\alpha_{\partial E} : \Sigma^\infty B_+ \to \mathrm{MTSO}(n-1)$ factors as

$$\Sigma^\infty B_+ \to \Sigma^\infty \mathbb{S}^0 \xrightarrow{\alpha_{\partial M}} \mathrm{MTSO}(n-1).$$

But there is an oriented nullbordism W of ∂M (of course, $W = M$ is a possible choice, but there is no reason to prefer this choice). This nullbordism induces a nullhomotopy of the composition

$$\Sigma^\infty \mathbb{S}^0 \xrightarrow{\alpha_{\partial M}} \mathrm{MTSO}(n-1) \to \Sigma\,\mathrm{MTSO}(n).$$

Thus we are in the above situation and hence we can define a map $\Sigma^\infty B_+ \to \mathrm{MTSO}(n)$, called α_E. Geometrically, this corresponds to gluing in the trivial bundle $B \times W$ into E along ∂E. If $\hat{E}$ denotes this new bundle, then $\alpha_E = \alpha_{\hat{E}}$. This geometric description, together with the homotopy equivalence $\alpha^{\mathrm{GMTW}} : \Omega B\,\mathrm{Cob}_n \simeq \Omega^\infty \mathrm{MTSO}(n)$ from [22], explains how to make the nullhomotopy precise. Note that this construction depends on the choice of W; thus α_E is well-defined only modulo maps of the form $\alpha_{B\times V}$ for constant bundles of closed manifolds. Therefore the map

$$\alpha_E : H^*(\mathrm{MTSO}(n)) \to H^*(B\,\mathrm{Diff}^*(M))$$

does not depend on the choice of W as long as we consider positive degrees $* > 0$.

(iii) If $A = \Sigma^k H\mathbb{Z}$, $y \in H^k(\mathrm{BSO}(n))$ and χ is the Euler class, let

$$x = \mathrm{th}(y\chi) \in H^k(\mathrm{MTSO}(n)).$$

There is a canonical nullhomotopy of $x \circ \eta$, induced by a nonzero cross-section of $L_n|_{\mathrm{BSO}(n-1)}$. Thus we can apply the above construction. This shows that, although there is no map $\alpha_E : \Sigma^\infty B_+ \to \mathrm{MTSO}(n)$ for a bundle with boundary, we can still define what ought to be the pullback $\alpha_E^*\,\mathrm{th}(y\chi)$, which should be the same as $f_!(y(T_v E)\chi(T_v E))$. Recall that the Becker–Gottlieb transfer $\mathrm{trf}_f : \Sigma^\infty B_+ \to \Sigma^\infty E_+$ also exists when $f : E \to B$ is a manifold bundle with boundary. One can show $\mathrm{trf}_f^*(y) = \alpha_E^*\,\mathrm{th}(y\chi)$.

(iv) If n is even and $x : \mathrm{MTSO}(n) \to K$ is the universal symbol class of the signature operator, the vanishing theorem implies that we can define the index of the signature on an arbitrary bundle of even dimension. On the other hand, for odd n, it turns out that the index of the signature of the boundary is an obstruction to define the index of the odd signature operator.

We can use the second example from above to define the MTW-map of any bundle with fixed boundary. Since, in the above notation, α_E is defined to be the MTW-map of the closed bundle $\hat{E}$, Theorem 1.1 (ii) is still true for the new MTW-map of a bundle with boundary. All consequences that were derived from Theorem 1.1 (ii) by formal computations are still true, namely Theorems 1.2, 6.1 and 6.2.

A. The component group of MTSO(n)

Here we sharpen a computation in [22, Section 3.1], concerning the group $\pi_0(\mathrm{MTSO}(n))$.

First we recall the definition of Reinhart's bordism group $\mathfrak{R}_n$; see [39]. An element of $\mathfrak{R}_n$ is represented by a closed oriented n-manifold M; two manifolds M_0 and M_1 represent the same element if there is an oriented cobordism N from M_0 to M_1 and a tangential, nowhere zero vector field v on N which is the inward normal vector field on M_0 and the outward normal vector field on M_1. There is a forgetful map $\mathfrak{R}_n \to \Omega_n^{\mathrm{SO}}$ to the ordinary oriented bordism group.

