Equivariant Twisted K-Theory, after Atiyah and Segal. Talk given at the Arbeitsgemeinschaft about twisted K-theory, Oberwolfach

Johannes Ebert

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1 The definition of equivariant twisted K-theory

Throughout the talk, G will be a compact Lie group and X a left G-space.

1.1 The definition of an equivariant twist

First of all, we need to clarify the notion of a G-equivariant stable projective bundle $P \to X$.
First of all, P is a fibre bundle with fiber a projective space of infinite dimension and structural group $\mathbb{P}U(\mathcal{H})$, endowed with the compact-open-topology.
Secondly, we need to impose an additional condition on the G-action on P. If $x \in X$ is a point and G_x its stabilizer, then there exists an open G_x-invariant neighborhood of x and and isomorphism of projective bundles $P|_U \cong U \times P_x$ of bundles with a G_x-action.

Finally, we shall only study *stable* bundles, i.e. bundles P such that $P \otimes L^2(G) \cong P$ as a projective G-bundle. This condition is analogous to the infinte-dimensionality of the bundles in the nonequivariant case.
To get an intuition for stable bundles, we shall study them in the case when X us a point.

Lemma 1.1.1. *The isomorphism classes of stable projective G-bundles is in bijection with the group* $\operatorname{Ext}(G; \mathbb{T})$ *of central extensions* $1 \to \mathbb{T} \to \tilde{G} \to G \to 1$.

Proof: Let $P \to *$ be a stable projective bundle. This is nothing else than a homomorphism $G \to \mathbb{P}\,\mathrm{GL}(H)$ for a Hilbert space H alias a projective representation of G. The pullback of

$$\begin{array}{ccc} & & \mathrm{GL}(H) \\ & & \downarrow \\ G & \longrightarrow & \mathbb{P}\,\mathrm{GL}(H) \end{array}$$

is the desired extension $\tilde{G}$. This gives $\Phi : \mathrm{Proj}_G(*) \to \mathrm{Ext}(G;\mathbb{T})$.
Conversely, let $\tilde{G} \to G$ be a central $\mathbb{T}$-extension. We want to create a stable projective representation of G out of this data. Choose an isomorphism of the central circle with $\mathbb{S}^1$. Let H be a stable Hilbert-representation of $\tilde{G}$, in other words, $H \otimes L^2(\tilde{G}) \cong H$ as $\tilde{G}$-modules, or, any irreducible representation of $\tilde{G}$ occurs with countable infinite multiplicity in H.
Restriction of the representation to the central circle gives a decomposition $H = \oplus_{n\in\mathbb{Z}} H_n$, where H_n is the subspace of H, on which the central circle acts by multiplication with z^n. Then H_n (unlike H) carries a projective representation of G, which is G-stable. The whole construction is unique up to isomorphism. The space $\mathbb{P}(H_1)$ is our sought-after projective bundle, and we have defined a map $\Psi : \mathrm{Ext}(G;\mathbb{T}) \to \mathrm{Proj}_G(*)$.
Clearly, $\Phi\circ\Psi = \mathrm{id}$. For $\Psi\circ\Phi$, we need the stability condition on the projective bundles. $\square$

1.2 The definition of K-theory

Let X be a left-G-space and let $P \to X$ be an equivariant projective bundle.
Assume that P is a stable equivariant projective bundle. Let $\mathcal{H}$ be a separable stable G-Hilbert space (i.e. $\mathcal{H} \otimes L^2(G) \cong \mathcal{H}$) and let $\hat{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$. Then we can form the associated $\mathbb{P}U(\mathcal{H})$-bundle Q:

$$Q_x := \mathbb{P}\,\mathrm{Iso}(\mathbb{P}(\mathcal{H}); P_x),$$

(no equivariance). G acts from the left on Q, and the bundle map $Q \to X$ is G-equivariant. We have seen that $\mathbb{P}U(\mathcal{H})$ (with compact-open topology) acts continuously on $\mathrm{Fred}^{(0)}(\hat{\mathcal{H}})$. Therefore we can form

$$\mathrm{Fred}(P) := Q \times_{\mathbb{P}U(\mathcal{H})} \mathrm{Fred}^0(\mathcal{H})$$

The G-action is seen in the best way in a more abstract setting. Let G and H be groups, let $\rho : G \to H$ be a homomorphism, let X be a G-space, let $\pi : Q \to X$ be a H principal bundle with an action by G from the left, making π G-equivariant and let V be an H-space. Then $Q \times_H V \to X$ is a G-equivariant map via $g[q;v] := [gq;v]$. At the first glance, the homomorphism ρ does not seem to enter the definition. But it is hidden. Let X be a point. Choose a point $1 \in Q$. Then the map

$$[q;v] =: [1\cdot q; v] = [1, qv] \mapsto qv$$

identifies the bundle with the G-space V (via ρ)!
now we define K-theory.

