On the topology of the moduli stack of stable curves

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Abstract

This note is an informal report on the joint paper [4] of the author with Jeffrey Giansiracusa, which grew out of the attempt to understand the topology of the moduli stack of stable curves. The main result is the construction of a map from the moduli stack to a certain infinite loop space, which is surjective on homology in a certain range. This shows the existence of a lot of torsion classes in the homology of $\overline{\mathfrak{M}}_g n$. We give a geometric description of some of the new torsion classes. Also, we give a new proof of an (old) theorem computing the second homology of the moduli stack.

The moduli space $\overline{\mathfrak{M}}_{g,n}$ of stable n-pointed curves of genus g is a compactification of the moduli space $\mathfrak{M}_{g,n}$ of smooth n-pointed curves. One adds a boundary $\partial\mathfrak{M}_{g,n}$ which contains singular curves of a certain type, namely *stable* ones. A singular curve C with n marked distinct smooth points $p_1, \ldots, p_n$ is called stable if all singularities are ordinary double points and if there is only a finite number of automorphisms of C which fix the p_i. Strictly speaking, due to the presence of automorphisms, one must study $\overline{\mathfrak{M}}_{g,n}$ as a *stack* and not as a space.
There is a coarse moduli space $\overline{\mathfrak{M}}_{g,n}^{coarse}$, which is the topological space usually referred to as the moduli space. There are two things to say about this coarse moduli space. First of all, the rational homology $H_*(\overline{\mathfrak{M}}_{g,n}^{coarse}; \mathbb{Q})$ is isomorphic to the rational homology of the stack $\overline{\mathfrak{M}}_{g,n}$ (a concept explained below). Also, $\overline{\mathfrak{M}}_{g,n}^{coarse}$ it is a projective variety of complex dimension $3g-3+n$ and its singularities are of a very mild type (quotients of domains in a complex vector space by a finite group action).
It follows that $\overline{\mathfrak{M}}_{g,n}^{coarse}$ is a rational homology manifold, in other words, Poincaré

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duality with rational coefficients holds. However, if one wants to study topological invariants finer than rational homology, one is forced to consider the stack $\overline{\mathfrak{M}}_{g,n}$. For example, the integral homology of the coarse moduli space is not well-behaved at all.

A few words on stacks

Let us say a few words about stacks and how they can be studied by methods of algebraic topology. We will mainly consider stacks in the category of complex manifolds. As an excellent first introduction into the subject we recommend [8]; he only treats differentiable stacks, but almost all ideas carry over without much change. By definition, a stack is a very abstract object ("a lax sheaf of groupoids on the site of complex manifolds"), so let us discuss a relatively simple example, which helps to clarify the concept. We consider the stack $\mathfrak{M}_{g,n}$, the moduli stack of smooth n-pointed curves of genus g (alias Riemann surfaces). Let X be a complex manifold. We have to say what the groupoid $\mathfrak{M}_{g,n}(X)$ is. An object is a triple (E, π, j), where E is a complex manifold, $\pi : E \to X$ is a proper, surjective holomorphic submersion all of whose fibers are connected Riemann surfaces of genus g. The last piece of data is a holomorphic embedding $j : X \times \{1, \ldots, n\} \to E$ such that $\pi \circ j$ is the projection onto X. If we forget about the complex structures, then Ehresmann's fibration theorem tells us that π is a differentiable fiber bundle with structure group $\mathrm{Diff}(F_g, (p_1, \ldots, p_n))$. However, the complex structures on the fibers $\pi^{-1}(x)$ can vary with x. Experience shows that this is the appropriate notion of a holomorphic family of Riemann surfaces. An isomorphism in the category $\mathfrak{M}_{g,n}(X)$ is the obvious thing: a biholomorphic map of the total spaces which commutes with the bundle maps and the embeddings.

Given a holomorphic map $f : Y \to X$, we obtain a functor $f^* : \mathfrak{M}_{g,n}(X) \to \mathfrak{M}_{g,n}(Y)$. For two composable morphisms f_1, f_2, we do not quite have an equality $(f_2 \circ f_1)^* = f_2^* \circ f_1^*$, but only up to "2-isomorphism". Finally, we can glue objects once we have a covering of a complex manifold and objects with suitably coherent isomorphisms on intersections.