Let $\mathrm{Eul}_n \subset \mathbb{Z}$ be the subgroup generated by all Euler numbers of oriented n-manifolds. Its values are

$$\mathrm{Eul}_n = \begin{cases} 0, & n \not\equiv 0 \pmod 2; \\ 2\mathbb{Z}, & n \equiv 2 \pmod 4; \\ \mathbb{Z}, & n \equiv 0 \pmod 4. \end{cases} \tag{A.1}$$

The first case is clear. The third case follows from $\chi(\mathbb{S}^{4k}) = 2$ and $\chi(\mathbb{CP}^{2k}) = 2k+1$. The second case is implied by $\chi(\mathbb{S}^{4k+2}) = 2$ and the congruence $\chi(M^{4k+2}) \equiv 0 \pmod 2$ which follows from Poincaré duality in a straightforward manner.

Theorem A.1. *The following assertions hold.*

(i) *There is an isomorphism* $\Phi : \mathfrak{R}_n \to \pi_0(\mathrm{MTSO}(n))$.

(ii) *There is a split exact sequence*

$$0 \to \mathbb{Z}/\mathrm{Eul}_{n+1} \to \pi_0(\mathrm{MTSO}(n)) \to \Omega_n^{\mathrm{SO}} \to 0. \tag{A.2}$$

Reinhart has shown the existence of the split exact sequence (A.2) with $\pi_0(\mathrm{MTSO}(n))$ replaced by $\mathfrak{R}_n$; see [39, Proposition (2)]. We will give an interpretation of the sequence (A.2) in terms of homotopy groups of spectra. Moreover, we will describe the splittings in slightly different terms (and give a much simpler proof of the splitting in the case $n \equiv 1 \pmod 4$).

We first construct Φ, using the classical Pontrjagin–Thom theorem. Then we construct the sequence (A.2), starting from a cofiber sequence of spectra, following [22, Section 3.1]. The proof of Theorem A.1 (i) is by comparing the sequence (A.2) to Reinhart's original sequence. Then we construct the splitting. If $n \not\equiv 1 \pmod 4$, this is easy and contained in [39]. If $n \equiv 1 \pmod 4$, we use the Kervaire semi-characteristic Kerv (whose definition we recall below) for the splitting. The bordism invariance of Kerv can be proven using index theory, but we offer also a geometric argument for the invariance that we could not find in the literature.

The classical Pontrjagin–Thom theorem gives a bordism theoretic description of the homotopy groups of Thom spectra and vice versa (this was the original application of the Pontrjagin–Thom construction). Here is the most general version of this correspondence.

Theorem A.2. *Let $V \to X$ be a stable vector bundle of rank $-n \in \mathbb{Z}$. If $-n > 0$, then $\pi_0(\mathbf{Th}(V)) = 0$. If $n \geq 0$, then the group $\pi_0(\mathbf{Th}(V))$ is isomorphic to the bordism group of triples (M^n, g, ϕ), where M^n is a closed smooth manifold, $g : M \to X$ a continuous map and $\phi : \nu(M) \cong g^*V$ a stable vector bundle isomorphism. Two triples (M_0, g_0, ϕ_0) and (M_1, g_1, ϕ_1) are bordant if there exists a bordism N from M_0 to M_1, a continuous map*

$h : N \to X$ such that $h|_{M_i} = g_i$ and a stable bundle isomorphism $\psi : \nu(N) \oplus \mathbb{R} \cong h^*V$ whose restriction to M_i is the isomorphism

$$\nu(N) \oplus \mathbb{R} \cong \nu(M_i) \overset{\phi_i}{\cong} g_i^* V.$$

Given a triple (M, g, ϕ), $c : M \to *$ the constant map, then the corresponding element in $\pi_0(\mathbf{Th}(V))$ is the composition

$$\Sigma^\infty \mathbb{S}^0 \xrightarrow{\mathrm{PT}_c} \mathbf{Th}(\nu(M)) \xrightarrow{g,\Phi} \mathbf{Th}(V).$$

A detailed proof of this well-known result can be found in [40, Chapter IV §7]. Of course, this also gives an interpretation of the groups $\pi_k(\mathbf{Th}(V)) \cong \pi_0(\mathbf{Th}(V - \mathbb{R}^k))$.