Definition 1.2.1. Let X be a G-space and let $P \to X$ be a G-equivariant twist. Then the equivariant twisted K-theory is

$$K^0_{P;G}(X) := \pi_0(\mathrm{Sect}(X; \mathrm{Fred}(P))^G).$$

Because the element

$$\begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \in \mathrm{Fred}^{(0)}(\hat{\mathcal{H}})$$

is fixed under $\mathbb{P}U(\mathcal{H})$, it forms a section of the bundle $\mathrm{Fred}(P)$, which is easily seen to be G-invariant. Therefore $\mathrm{Sect}(X; \mathrm{Fred}(P))^G$ has a distinguished base-point and we may speak about higher homotopy groups.

$$K^{-i}_{P;G}(X) := \pi_i(\mathrm{Sect}(X; \mathrm{Fred}(P))^G).$$

Bott periodicity will follow from the C^*-algebraic definition and can be used to define the K-groups for all integers.
It is instructive to calculate the twisted equivariant groups when X is a point. Let $\mathbb{T} \to \tilde{G} \to G$ be a central extension and let $\mathcal{H}$ be a stable $\tilde{G}$-module as in the proof of the lemma. Let $P := \mathbb{P}(\mathcal{H})$ with the associated G-action. Then the space $\mathrm{Fred}(P)$ (we do not need to consider the space $\mathrm{Fred}^{(0)}$, because X is a point) is just the space of all Fredholm operators of $\mathcal{H}$, with the conjugation as G-action. Thus

$$K^0_{G;P}(*) \cong \pi_0(\mathrm{Fred}(\mathcal{H})^{\tilde{G}}).$$

If the extension is trivial, $\tilde{G} = G \times \mathbb{T}$, then we obtain the untwisted equivariant K-theory of a point, which is isomorphic to the representation ring of G.

1.3 The algebraic definition

There is an alternative definition of the K-theory in terms of algebraic K-theory. Let $\mathcal{K}$ be the C^*-algebra (without unit) of compact operators on $\mathcal{H}$. Because $\mathbb{P}U(\mathcal{H})_{c.o.}$ acts continuously on $\mathbb{K}$, it makes sense to study the bundle of C^*-algebras $Q \times_{\mathbb{P}U(\mathcal{H})}$. again, it is a G-equivariant bundle. This makes the algebra of sections $X \to \times_{\mathbb{P}U(\mathcal{H})}$ into a C^*-algebra (if X is compact) with an G-action. Call this C^+-algebra $\Gamma(\mathcal{K}_P)$. We can talk about the G-equivariant K-theory of this algebra.
Let A be a C^*-algebra with an action α of G ($G \to \mathrm{Aut}(A)$ is continuous in the topology of pointwise norm-convergence). Then there is a construction of a C^*-algebra $C^*(G; A, \alpha)$, and one defines the equivariant K-theory

$$K^G_i(A) := K_i(C^*(G; A)).$$

Note that the right-hand side is the usual K-theory of a C^*-algebra without G-action.

Proposition 1.3.1. *There is a natural homomorphism*

$$K^{-n}_{G;P}(X) \cong K^G_n(\Gamma(\mathcal{K}_P)$$

for all $n \geq 0$.

We will not sketch the proof of the proposition, but we will note a corollary. Because the equivariant K-theory of a $G - C^*$-algebra is the usual K-theory of a C^*-algebra, Bott periodicity holds for equivariant twisted K-theory.

2 The classification of equivariant twists

If X is a G-space, then we can form the Borel construction $E(G;X) := EG \times_G X$. It is a bundle with fiber X on BG. The equivariant cohomology groups of X are defined as

$$H^*_G(X) := H^*(E(G;X)).$$

Now we define $\mathrm{Pic}_G(X)$ as the group of G-isomorphism classes of complex G-line bundles on X and $\mathrm{Proj}_G(X)$ as the group of G-isomorphism classes of stable G-projective bundles on X (it is a group under the tensor product, with neutral element $X \times \mathbb{P}(L^2(G) \otimes \ell^2)$). If $L \to X$ is a G-line bundle, then we can form the line bundle (without G-action)

$$E(G;L) \to E(G;X),$$

and a similar construction exists for projective bundles. This gives natural transformations of functors $G - \mathcal{TOP} \to \mathcal{SET}$

$$\mathrm{Pic}_G(X) \to \mathrm{Pic}(E(G;X))$$

and

$$\mathrm{Proj}_G(X) \to \mathrm{Proj}(E(G;X)),$$

and we want to show that both transformations are bijective.

Theorem 2.0.1. *1.* $\mathrm{Pic}_G(X) \cong H^2(E(G;X);\mathbb{Z})$,

2. $\mathrm{Proj}_G(X) \cong H^3(E(G;X);\mathbb{Z})$.

If X is a point, then the theorem specializes to

Corollary 2.0.2. *1.* $\mathrm{Hom}(G;\mathbb{T}) \cong H^2(BG;\mathbb{Z})$

2. $\mathrm{Ext}(G;\mathbb{T}) \cong H^3(BG;\mathbb{Z})$.