It is a standard remark that the stack $\mathfrak{M}_{g,n}$ is not representable, i.e. that there does not exists a manifold M such that for any X, the groupoid $\mathfrak{M}_{g,n}(X)$ is equivalent to the set of holomorphic maps $X \to M$. However, in a certain precise sense, $\mathfrak{M}_{g,n}$ is not too far from being representable. The statement is formal, but the proof is not - it relies on Teichmüller theory (or geometric invariant theory, for those who like schemes). Let $\mathcal{T}_{g,n}$ be the Teichmüller space of n-pointed Riemann surfaces of genus g; it is a complex $3g - 3 + n$-dimensional complex manifold which is homeomorphic to $\mathbb{C}^{3g-3+n}$. Over $\mathcal{T}_{g,n}$, there is a universal family of Riemann surfaces, which gives an object in $\mathfrak{M}_{g,n}(\mathcal{T}_{g,n})$ which is, by abstract nonsense, a morphism of stacks $p : \mathcal{T}_{g,n} \to \mathfrak{M}_{g,n}$. This is an "atlas". The meaning of this phrase is that, whenever we have a complex manifold X and an object in $\mathfrak{M}_{g,n}(X)$ (alias a map $f : X \to \mathfrak{M}_{g,n}$), then we can find an open covering $(U_i)_{i \in I}$ of X, such that the restriction $f|_{U_i}$ admits a lift to

$\mathcal{T}_{g,n}$. This is not hard to show (if and only if one knows Teichmüller theory): $\mathcal{T}_{g,n}$ is a classifying space for objects in $\mathfrak{M}_{g,n}$ with an additional piece of data: a homotopical trivialization of the underlying fiber bundle. For a general family of Riemann surfaces, such trivializations locally exist (by Ehresmann's theorem). The atlas $\phi : \mathcal{T}_{g,n} \to \mathfrak{M}_{g,n}$ has some additional properties which qualify the stack $\mathfrak{M}_{g,n}$ as a *complex-analytic Deligne-Mumford stack* or as a *complex orbifold* (which means the same).
To define the stack $\overline{\mathfrak{M}}_{g,n}$, we take not only holomorphic submersions as in the definition of $\mathfrak{M}_{g,n}$, but also suitably defined families of stable curves. This is also a Deligne-Mumford stack, but the construction of the atlas is far more technical as in the case of $\mathfrak{M}_{g,n}$. The reader is advised to consult either [3] and [9] (for an algebraic construction) or [14] for a differential-geometric perspective.

Homotopy theory of stacks

How do we extract homotopy theoretic information out of a stack? The following method makes sense in a more general situation, namely if we deal with topological stacks. Such a topological stack is a lax sheaf of groupoids on the site of topological spaces which admits an atlas (defined similarly as before). Let Stacks^{cpl} be the category of complex-analytic stacks and Stacks^{top} be the category of topological stacks. Given the very definition of a stack, one expects a functor $\mathrm{Stacks}^{top} \to \mathrm{Stacks}^{cpl}$, but that does not happen. Let $\mathfrak{X}$ be a topological stack. Of course, we can restrict the sheaf defining $\mathfrak{X}$ to the subcategory of complex manifolds, but there is no reason why there should exist a complex-analytic atlas! Instead, there is a functor $\phi : \mathrm{Stacks}^{cpl} \to \mathrm{Stacks}^{top}$ which extends the "underlying topological space functor" from complex manifolds to spaces. This is defined using an atlas, but it is a canonical construction whose result does not depend on that choice. However, given an analytic stack $\mathfrak{X}$, it may be very hard to describe the sheaf $\phi(\mathfrak{X})$ explicitly. There are also differentiable stacks and similar remarks apply to this notion.