Now we apply Theorem A.2 to $\mathrm{MTSO}(n)$. We learn that $\pi_0(\mathrm{MTSO}(n))$ is the bordism group of oriented n-manifolds, where M_0 and M_1 are considered to be bordant if and only if there exists an oriented bordism N between them, an oriented n-dimensional vector bundle V on N and a stable bundle isomorphism $TN \cong V \oplus \mathbb{R}$. If two oriented manifolds M_0 and M_1 are bordant in Reinhart's sense, then they are bordant in $\pi_0(\mathrm{MTSO}(n))$: if x is a vector field on the bordism, then we take $V \subset TN$ to be the complement. Thus there is a surjective homomorphism

$$\Phi : \mathfrak{R}_n \to \pi_0(\mathrm{MTSO}(n)).$$

Remark A.3. It is possible to show that Φ is injective by means of obstruction theory. This, however, is nontrivial. Furthermore, the arguments going into this are very similar (but not identical) to Reinhart's arguments. Therefore we chose to use Reinhart's results to prove that Φ is an isomorphism.

There are several maps which relate the spectra $\mathrm{MTSO}(n)$ for different values of n.

- The obvious bundle isomorphism $L_{n+1}|_{\mathrm{BSO}(n)} \cong L_n \oplus \mathbb{R}$ induces a map of spectra $\eta : \mathrm{MTSO}(n) \to \Sigma\,\mathrm{MTSO}(n+1)$.
- The inclusion $-L_{n+1} \to \underline{0}$ of stable vector bundles on $\mathrm{BSO}(n+1)$ yields a spectrum map $\omega : \mathrm{MTSO}(n+1) \to \Sigma^\infty\,\mathrm{BSO}(n+1)_+$.
- The Madsen–Tillmann–Weiss map of the oriented $\mathbb{S}^n$-bundle $\mathrm{BSO}(n) \to \mathrm{BSO}(n+1)$ is a map $\beta : \Sigma^\infty\,\mathrm{BSO}(n+1)_+ \to \mathrm{MTSO}(n)$.

Proposition A.4. *The maps η, ω and β form a cofibration sequence*

$$\mathrm{MTSO}(n+1) \xrightarrow{\omega} \Sigma^\infty\,\mathrm{BSO}(n+1)_+ \xrightarrow{\beta} \mathrm{MTSO}(n) \xrightarrow{\eta} \Sigma\,\mathrm{MTSO}(n+1). \tag{A.3}$$

Proof. See [20, Section 2] and [22, Section 3]. □

The (homotopy) colimit of the sequence

$$\mathrm{MTSO}(0) \xrightarrow{\eta} \Sigma\,\mathrm{MTSO}(1) \xrightarrow{\eta} \Sigma^2\,\mathrm{MTSO}(2) \to \cdots$$

is the universal Thom spectrum $\widetilde{\mathrm{MSO}}$, the Thom spectrum of the universal 0-dimensional stable vector bundle $-L \to \mathrm{BSO}$ (which becomes $\mathbb{R}^n - L_n$ when restricted to $\mathrm{BSO}(n)$). The usual universal Thom spectrum MSO is the Thom spectrum of $L \to \mathrm{BSO}$. The spectra $\widetilde{\mathrm{MSO}}$ and MSO are homotopy equivalent: Let $\iota : \mathrm{BSO} \to \mathrm{BSO}$ be the inversion map, such that $\iota^* L = -L$. The map ι is covered by a bundle map $j : -L \to L$ which induces a homotopy equivalence $\mathbf{Th}(j) : \widetilde{\mathrm{MSO}} \to \mathrm{MSO}$.

The long exact homotopy sequence induced by (A.3) shows that the map

$$\eta_* : \pi_i(\mathrm{MTSO}(n)) \to \pi_i(\Sigma\, \mathrm{MTSO}(n+1))$$

is an epimorphism if $i \leq 0$ and an isomorphism if $i < 0$. Thus we have that the inclusion $\Sigma^n \mathrm{MTSO}(n) \to \widetilde{\mathrm{MSO}}$ yields an isomorphism

$$\pi_i(\mathrm{MTSO}(n)) \cong \pi_{n+i}(\widetilde{\mathrm{MSO}}) \cong \pi_{n+i}(\mathrm{MSO}) \cong \Omega_{n+i}^{\mathrm{SO}}$$

with the oriented bordism group for all $i < 0$. In particular $\pi_{-1}(\mathrm{MTSO}(n+1)) \cong \Omega_n^{\mathrm{SO}}$. Therefore (A.3) induces an exact sequence

$$\text{(A.4)} \quad \pi_0(\mathrm{MTSO}(n+1)) \to \pi_0(\Sigma^\infty \mathrm{BSO}(n+1)_+) = \mathbb{Z} \\ \to \pi_0(\mathrm{MTSO}(n)) \to \Omega_n^{\mathrm{SO}} \to 0.$$

In terms of the bordism-theoretic interpretation of $\pi_0(\mathrm{MTSO}(n))$, the maps in (A.4) have the following interpretation:

Proposition A.5. *The following assertions hold.*

(i) $\pi_0(\mathrm{MTSO}(n+1)) \to \mathbb{Z}$ *sends the bordism class of an oriented* $(n+1)$*-manifold* M *to its Euler number* $\chi(M)$.