There are not too many interesting extensions of a compact Lie group: A theorem of Borel asserts that $H^{odd}(BG;\mathbb{Q}) = 0$ and consequently, $H^3(BG;\mathbb{Z})$ is finite[1].
A more elaborate argument shows that for G connected, $H^3(BG;\mathbb{Z}) \cong \mathrm{Ext}(\pi_1(G);\mathbb{Z}) \cong \mathrm{Ext}(Tors(\pi_1(G));\mathbb{Z})$. Namely, let $\pi = \pi_1(G)$ and consider the map $G \to K(\pi;1)$, which is 2-connected and induces a 3-connected $BG \to K(\pi;2)$. Thus $H^3(BG;\mathbb{Z}) \cong H^3(K(\pi;2);\mathbb{Z}) \cong \mathrm{Ext}(\pi_1(G);\mathbb{Z})$. More geometrically, an extension $\mathbb{T} \to \tilde{G} \to G$ gives us the sequence

$$0 = \pi_2(G) \to \pi_1(\mathbb{T}) = \mathbb{Z} \to \pi_1(\tilde{G}) \to \pi_1(G) \to 0$$

from the long exact homotopy sequence.

The proof of Theorem 2.1.3 needs some preparation, which will be the subject of the next section.

2.1 some homological algebra

Let $X_\bullet$ be a simplicial space (i.e.: a simplicial object in the category of spaces) and let A be an abelian topological group. We are going to define the hypercohomology $\mathbb{H}(X_\bullet; sh(A))$ of $X_\bullet$ with coefficients in the sheaf of A-valued continuous functions. Let $sh(A) \to \mathcal{F}^{p,0} \to \mathcal{F}^{p,1} \to \mathcal{F}^{p,2} \dots$ be an injective resolution of the sheaf $sh(A)$ on X_p.
There is a natural construction for an injective resolution of a sheaf on a space, so that after taking global sections ($C^{p,q} := \Gamma(X_p;\mathcal{F}^{p,q})$, we have maps $C^{p,q} \to C^{p+1,q}$ given as the sum $\sum_{i=0}^{p}(-1)^i d_i^*$ of the maps induced by the simplicial structural maps.
The result of the whole construction is a double complex with components $C^{p,q}$. The cohomology of its total complex is by definition the cohomology $\mathbb{H}^*(X_\bullet; sh(A))$.
There is a spectral sequence

$$E_1^{p,q} = H^q(X_p, sh(A)) \to \mathbb{H}^{p+q}(X_\bullet; sh(A)).$$

If either X or A is discrete, then we obtain the usual cohomology of the geometric realization of $X_\bullet$ with coefficients in A. The example of a simplicial space which is most interesting to us arises from a continuous action of the topological group G on the space X. More generally, let us assume that $\mathcal{C}$ is a topological category (with discrete object set) and that $F : \mathcal{C} \to \mathcal{TOP}$ is a contravariant continuous functor. The *transport category* $\mathcal{C} \int F$ has object set

$$\mathfrak{Ob}(\mathcal{C}\int F) := \coprod_{c \in \mathfrak{Ob}(\mathcal{C})} F(c),$$

[1]**Proof:** The Leray-Serre sequence of the fibration $BG_0 \to BG \to \pi_0(G)$ shows that $H^*(BG;\mathbb{Q}) = H^*(BG_0;\mathbb{Q})^{\pi_0 G}$, which reduces the proof to the case of connected G. Choose a maximal torus $T \subset G$. Then the Euler number $\chi(G/T)$ is the order of the Weyl group W (Hopf-Samelson). Consider the oriented fiber bundle $G/T \to BT \to BG$. A transfer argument (Becker-Gottlieb) shows that the induced map $H^*(BG;\mathbb{Q}) \to H^*(BT;\mathbb{Q})$ is injective. The knowledge of the cohomology of BT finishes the proof.

the set of all pairs (c, x) with $c \in \mathfrak{Ob}(\mathcal{C})$ and $x \in F(c)$. A morphism $(c, x) \to (d, y)$ is a $g : c \to d$ in $\mathcal{C}$, such that $F(g)^*(y) = x$. Composition of morphisms is in the obvious way[2]. Because the category $\mathcal{C} \int F$ has a topology, its *nerve* is a simplicial space.
We can consider the group G as a category with only one object, while the morphisms are the group elements. Then a contravariant functor $X : G^{op} \to \mathcal{TOP}$ is the same as a left G-space X.
We are interested in the simplicial space $N_\bullet(G^{op} \int X)$.
The object space of the transport category is the space X, while the morphism space $\mathfrak{Mor}_{G \int X}(x, y)$ is the set of all $g \in G$ such that $gx = y$, with the obvious composition. The pth space $N_p(G^{op} \int X)$ of the nerve is $G^p \times X$, with the face maps $d_i : G^p \times X \to gG^{p-1} \times X$ given by

$$d_p(g_1, \dots g_p; x) = (g_1, \dots g_{p-1}, g_p x);$$

$$d_i(g_1, \dots g_p; x) = (g_1, \dots g_i g_{i+1}, \dots, g_p; x);$$

$$d_0(g_1, \dots g_p; x) = (g_2, \dots g_{p-1}, g_p, x).$$

It is well-known that the geometric realization of $N_\bullet G \int X$ is homotopy-equivalent to the Borel space $E(G; X) := EG \times_G X \to BG$.
The following is important.