Given an atlas $X_0 \to \mathfrak{X}$ of a topological stack, the pullback $X_1 := X_0 \times_{\mathfrak{X}} X_0$ is again a space and there are suitable maps which define a topological groupoid $\mathbb{X}$ with object space X_0 and morphism space X_1. If $\mathfrak{X} = X /\!\!/ G$ is a quotient stack, then we obtain the translation groupoid of the group action $G \curvearrowright X$.
The following definition seems to be folklore.

Definition 0.1. Let $\mathfrak{X}$ be a topological stack and let $\mathbb{X}$ be the groupoid arising from an atlas of $\mathfrak{X}$. Then the *homotopy type* $\mathrm{Ho}(\mathfrak{X})$ of the stack $\mathfrak{X}$ is the homotopy type of $B\mathbb{X}$.

This definition has the obvious disadvantage that it is not clear that $\mathrm{Ho}(\mathfrak{X})$ is independent of the choice of the atlas. But in fact, it is.

Theorem 0.2. *The homotopy type* $\mathrm{Ho}(\mathfrak{X})$ *does not depend on the choice of the atlas. Moreover, it extends to a functor from the category of topological stacks to the homotopy category of spaces.*

The proof can be found in [4] and it is built on ideas from [12]. The second sentence is a quite strong statement, because it asserts that two different atlases do not merely give homotopy equivalent classifying spaces but also that all homotopy equivalences arising from different choices are mutually compatible. If $\mathfrak{X} = X /\!\!/ G$ is a global quotient stack, then the homotopy type is the Borel-construction:

$$\mathrm{Ho}(X /\!\!/ G) = EG \times_G X.$$

A special case is the moduli space $\mathfrak{M}_{g,n}$, because the it is equivalent to the quotient of the Teichmüller space by the mapping class group[1] Γ_g^n. Because the $\mathcal{T}_{g,n}$ is contractible, we conclude that $\mathrm{Ho}(\mathfrak{M}_{g,n}) = B\Gamma_g^n$.
One can show that the homotopy type has the right (co)homology groups - there is a natural definition of the cohomology of a stack in terms of homological algebra and the result is that this homology is the same as the homology of the homotopy type. However, this remark does not apply to any homotopy-invariant functor, for example not to complex K-theory. Any good notion of complex K-theory should satisfy $K(X /\!\!/ G) = K_G(X)$ if G is a compact Lie group. But it is well-known that $K_G(X)$ and $K(EG \times_G X)$ are usually not isomorphic, see [2].

Pontrjagin-Thom maps

Let $f : M^m \to N^n$ be a proper smooth map of smooth manifolds, of codimension $d = n - m$ (which can be negative). The *normal bundle* is the stable vector bundle $\nu(f) := f^*TN - TM$ of virtual dimension d on M. As a stable vector bundle, it has a Thom spectrum $M^{\nu(f)}$. The Pontrjagin-Thom construction yields a stable homotopy class

$$\mathrm{PT}_f : \Sigma^\infty N_+ \to M^{\nu(f)}.$$

These Pontrjagin-Thom maps can be used to define umkehr maps in cohomology, once the normal bundle $\nu(f)$ is oriented. One defines f as the composition

$$H^*(M) \cong H^*(\Sigma^\infty M_+) \cong H^{*+d}(M^{\nu(f)}) \to H^{*+d}(\Sigma^\infty N_+) = H^{*+d}(N).$$

If we want to define umkehr maps also in the context of stacks, we need to construct Pontrjagin-Thom maps in the category of stacks. The problem is that the Whitney embedding theorem does not hold for stacks. But one can find a way around and we can define the Pontrjagin-Thom map if $f : \mathfrak{X} \to \mathfrak{Y}$ is a representable proper map between complex-analytic stacks and $\mathfrak{Y}$ satisfies some mild technical conditions (this condition is satisfied for all orbifolds).

[1] This notation is traditional in the theory of mapping class groups. The group usually denoted by $\Gamma_{g.n}$ is closely related, but different.