(ii) $\mathbb{Z} \to \pi_0(\mathrm{MTSO}(n))$ *sends* 1 *to the bordism class of* $\mathbb{S}^n$.

(iii) $\pi_0(\mathrm{MTSO}(n)) \to \Omega_n^{\mathrm{SO}}$ *is the forgetful map.*

Proof. Only the first claim needs a further justification. If $f : E \to B$ is an oriented $(n+1)$-manifold bundle, then the composition

$$\Sigma^\infty B_+ \xrightarrow{\alpha_E} \mathrm{MTSO}(n+1) \xrightarrow{\omega} \Sigma^\infty \mathrm{BSO}(n+1)_+$$

is the composition of the Becker–Gottlieb transfer $\Sigma^\infty B_+ \to \Sigma^\infty E_+$ (see [15]) with the classifying map $\Sigma^\infty E_+ \to \Sigma^\infty \mathrm{BSO}(n)_+$ of $T_v E$. Therefore, if $B = *$ and $n+1 > 0$, then the induced map on $\mathbb{Z} = \pi_0 \Sigma^\infty \mathbb{S}^0 \to \pi_0(\Sigma^\infty \mathrm{BSO}(n+1)_+) = \mathbb{Z}$ is the multiplication by the Euler number $\chi(M)$ of M, by [15, Theorem 2.4]. In particular, this argument shows that the Euler number is a bordism invariant (Reinhart showed this statement for $\mathfrak{R}_n$). □

Thus the exact sequence (A.4) induces

$$0 \to \mathbb{Z}/\,\mathrm{Eul}_{n+1} \to \pi_0(\mathrm{MTSO}(n)) \to \Omega_n^{\mathrm{SO}} \to 0;$$

the values of Eul_n are given by (A.1).

Proof of Theorem A.1. There is a diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/\operatorname{Eul}_{n+1} & \longrightarrow & \mathfrak{R}_n & \longrightarrow & \Omega_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathbb{Z}/\operatorname{Eul}_{n+1} & \longrightarrow & \pi_0(\mathrm{MTSO}(n)) & \longrightarrow & \Omega_n^{\mathrm{SO}} & \longrightarrow & 0. \end{array}$$

The lower row is (A.4), the upper row has been constructed by Reinhart [39, Proposition (2)]. The first map in the upper row sends 1 to the bordism class of the sphere and so the first square commutes, compare Proposition A.5. The second map in the upper row is the forgetful map and so the second square is commutative as well, again by Proposition A.5. The 5-Lemma finishes the proof. □

The sequence (A.4) splits. This was shown by Reinhart, but his proof for the case $n \equiv 1 \pmod 4$ is complicated since it relies on Wall's work on the structure of Ω_n^{SO}. Here we give a direct argument, using the Kervaire semicharacteristic. But first we recall Reinhart's splitting of $\mathbb{Z}/\operatorname{Eul}_{n+1} \to \pi_0(\mathrm{MTSO}(n))$ for even n.

If M^{4m+2} is a closed oriented manifold, then the Euler number is even. If M^{4m} is an oriented manifold, then $\operatorname{sign}(M) + \chi(M) \equiv 0 \pmod 2$ is immediate from the definition of the signature and the Euler number and from Poincaré duality. Thus we can define splittings of (A.2) in these cases by

$$[M] \mapsto \begin{cases} \frac{1}{2}\chi(M) & \text{for } n \equiv 2 \pmod 4, \\ \frac{1}{2}(\operatorname{sign}(M) + \chi(M)) & \text{for } n \equiv 0 \pmod 4. \end{cases}$$

If M is an oriented closed $(4m+1)$-dimensional manifold, then the *Kervaire semi-characteristic* $\operatorname{Kerv}(M) \in \mathbb{Z}/2$ is defined to be

$$\operatorname{Kerv}(M) := \sum_{i \geq 0} b_{2i}(M) = \sum_{i=0}^{2m} \dim b_i(M),$$

where b_i is the real Betti number of M. By the proposition below, $\operatorname{Kerv}(M)$ defines a homomorphism $\pi_0(\mathrm{MTSO}(4m+1)) \to \mathbb{Z}/2$. We were not able to find a proof of the following simple observation in the literature.