Lemma 2.1.1. *If the group G is compact, then $\mathbb{H}^p(N_\bullet G \int X; sh(\mathbb{R})) = 0$ for all $p > 0$.*

Proof: Because the sheaf $sh(\mathbb{R})$ is fine, the term $E_1^{p,q}$ is trivial if $q > 0$ and the spectral sequence collapses at E_2. The group $E_1^{p,0}$ is $C(G^p \times X; \mathbb{R}) \cong C(G^p; C(X; \mathbb{R}))$. The locally convex complete topological vector space $V := C(X, \mathbb{R})$ has a continuous G-action. The differential $\delta : C(G^{p-1}; V) \to C(G^p; V)$ is of the form

$$(\delta f)(g_1, \dots g_p)(x) = \sum_{i=0}^{p} (-1)^i f \circ d_i(g_1, \dots g_p)(x) =$$

$$f(g_2, g_3, \dots g_p)(x) - f(g_1 g_2, g_3, \dots g_p)(x) + \dots (-1)^p f(g_1, \dots g_{p-1})(g_p x),$$

which is exactly the differential computing the continuous group cohomology $H_{cts}(G; V)$. It is (roughly) the right derived functor[3] of the left-exact functor $G-Vect \to Vect$; $V \mapsto V^G$, which sends a locally convex topological complete G-vector space to its invariant subspace. If G is compact, then the invariant integration can be used to show that the functor $V \mapsto V^G$ is even exact (trivial!). Thus the derived functors vanish, and we conclude that

[2] we shall denote the composition of two morphisms $f : c \to d$ and $g : d \to e$ in a category always by $g \circ f$, pretending that they are maps of sets.

[3] Some problems arise because the category of continuous G-modules is not abelian. The easiest way to overcome the difficulties is to write down the standard resolution of a G-module V explicitly. Exactness can be checked immediately, and the G-invariant subcomplex is the complex which we consider.

$$\mathbb{H}^0(N_\bullet G \int X; sh(\mathbb{R})) = C(G\backslash X; \mathbb{R})$$

,
while the higher groups vanish.

□

We introduce the notation $H^p_G(X; A) := \mathbb{H}^p(N_\bullet G \int X; sh(A))$

Corollary 2.1.2. *If G is compact, then $H^{p+1}_G(X; \mathbb{Z}) \cong H^p_G(X; \mathbb{T})$.*

This follows immediately from the proposition and from the use of the short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 0$. □

We are now going to prove

Theorem 2.1.3. *Let G be a compact Lie group.*

1. $\mathrm{Pic}_G(X) \cong H^1_G(X; \mathbb{T})$,
2. $\mathrm{Proj}_G(X) \cong H^2_G(X; \mathbb{T})$.

If X is a point, then the theorem specializes to

Corollary 2.1.4. 1. $\mathrm{Hom}(G; \mathbb{T}) \cong H^1(BG; \mathbb{T})$

2. $\mathrm{Ext}(G; \mathbb{T}) \cong H^2(BG; \mathbb{T})$.

2.2 Proof of Theorem 2.1.3, the line bundle case

Now we can prove the theorem. First we study the line bundle case, which is much easier. Recall that the E_1-term of the spectral sequence is

$$\begin{array}{ccc} \mathrm{Proj}(X) & \mathrm{Proj}(G \times X) & \mathrm{Proj}(G^2 \times X) \\ \\ \mathrm{Pic}(X) & \mathrm{Pic}(G \times X) & \mathrm{Pic}(G^2 \times X) \\ \\ C(X; \mathbb{T}) & C(G \times X; \mathbb{T}) & C(G^2 \times X; \mathbb{T}) \end{array}$$

and the differentials are the alternating sums of the maps induced by the face maps. More precisely,

$$C(X; \mathbb{T}) \to C(G \times X; \mathbb{T}); f \mapsto ((g, x) \mapsto f(x) - f(gx);$$

$$C(G \times X; \mathbb{T}) \to C(G^2 \times X; \mathbb{T}); f \mapsto ((g, h, x) \mapsto f(h, x) - f(gh, x) + f(g, hx);$$

and the same formulae apply to the Pic and Proj groups, with suitable interpretation.