0.1 Homotopy theory of smooth moduli spaces

The Pontrjagin-Thom construction played a crucial role in the modern homotopy theory of the moduli space $\mathfrak{M}_{g,n}$ which was developed by Tillmann, Madsen and Weiss [10], [11]. They studied the universal surface bundle[2] $\pi : \mathfrak{M}_{g,1} \to \mathfrak{M}_{g,0}$. The stable normal bundle $\nu(\pi)$ can be identified with the inverse of the vertical tangent bundle $T_v\pi$; the classifying map of $T_v\pi$ is a map $\mathfrak{M}_{g,1} \to BU(1)$. Thus the Pontrjagin-Thom construction yields a map

$$\alpha : \mathfrak{M}_g \to \Omega^\infty BU(1)^{-L}.$$

The main theorem of [11] is that α induces an isomorphism in integral homology in degrees $k \leq (g-2)/2$. A crucial ingredient of the proof is Harer's stability theorem [6] which says that $H_k(\mathfrak{M}_g;\mathbb{Z})$ does not depend on g if $k \leq (g-2)/2$. We will see below (see 0.4) that the homology of $\overline{\mathfrak{M}}_{g,n}$ does not satisfy any kind of stability. Therefore we cannot expect a result as elegant as the Madsen-Weiss theorem for $\overline{\mathfrak{M}}_{g,n}$.

The surjectivity theorem

There are several natural morphisms between the moduli stacks of stable curves. Namely, there are maps

1. $\xi_{g,n} : \overline{\mathfrak{M}}_{g-1,n+2} \to \overline{\mathfrak{M}}_{g,n}$
2. $\theta_{h,k} : \overline{\mathfrak{M}}_{h,k+1} \times \overline{\mathfrak{M}}_{g-h,n-k+1} \to \overline{\mathfrak{M}}_{g,n}$,
3. $\pi : \overline{\mathfrak{M}}_{g,n+1} \to \overline{\mathfrak{M}}_{g,n}$
4. $\sigma_* : \overline{\mathfrak{M}}_{g,n} \to \overline{\mathfrak{M}}_{g,n}; \sigma \in \Sigma_n$,

given by: identifying two smooth points to a node (1 and 2), forgetting the last point (3) or permuting the n marked points (4). These morphisms are representable morphisms of complex-analytic stacks; ξ and $\theta_{h,k}$ are proper immersions of codimension 1 and π can be interpreted as the universal family of stable curves (it has codimension -1). We are particularly interested in the morphisms ξ and $\theta_{h,k}$ and study their Pontrjagin-Thom maps. The normal bundles of these morphisms are easy to describe.
There are certain natural complex line bundles on $\overline{\mathfrak{M}}_{g,n}$: if $(C, p_1, \ldots, p_n)$ is an n-pointed stable curve, then p_i is a smooth point and hence $T_{p_i}C$ is defined; this gives line bundles $L_i \to \overline{\mathfrak{M}}_{g,n}$, $i = 1, \ldots, n$.
The normal bundle of ξ is $L_{n+1} \otimes L_{n+2}$ and the normal bundle of $\theta_{h,k}$ is $L_{k+1} \otimes L_{n-k+1}$ (exterior tensor product). The morphism ξ is Σ_2-invariant and therefore induces $\tilde{\xi} : \overline{\mathfrak{M}}_{g-1,n+2} /\!\!/ \Sigma_2 \to \overline{\mathfrak{M}}_{g,n}$.

Let now $N(2) \subset U(2)$ be the normalizer of the standard maximal torus; there is a homomorphism $N(2) \to U(1)$ which multiplies the nonzero matrix entries.

[2]The case $n > 0$ can easily be reduced to $n = 0$ using Harer stability.

This induces a line bundle $V \to BN(2)$. The normal bundle of $\tilde{\xi}$ admits a bundle map and thus we obtain

$$\mathrm{PT}_{\tilde{\xi}} : \mathrm{Ho}(\overline{\mathfrak{M}}_{g,n}) \to \Omega^\infty \Sigma^\infty BN(2)^V.$$

Similarly, the normal bundle of $\theta_{h,0}$ admits a bundle map to the universal line bundle $L \to BU(1)$ and we obtain

$$\mathrm{PT}_{\theta_{h,0}} : \mathrm{Ho}(\overline{\mathfrak{M}}_{g,n}) \to \Omega^\infty \Sigma^\infty BU(1)^L.$$

The main result of [4] is the following.