Proposition A.6. *The Kervaire semi-characteristic of an oriented* $(4m+1)$*-manifold* M *only depends on its bordism class in* $\pi_0(\mathrm{MTSO}(4m+1))$ *(or equivalently, its Reinhart bordism class).*

This proposition implies immediately that $\mathbb{Z}/2 \to \pi_0(\mathrm{MTSO}(4m+1))$ is split by $[M] \mapsto \operatorname{Kerv}(M)$.

Proof. The first proof uses index theory of real operators [14]. The symbol class of the real signature operator is a class in $KO^{-1}(\mathrm{MTSO}(4m+1))$ and the index in $KO^{-1} \cong \mathbb{Z}$ can be identified with Kerv.

Here is an elementary proof. It is enough to show the following: If N^{4m+2} is a connected oriented manifold with boundary M and if there is a nowhere vanishing vector field on N which is normal to the boundary, then $\mathrm{Kerv}(M) = 0$. Clearly, the double dN of N is closed and has a vector field without zeroes; thus $\chi(dN) = 0$ and therefore $\chi(N) = 0$. Let A be the image of $H^{2m+1}(M,N) \to H^{2m+1}(N)$.

Look at the long exact sequence of the pair (N, M) in real cohomology:

$$0 \to H^0(N,M) \to H^0(N) \to H^0(M) \to \cdots \to H^{2m}(M) \to H^{2m+1}(N,M) \to A \to 0.$$

We compute (in $\mathbb{Z}/2$)

$$\begin{aligned} 0 &= \sum_{i=0}^{2m+1} b_i(N;M) + \sum_{i=0}^{2m} b_i(N) + \sum_{i=0}^{2m} b_i(M) + \dim A \\ &= \sum_{i=2m+1}^{4m+2} b_i(N) + \sum_{i=0}^{2m} b_i(N) + \sum_{i=0}^{2m} b_i(M) + \dim A \\ &= \chi(N) + \mathrm{Kerv}(M) + \dim A \\ &= \mathrm{Kerv}(M) + \dim A. \end{aligned}$$

By Poincaré duality, the cup product pairing on A is skew-symmetric and nondegenerate, thus $\dim A \equiv 0 \pmod 2$. □

References

[1] *J. F. Adams*, On Chern characters and the structure of the unitary group, Proc. Cambridge Philos. Soc. **57** (1961), 189–199.
[2] *J. F. Adams*, Chern characters revisited, Illinois J. Math. **17** (1973), 333–336.
[3] *J. F. Adams*, Stable homotopy and generalised homology, reprint of the 1974 original, Chicago Lectures Math., University of Chicago Press, Chicago 1995.
[4] *D. W. Anderson*, The real K-Theory of classifying spaces, Proc. Natl. Acad. Sci. USA **51** (1964), 634–636.
[5] *M. F. Atiyah*, The signature of fibre bundles, in: Global analysis, Univ. Tokyo Press, Tokyo (1969), 73–84.
[6] *M. F. Atiyah* and *I. G. Macdonald*, Introduction to commutative algebra, Addison-Wesley Publishing Company, Reading 1969.
[7] *M. F. Atiyah*, *V. K. Patodi* and *I. M. Singer*, Spectral asymmetry and Riemannian geometry I, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69.
[8] *M. F. Atiyah*, *V. K. Patodi* and *I. M. Singer*, Spectral asymmetry and Riemannian geometry III, Math. Proc. Cambridge Philos. Soc. **79** (1976), 71–99.
[9] *M. F. Atiyah* and *G. Segal*, Equivariant K-theory and completion, J. Differential Geom. **3** (1969), 1–18.
[10] *M. F. Atiyah* and *G. Segal*, Twisted K-theory, Ukr. Mat. Visn. **1** (2004), 287–330.
[11] *M. F. Atiyah* and *I. M. Singer*, The index of elliptic operators III, Ann. of Math. **87** (1968), 546–604.
[12] *M. F. Atiyah* and *I. M. Singer*, Index theory for skew-adjoint Fredholm operators, Publ. Math. IHES **37** (1969), 5–26.
[13] *M. F. Atiyah* and *I. M. Singer*, The index of elliptic operators IV, Ann. of Math. **93** (1971), 119–138.
[14] *M. F. Atiyah* and *I. M. Singer*: The index of elliptic operators V, Ann. of Math. **93** (1971), 139–149.
[15] *J. C. Becker* and *D. H. Gottlieb*, The transfer map and fiber bundles, Topology **14** (1975), 1–12.
[16] *A. Borel* and *J. P. Serre*, Groupes de Lie et puissances réduites de Steenrod, Amer. J. Math. **75** (1953), 409–448.
[17] *U. Bunke*, Index theory, eta forms, and Deligne cohomology, Mem. Amer. Math. Soc. **198**, American Mathematical Society, Providence 2009.
[18] *J. Ebert*, Algebraic independence of generalized Miller-Morita-Mumford classes, Algebr. Geom. Topol. **11** (2011), 69–105.
[19] *D. Freed*, *M. Hopkins* and *C. Teleman*, Loop groups and twisted K-theory I, J. of Topology **4** (2011), 737–798.
[20] *S. Galatius*, Mod 2 homology of the stable spin mapping class group, Math. Ann. **334** (2006), 439–455.