Lemma 2.2.1. $E_2^{1,0} = H^1_{cts}(G; C(X; \mathbb{T}))$ *is isomorphic to the group* $\mathrm{Act}(G; X \times \mathbb{T})$ *of isomorphism classes of actions of* G *on the circle bundle* $X \times \mathbb{T}$.

Proof: An action on the trivialized bundle associates a function $X \to \mathbb{T}$ for any group element g. This gives the function $f \in C(G \times X; \mathbb{T})$. Changing the trivialization by another function h (or application of an isomorphism of actions) changes the function to $f + \partial h$. f is closed, because we started with an action.
Given a cocycle f, we can define the action by $g(x, t) := (gx, f(g, x) + t)$. It is easily checked that these construction give bijections. □

Lemma 2.2.2. $E_2^{2,0} = H^2_{cts}(G; C(X; \mathbb{T}))$ *is the group of all group extensions* $C(X; \mathbb{T}) \to \tilde{G} \to G$ *which admit a continuous cross-section (of spaces).*

This is classical, see: [2], Theorem 3.12 on p.93.

Lemma 2.2.3. $E_2^{0,1}$ *is the group* Pic_ρ *of those line bundles* L *on* X *which admit a bundle map* $G \times L \to L$ *covering the action map* $\rho : G \times X \to X$.

Proof: The differential $d_1 : E_1^{1,0} \to E_1^{1,1}$ maps the line bundle L to $L \otimes \mu^* L^{-1}$. □
Thus we obtain an exact sequence

$$0 \to \mathrm{Act}(G; X \times \mathbb{T}) \to H^1_G(X; \mathbb{T}) \to \mathrm{Pic}_\rho \to H^2_{cts}(G; C(X; \mathbb{T}))$$

from the spectral sequence. But there is another sequence. Let $L \in \mathrm{Pic}_\rho$ be a line bundle admitting a bundle map $\mu : G \times L \to L$ over the action map $\rho : G \times X \to X$. Choose one such bundle map. Try to turn μ into an action. We want to have $\mu(gh, l) = \mu(g, \mu(h, l))$, and the deviation between both expressions defines a cocycle in $C(G^2; C(X; \mathbb{T}))$. It turns out that the cohomology class of this cocycle does not depend on the choice of μ, and that one can turn μ into an action if and only if the cohomology class in $H^2_{cts}(G; C(X; \mathbb{T}))$ vanishes. Moreover, the resulting map $\mathrm{Pic}_\rho \to H^2_{cts}(G; C(X; \mathbb{T}))$ can be identified with the diferential.
This argument leads to the exact sequence

$$0 \to \mathrm{Act}(G; X \times \mathbb{T}) \to \mathrm{Pic}_G(X) \to \mathrm{Pic}_\rho \to H^2_{cts}(G; C(X; \mathbb{T})),$$

which intertwines with the sequence above. An application of the 5-lemma finishes the proof. □

2.3 The proof for projective bundles; homological part

The projective bundle case is much more difficult. The proof has two main steps. First, one proves that the map $\mathrm{Proj}_G(X) \to H^3_G(X; \mathbb{Z})$ is injective. In the second step, we prove that there exists a G space $\mathcal{P}$ and a stable projective G-bundle on $\mathcal{P}$, such that the composition

$$[X; \mathcal{P}]^G \to \mathrm{Proj}_G(X) \to H^3_G(X; \mathbb{Z})$$

is an isomorphism, and the surjectivity of the classifing map follows.
For the first step, one employs the spectral sequence again.

2.4 The proof for projective bundles, the homotopical part

To construct the space $\mathcal{P}$, we first choose a closed subgroup $H \subset G$ and construct a space $\mathcal{P}_H$. We have seen that stable projective representations of H are in bijection with $\operatorname{Ext}(H;\mathbb{T})$. Set

$$\mathcal{P}_H := \coprod_{\mathcal{H}\in \operatorname{Ext}(H;\mathbb{T})} B((\mathbb{P}U(\mathcal{H}))^H)$$

Lemma 2.4.1. *There is an exact sequence of groups*

$$\mathbb{P}U(\ell^2) \to (\mathbb{P}U(\mathcal{H}))^H \to \operatorname{Hom}(H;\mathbb{T}).$$

Proof: Let V be an arbitrary H-module. Then it is clear that

$$\mathbb{P}U(V) = \mathbb{P}\{u \in U(V) | \forall h \in H \exists \phi_u(h) \in \mathbb{C} : huh^{-1} = \phi_u(h)u\}$$

Given u, then $\phi_u(h_0h_1) = \phi_u(h_0)\phi_u(h_1)$, so $\phi_u \in \operatorname{Hom}(H;\mathbb{T})$. Clearly, ϕ_u is a projective invariant and descends to a continuous map $\mathbb{P}U(V) \to \operatorname{Hom}(H;\mathbb{T})$. On the other hand, $\phi_{u_0u_1}(h)u_0u_1 = hu_0h^{-1}hu_1h^{-1} = \phi_{u_0}u_0\phi_{u_1}u_1 = \phi_{u_0}\phi_{u_1}u_0u_1$. Thus $u \mapsto \phi_u$ is a homomorphism $\mathbb{P}U(V) \to \operatorname{Hom}(H;\mathbb{T})$. We show that for a stable representation V, the kernel of ϕ is connected and has the homotopy type of $\mathbb{CP}^\infty$.
The kernel of ϕ is the projective space of the space of all honest $\tilde{H}$-equivariant maps. Because V was assumed to be stable, it follows that the homotopy type is $\mathbb{P}U(\ell^2)$.