Theorem 0.3. *The map* $\mathrm{PT}_{\tilde{\xi}}$ *induces an epimorphism in homology with field coefficients in degrees* $k \leq (g-2)/4$.
The map $\mathrm{PT}_{\theta_{h,0}}$ *induces an epimorphism in homology with field coefficients in degrees* $i \leq (g-2)/2(h+1)$.

The proof is based on the Harer-Ivanov stability theorem for the homology of the mapping class groups, on the Barratt-Priddy-Quillen theorem relating symmetric groups to infinite loop spaces and on the computation of the homology of the infinite loop space of the suspension spectrum of a space X in terms of the homology of X and the Dyer-Lashof algebra.
In section **??** below we will discuss the geometric meaning of some of the torsion classes provided by this theorem.
There is an important family of subrings $\mathcal{R}^*(\overline{\mathfrak{M}}_{g,n}) \subset H^*(\overline{\mathfrak{M}}_{g,n};\mathbb{Q})$, the *tautological rings*, which is the smallest system of subalgebras which contain the classes $c_1(L_i)$ and which are closed under pullback and umkehr homomorphisms by the natural maps. It is easy to see that the Pontrjagin-Thom maps above map rational cohomology into the tautological ring. Therefore we can consider the cohomology classes induced by the Pontrjagin-Thom maps as an integral refinement of the tautological rings

The low-dimensional homology groups of $\overline{\mathfrak{M}}_g$

In this section, we present a short proof of the following theorem, which was first proven by Arbarello and Cornalba [1].

Theorem 0.4. *If* $g > 4$, *then* $H_2(\overline{\mathfrak{M}}_g;\mathbb{Z})$ *is a free Abelian group of rank* $2+[g/2]$.

Arbarello and Cornalba showed used methods from algebraic geometry and their argument showed the apparently sharper result that any complex line bundle on $\overline{\mathfrak{M}}_g$ has a unique holomorphic structure. But the classical fact that $\pi_1(\overline{\mathfrak{M}}_g) = 0$ and an easy Hodge-theoretic argument shows that Theorem 0.4 implies that as well.
The proof of 0.4 is based on the differential-topological notion of a Lefschetz fibration. In this framework, it is also easy to see that $\pi_1(\overline{\mathfrak{M}}_g) = 0$, using Dehn's theorem that Dehn twists generate the mapping class group. Consider the stack $\overline{\mathfrak{M}}_g$ as a differentiable stack. It quite difficult to describe this differentiable stack

explicitly as a sheaf because a family of stable curves is not a bundle and when we pull back a family with an arbitrary smooth map, the resulting space becomes highly singular.
But "up to concordance", the differentiable stack $\overline{\mathfrak{M}}_g$ is not too hard to understand. The notion of Lefschetz fibration is an old one in algebraic geometry, I learnt the following formulation from [5].

Definition 0.5. A *Lefschetz fibration* is a tuple (p, S, U, L, q), where $p : E^{k+2} \to B^k$ is a smooth map, $S \subset E$ is the subset of critical points of p and it is a submanifold of real codimension 4. One requires that $p|_S$ is an immersion with normal crossings. The normal bundle U of S in E is endowed with a complex structure and an embedding $j : U \to E$ as a tubular neighborhood; $L \to S$ is a complex line bundle, endowed with an immersion $i : L \to B$. $q : U \to L$ is a nondegenerate quadratic form and $p \circ j = i \circ q$. Finally, the fibers of p are oriented, connected stable surfaces.