[21] *S. Galatius*, *I. Madsen* and *U. Tillmann*, Divisibility of the stable Miller-Morita-Mumford classes, J. Am. Math. Soc. **19** (2006), 759–779.
[22] *S. Galatius*, *I. Madsen*, *U. Tillmann* and *M. Weiss*, The homotopy type of the cobordism category, Acta Math. **202** (2009), 195–239.
[23] *P. Gilkey*, Invariance theory, the heat equation and the Atiyah-Singer index theorem, Publish or Perish, 1984.
[24] *A. Hatcher* and *N. Wahl*, Stabilization for mapping class groups of 3-manifolds, Duke Math. J. **155** (2010), 205–269.
[25] *F. Hirzebruch*, Neue topologische Methoden in der algebraischen Geometrie, Springer-Verlag, Berlin 1962.
[26] *M. Karoubi*, K-Theory. An introduction, reprint of the 1978 edition, Springer-Verlag, Berlin 2008.
[27] *N. H. Kuiper*, The homotopy type of the unitary group of Hilbert space, Topology **3** (1965), 19–30.
[28] *W. Lück* and *A. Ranicki*, Surgery obstructions in fiber bundles, J. Pure Appl. Algebra **81** (1992), 139–189.
[29] *H. B. Lawson* and *M. Michelsohn*, Spin geometry, Princeton University Press, 1989.
[30] *I. Madsen* and *U. Tillmann*, The stable mapping class group and $\mathbb{CP}^{\infty}_{-1}$, Invent. Math. **145** (2001), 509–544.
[31] *I. Madsen* and *M. Weiss*, The stable moduli space of Riemann surfaces: Mumford's conjecture, Ann. of Math. **165** (2007), 843–941.
[32] *W. Meyer*, Die Signatur von Faserbündeln und lokalen Koeffizientensystemen, Bonn. Math. Schr. **53** (1972).
[33] *E. Y. Miller*, The homology of the mapping class group, J. Differential Geom. **24** (1986), 1–14.
[34] *J. W. Milnor* and *J. C. Moore*, On the structure of Hopf algebras, Ann. of Math. **81** (1965), 211–264.
[35] *J. W. Milnor* and *J. Stasheff*, Characteristic classes, Ann. of Math. Stud. **76**, Princeton University Press, Princeton 1974.
[36] *S. Morita*, Characteristic classes of surface bundles, Invent. Math. **90** (1987), 551–577.
[37] *D. Mumford*, Towards an enumerative geometry of the moduli space of curves, in: Arithmetic and geometry, Vol. II, Progr. Math. **36**, Birkhäuser, Boston (1983), 271–328.
[38] *G. Pappas*, Integral Riemann-Roch theorem, Invent. Math. **170** (2007), 455–481.
[39] *B. L. Reinhart*, Cobordism and the Euler number, Topology **2** (1963), 173–177.
[40] *Y. B. Rudyak*, On Thom spectra, orientability and cobordism, Springer-Verlag, 1998.
[41] *U. Tillmann*, On the homotopy of the stable mapping class group, Invent. Math. **130** (1997), 257–275.

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