Let G be a topological group. The orbit category of G is the topological category $\mathcal{O}$, whose objects are the transitive G-spaces and whose morphisms are the G-equivariant continuous maps. A G-space Y gives a contravariant functor $F_Y : \mathcal{O} \to \mathcal{TOP}$ via $S \mapsto C(S;Y)^G$. Note that $F_Y(G) \cong Y$ as a space. We want to reconstruct the G-space (up to homotopy) from the functor F_Y. Let $F : \mathcal{O} \to \mathcal{TOP}$ be (continuous) functor. We form the topological category $\mathcal{O} \int F$. Its object space is

$$\coprod_{S \in \mathfrak{Ob}(\mathcal{O})} S \times F(S),$$

and a morphism $(s_0, y_0) \to (s_1, y_1)$ is a morphism $\theta : S_0 \to S_1$, such that $\theta(s_0) = s_1$ and $\theta^* y_1 = y_0$. Then we form the classifying space $B(\mathcal{O} \int F)$ of this category. The category $\mathcal{O} \int F$ has a G-action (i.e.: $g(s, y) := (gs, x)$). Thus $B(\mathcal{O} \int F)$ is a G-space, and one can show that $B(\mathcal{O} \int F)^H$ contains $F(G/H)$. Moreover, $F(G/H) \to B(\mathcal{O} \int F)^H$ is a homotopy equivalence.

3 Examples

As an example, we study the action of G on itself by conjugation. Call this G-space G_{conj}. For simplicity, let us assume that G is *connected*.

We want to compute the groups $H^k_G(G_{conj})$ for $k \leq 3$. This is done by the Leray-Serre spectral sequence of the fibration $G \to E(G; G_{conj}) \to BG$. The computation is greatly simplified by the existence of the fixed point 1 of the action. This produces a section $BG \to E(G;G)$ of the fibration.
Algebraically, we obtain a retraction of the Leray-Serre spectral sequence $E_r^{p,q}$ to the constant spectral sequence with $E_2^{p,q} = H^p(BG)$ for $q = 0$ and $= 0$ if $q > 0$. Another algebraic consequence is that all differentials $d_r : E_r^{p,r-1} \to E_r^{p+r,0}$ are zero!

An interesting and easy case occurs when G is semisimple. A compact Lie group is semisimple if and only if $\pi_1(G)$ is finite.
It follows that $H^1(G) = H^1(BG) = H^2(BG) = 0$.
Hence, the E_2-term of the Leray-Serre sequence is

$$\begin{array}{ccccc}
H^3(G) & H^2(BG;H^3(G)) & 0 & H^3(BG;H^3(G)) & H^4(BG;H^3(G)) \\
H^2(G) & 0 & H^2(BG;H^2(G)) & H^3(BG;H^2(G)) & H^4(BG;H^2(G)) \\
0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & H^3(BG) & H^4(BG)
\end{array}$$

with the only possibly nonzero differential with source in degree ≤ 3 indicated. It follows that $H^2_G(G) = H^2(G)$.
For $H^3_G(G)$, we look at the exact sequence

$$0 \longrightarrow H^3(BG) \longleftrightarrow H^3_G(G_{conj}) \longrightarrow H^3(G) \longrightarrow H^2(BG; H^2(G))$$

(The last group is always finite). Question: can one describe the differential for a simple group?
The splitting of the sequence is induced from the retraction of the Leray-Serre spectral sequence. In particular,

$$H^3_G(G_{conj}) \to H^3(BG) \oplus H^3(G) = \mathrm{Ext}(G; \mathbb{T}) \oplus H^3(G),$$

which sends an equivariant cohomology class to its restriction to the fixed point 1 and the underlying nonequivariant class, is injective with finite cokernel.
Moreover, if G is in addtion simply-connected, then $H^2(G) = H^3(BG) = 0$ (because $\pi_2(G) = 0$) and the result becomes much simpler.