For all $x \in B$, the nodes of the fiber $p^{-1}(x)$ are the points of $S \cap p^{-1}(x)$. Any component of S has a *type* $i \in \{0, 1, \ldots [g/2]\}$ (g is the genus of the fibers). Namely, a node can either be nonseparating ($i = 0$) or it can separate the surface into two parts of genus h and $g - h$ (if $h \leq g - h$, the type is h).
One can show that for any smooth manifold B, the set of concordance classes of Lefschetz fibrations is in bijection with the set of homotopy classes $[B; \mathrm{Ho}(\overline{\mathfrak{M}}_g]$. Details will appear elsewhere.
A Lefschetz fibration over a 1-manifold is nothing else than an oriented surface bundle; Lefschetz fibrations over oriented surfaces are also not hard to describe. If F is a surface of genus g and $c \subset F$ a simple closed curve of type i (this is defined analogously to the type of a node), then there exists a Lefschetz fibration $p : E \to \mathbb{D}^2$ such that S consists of a single point s, $p(s) = 0$ of type i and the restriction $E|_{\mathbb{S}^1} \to \mathbb{S}^1$ is an oriented surface bundle whose monodromy is the Dehn twist around the curve c. If $E \to B$ is a Lefschetz fibration over a surface, then it is determined by the isomorphism class of the surface bundle $E|_{B \setminus p(S)}$ and by the monodromies around the points of $p(S)$.

Now we are ready for the proof of Theorem 0.4. We use the oriented bordism group $\Omega_2(\overline{\mathfrak{M}}_g)$ of Lefschetz fibrations, which is isomorphic to $H_2(\overline{\mathfrak{M}}_g; \mathbb{Z})$. We will establish an exact sequence

$$0 \longrightarrow \Omega_2(\mathfrak{M}_g) \longrightarrow \Omega_2(\overline{\mathfrak{M}}_g) \xrightarrow{\delta} \mathbb{Z}^{[g/2]+1} \longrightarrow 0. \tag{0.6}$$

The homomorphism δ is obtained by counting the singularities of a Lefschetz fibration, according to their type and with a sign which stems from orientation issues. This is invariant under oriented bordism.
To show that δ is surjective, we need to construct a Lefschetz fibration on an oriented surface with a single singularity of prescribed type. Take a Lefschetz fibration $E \to \mathbb{D}^2$ with a singularity of type i. The surface bundle $E|_{\mathbb{S}^1}$ is nullbordant in $\mathfrak{M}_g$, because $H_1(\mathfrak{M}_g; \mathbb{Z}) = 0$ for $g > 3$; this is a classical theorem by Powell [13]. Now take any nullbordism and glue in E. The result is a Lefschetz

fibration with a single singularity.
An old theorem of Harer [7] states that $\Omega_2(\mathfrak{M}_g;\mathbb{Z}) \cong \mathbb{Z}$ if $g > 4$; an isomorphism is given by the following procedure: Take $a \in \Omega_2(\mathfrak{M}_g)$, which can be represented by an oriented closed surface M and a surface bundle $E \to M$. The signature of the oriented 4-manifold E is divisible by 4 and the assignment $[E \to M] \mapsto \frac{1}{4} sign(E)$ is an isomorphism. It follows immediately that the map $\Omega_2(\mathfrak{M}_g) \to \Omega_2(\overline{\mathfrak{M}}_g)$ induced by the inclusion is injective: if a surface bundle $E \to B$ is nullbordant when considered as a Lefschetz fibration, the manifold E is nullbordant and hence has signature 0. This show exactness of the sequence 0.6 on the left.
Exactness in the middle is shown by a simple surgery argument. If $E \to M$ is a Lefschetz fibration with $\delta(E \to M) = 0$, then the singular points of S of type i occur in pairs with opposite sign. If s_1, s_2 is such a pair, then we can cut out small discs in M around $p(s_1)$ and $p(s_2)$. The restriction of E to the boundary of any of the two discs is a surface bundle and both are isomorphic (but the base has opposite orientations). Thus they are concordant as bundles over a cylinder. This cylinder can be glued in in and we obtain a new Lefschetz fibration, with the number of singularities reduced by 2. It represents the same bordism class as the original Lefschetz fibration. This finishes the proof of Theorem 0.4.