Another example is the unitary group U_m, $m \geq 2$. It is not semisimple, but the spectral sequence is not difficult to compute because we know the cohomology.

$$H^*(U_m) \cong \Lambda(y_1, y_2, \dots y_m); \deg(y_i) = 2i - 1;$$

$$H^*(BU_m) \cong \mathbb{Z}[x_1, x_2, \dots x_m]; \deg(x_i) = 2i.$$

The spectral sequence is

$$\begin{array}{ccccc} \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z}^2 \\ \\ 0 & 0 & 0 & 0 & 0 \\ \\ \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z}^2 \\ \\ \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z}^2 \end{array}$$

All differentials in the indicated domain are trivial, because of the existence of a fixed point. Thus

$$0 \to H^2(BU_m; H^1(U_m)) \to H^3_{U_m}(U_m)) \to H^3(U_m) \to 0$$

is exact. The inclusion $U_1 \to U_m$ ($z \mapsto diag(z, 1, 1, \dots)$) induces an isomorphism on H^1 and on $H^2(B...)$. Thus this inclusion determines

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^2(BU_m; H^1(U_m)) & \longrightarrow & H^3_{U_m}(U_m)) & \longrightarrow & H^3(U_m) & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^2(BU_1; H^1(U_1)) & \xrightarrow{\cong} & H^3_{U_1}(U_1)) & \longrightarrow & H^3(U_1) = 0 & \longrightarrow & 0 \end{array}$$

which is a splitting of the exact sequence.

3.1 Relation with loop groups

We have seen a classification of all equivariant twists on a compact Lie group G acting on itself by conjugation, but have not seen a singke example of a twist. We do this now.

Definition 3.1.1. Let G be a Lie group. Then the *loop group* $\mathcal{L}G$ of G is the group of all smooth maps $\mathbb{S}^1 \to G$, under pointwise multiplication.

It is not at all clear what lops groups have to do with equivariant twists on G.
Let $\mathcal{P}G$ be the space of all smooth maps $f : \mathbb{R} \to G$, such that $t \mapsto f(t + 2\pi)f(t)^{-1}$ is constant. The loop group $\mathcal{L}G$ acts from the right on $\mathcal{P}G$ by pointwise multiplication. The map $\mathcal{P} \to G$; $f \mapsto f(2\pi)f(0)^{-1}$ makes $\mathcal{P}$ into a $\mathcal{L}G$-principal bundle.
Moreover, G acts from the left on $\mathcal{P}$, and the computation $gf \mapsto gf(2\pi)f(0)^{-1}g^{-1}$ shows

that this is a G-equivariant principal bundle over G_{conj}.

Because $\mathcal{P}G \to G$ is equivariant, we can form the Borel consruction. The result is the $\mathcal{L}G$ principal bundle $EG \times_G \mathcal{P}G \to EG \times_G G$. It has a classifying map

$$\lambda : E(G; G_{conj} \to B\mathcal{L}G.$$

A surprising result is the following:

Proposition 3.1.2. *If G is connected, then λ is a homotopy equivalence.*

Proof: we construct a homotopy commutative diagram of fibrations

$$\begin{array}{ccc} G & \xrightarrow{\simeq} & B\Omega G \\ \downarrow{\scriptstyle j} & & \downarrow \\ E(G; G_{conj} & \xrightarrow{\lambda} & B\mathcal{L}G \\ \downarrow & & \downarrow \\ BG & \xrightarrow{=} & BG, \end{array}$$

The composition $\lambda \circ j$ is the classifying map for the original bundle $\mathcal{P}G \to G$. We construct a lift of this map to $B\Omega G$. This lift exists because $\mathcal{P}G$ has a (nonequivariant) reduction of the structural group to the subgroup $\Omega G \subset \mathcal{L}G$. This is seen as follows. Let $QG := \{f : \mathbb{R} \to G | f(2\pi + t)f^{-1}(t) \equiv const; f(0) = 1\}$. This is an ΩG-principal bundle and $QG \times_{\Omega G} \mathcal{L}G \to G$ equals $\mathcal{P}G \to G$.

It remains to show that the diagram

$$\begin{array}{ccc} E(G; G_{conj}) & \longrightarrow & B\mathcal{L}G \\ \downarrow & & \downarrow \\ BG & \xrightarrow{\text{id}} & BG \end{array}$$

is commutative (up to homotopy). This will be accomplished by a comparison of the two G-principal bundles on $E(G; G_{conj})$ given by the two compositions. The bundle classified by $E(G; G_{conj}) \to BG$ is the bundle

$$EG \times_G (G_{conj} \times G_{tr})$$

(G_{tr} means G with the translation action), while the bundle classified by $E(G; G_{conj}) \to B\mathcal{L}G \to BG$ is the bundle

$$EG \times_G (\mathcal{P}G \times_{\mathcal{L}G} G).$$

We must show that $G_{conj} \times G_{tr} \cong \mathcal{P}G \times_{\mathcal{L}G} G$ (as G-spaces).
An appropriate map $\mathcal{P}G \times_{\mathcal{L}G} G \to \mathcal{P}G/\Omega G$ is given by $[f, g] \mapsto (f(2\pi)f(0)^{-1}; f(0)g^{-1})$.