Remark 0.7. *The components of δ give cohomology classes $\delta_i \in H^2(\overline{\mathfrak{M}}_g;\mathbb{Z})$. They are related to our Pontrjagin-Thom maps as follows. Set $i = 0$, the other cases are similar. The Thom class of V is an element $u \in H^2(BN(2)^V;\mathbb{Z})$; it is suspended to $u' \in H^2(\Omega^\infty\Sigma^\infty BN(2)^V;\mathbb{Z})$. The class $\mathrm{PT}^*_{\tilde{\xi}} u'$ is precisely δ_0.*

An interesting class in $H^3(\overline{\mathfrak{M}}_g;\mathbb{F}_2)$

Theorem 0.3 states that the map $\overline{\mathfrak{M}}_{g,n} \to \Omega^\infty\Sigma^\infty BN(2)^V$ induces a surjection in homology with field coefficients. Equivalently, the map in cohomology with field coefficients is injective. Here we describe one of the torsion classes in $H^3(\overline{\mathfrak{M}}_{g,n};\mathbb{F}_2)$ geometrically. It is not hard to see that $H^*(BN(2);\mathbb{F}_2) \cong bF_2[x_1, x_2, w]/(w^3 = 0)$, where x_i is the image of (mod 2 reduction of) the Chern class $c_i \in H^(BU(2);\mathbb{F}_2)$ under the map induced from the inclusion $N(2) \subset U(2)$. The class $w\imath H^1(BN(2);\mathbb{F}_2)$ comes from $BN(2) \to B\pi_0(N(2)) = B\mathbb{Z}/2$. Furthermore, the Euler class of the vector bundle V is $x_1 + w^2$. The Thom isomorphism is an isomorphism $\mathrm{th} : H^*(BN(2)) \cong H^{*+2}(BN(2)^V)$. Therefore, $H^3(BN(2)^V;\mathbb{F}_2) \cong \mathbb{F}_2$ and $\mathrm{th}(w)$ is a generator.
There is an (injective) homomorphism (of graded vector spaces, not of rings) $\sigma : H^*(BN(2)^V;\mathbb{F}_2) \to H^*(\Omega^\infty\Sigma^\infty BN(2)^V;\mathbb{F}_2)$, the cohomology suspension, and we want to describe $\mathrm{PT}^*_{\tilde{xi}}\,\sigma(\mathrm{th}(w)) \in H^3(\overline{\mathfrak{M}}_{g,n};\mathbb{F}_2)$. By 0.3, this is nonzero (if $g \geq 14$. By the universal coefficient theorem and by 0.4, $H^3(\overline{\mathfrak{M}}_{g,n};\mathbb{F}_2) \cong \mathrm{Hom}(H_3(\overline{\mathfrak{M}}_{g,n};\mathbb{Z});\mathbb{F}_2)$, and the latter is isomorphic to $\mathrm{Hom}(\Omega_3(\overline{\mathfrak{M}}_{g,n});\mathbb{F}_2)$.
Assume that B is a closed oriented 3-manifold and that $p : E \to B$ is a Lefschetz fibration which represents an element in $\Omega_3(\overline{\mathfrak{M}}_{g,n})$. Let $S \subset E$ be the singular locus, a 1-dimensional submanifold and let $S_0 \subset S$ be the open and closed subspace of singular points which are of type 0. Clearly, S_0 is a disjoint

union of a finite number of circles $C_1, \dots C_k$. On any circle C_i, there is a twofold covering $q_i : \tilde{C}_i \to C_i$. Namely, for any $x \in C_i$, there exists a neighborhood $U \subset p^{-1}(p(x))$ such that $U \setminus x$ has precisely two components. These components are the elements of the fiber $q_i^{-1}(x)$.
Recall that there are exactly two equivalence classes of twofold coverings on a circle. Let $a_i = 1$ if q_i is nontrivial and $a_i = 0$ if it is trivial. Define

$$\lambda(E \to B) := \sum_{i=1}^{k} a_i \in \mathbb{F}_2.$$

It is not hard to see that this is an additive bordism-invariant $\lambda : \Omega_3(\overline{\mathfrak{M}}_{g,n}) \to \mathbb{F}_2$ and hence a cohomology class $\lambda' \in H^3(\overline{\mathfrak{M}}_{g,n}; \mathbb{F}_2)$. More or less by unwinding the definitions, one can show that

$$\lambda' = \mathrm{PT}^*_{\tilde{\xi}} \, \sigma(\mathrm{th}(w)).$$

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