If G is connected, then $G \to B\Omega G$ is a homotopy-equivalence, and the proposition follows. □

Now it is clear how to construct examples of equivariant twists. We need a projective representation $\rho : \mathcal{L}G \to \mathbb{P}U(H)$ for some Hilbert space. We should assume that the restriction to the constant loops $G \subset \mathcal{L}G$ gives a stable projective representation of G. Once ρ is given, we form

$$\mathcal{P} \times_{\mathcal{L}G} \mathbb{P}(\mathcal{H}) \to G.$$

By construction, this is a G-equivariant projective bundle on G; and our next task is the determination of the invariant in $H^3_G(G_{conj})$ of this bundle.

The projective representation ρ determines an extension of the loop group, and its underlying line bundle gives a class $u_\rho \in H^2(\mathcal{L}G)$. Let us assume for simplicity that G is connected and semisimple. Under this circumstance, we have seen that

$$H^3_G(G) \to H^3(G) \oplus H^3(BG)$$

is injective. The first component is given by forgetting of the G-equivariance of the bundle, and the second component is obtained by the restriction of the extension of $\mathcal{L}G$ to the constant loops G. we do not bother about the second component here.

It is clear that the invariant in $H^3(G)$ is given by the homotopy class

$$G \longrightarrow B\mathcal{L}G \xrightarrow{B\rho} B\mathbb{P}U(\mathcal{H}) \simeq K(\mathbb{Z};3).$$

and we need to relate the cohomology classes $u \in H^2(\mathcal{L}G)$ and $B\rho \in H^3(B\mathcal{L}G)$. This is done by the next lemma

Lemma 3.1.3. *Let γ be a topological group and let $\rho : \Gamma \to \mathbb{P}U(H)$ be a representation. Let u be the first Chern class of the line bundle defined by the extension and $v \in H^3(B\Gamma)$ be the class given by $B\rho$. Then the following holds:*

1. $u \in H^2(\Gamma;\mathbb{Z})^\Gamma$;
2. $d_2 u = 0 \in H^2(B\Gamma; H^1(\Gamma)$ *(differential in the Leray-Serre spectral sequence for $\Gamma \to E\Gamma \to B\Gamma$);*
3. $d_3(u) = v \in H^3(B\Gamma)$.

Proof: By naturality, it suffices to check the statements for the case $\Gamma = \mathbb{P}U(H)$. There it is a well-known standard result about the spectralsequence ogf the path-loop fibration of a space. □

Thus u lies in the domain of a partially defined map $H^2(\mathcal{L}G) \to H^3(B\mathcal{L}G)$.

Let G be semisimple, and connected. We have seen that it suffices to determine the class of $\mathcal{P} \times_{\mathcal{L}G} \mathbb{P}(\mathcal{H})$ as a nonequivariant bundle and the class of the restriction to the point $1 \in G$ in $\mathrm{Ext}(G;\mathbb{T})$.

The latter is very easy: There is an inclusion homomorphism $j : G \to \mathcal{L}G$ by regarding

elements of G as constant loops, and the restriction of the projective G-bundle $\mathcal{P} \times_{\mathcal{L}G} \mathbb{P}(\mathcal{H})$ to $1 \in G$ is just the projective representation $\rho \circ j$ of G.
[IS THIS A STABLE REPRESENTATION; AND IF SO: WHY???]
The representation ρ defines a central extension $\tilde{\mathcal{L}G}$ of $\mathcal{L}G$, and the class in $\mathrm{Ext}(G; \mathbb{T})$ is just the extension of G obtained by pulling back $\tilde{\mathcal{L}G}$ via j.
Thus we are left with the problem of determining the class of the bundle as a non-equivariant bundle.
Clearly,

$$G \to B\mathcal{L}G \to B\mathbb{P}U(\mathcal{H}) \simeq K(\mathbb{Z}; 3)$$

classifies the non-equivariant cohomology class of $\mathcal{P} \times_{\mathcal{L}G} \mathbb{P}(\mathcal{H})$, where the first map map classifies $\mathcal{P}G \to G$ and the second map is induced from ρ.
Let us compute the low-dimensional cohomology of $\mathcal{L}G$ and $B\mathcal{L}G$. There is an exact sequence of groups $\Omega G \to \mathcal{L}G \to G$, which is split and shows us that $\mathcal{L}G$ and $G \times \Omega G$ are homeomorphic. Caution: They are *not* isomorphic as groups and we cannot conclude that $B\mathcal{L}G \simeq BG \times B\Omega G$
Proof: Recall that $\mathcal{P}G \to G$ is a G-equivariant $\mathcal{L}G$-bundle. Thus we can form the (nonequivariant) $\mathcal{L}G$-bundle

$$EG \times_G \mathcal{P}G \to EG \times_G G_{conj},$$

with classifying map $E(G; G_{conj}) \to B\mathcal{L}G$. The restriction to the fiber G of $E(G; G_{conj}) \to BG$ is the original bundle $\mathcal{P}G \to G$, with the G-action forgotten.
Next

References

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