HIGHER COHERENCES FOR EQUIVARIANT K-THEORY

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Abstract. Let G be a compact Lie group. We show that concepts of operator theory can be used to define an E_∞-ring spectrum representing G-equivariant K-theory. In addition we construct an E_∞-model for the G-equivariant Atiyah-Bott-Shapiro orientation $MSpin^c \to K$.

1. Introduction

About twenty-five years ago May, Quinn and Ray introduced the concept of E_∞-ring spectra [16, Chapter IV]. The definition was motiviated by the fact that there is no way to construct an internal smash product on the category of ordinary spectra and functions between them in such a way that the smash product would equip this category of spectra with a symmetric monoidal product. Of course there is the well-defined smash product on the homotopy category of spectra. An E_∞-structure on a commutative "homotopy ring spectrum" R or on a module M over it essentially guarantees that the homotopy multiplications $R \wedge R \to R$ and $R \wedge M \to M$ satisfy "all relevant algebraic relations". For example, E_∞-structures allow to define the smash product of two E_∞-module spectra over an E_∞-ring spectrum which then again is an E_∞-module spectrum over the E_∞-ring spectrum which then again will be an E_∞-module spectrum over the E_∞-ring spectrum.

Recently however several people suceeded in defining a symmetric monoidal smash product on certain categories of spectra. Of course, for doing so one needs to put some extra structure on the spectra. In particular one should mention here the category of S-modules invented by Elmendorf, Kriz, Mandell and May, and the category of symmetric spectra invented by Smith. The two approaches both have their specific advantages and disadvantages. The definition of a commutative symmetric ring spectrum actually is quite easy to give. However, if one is interested in the associated model category structure on the category of symmetric spectra or the category of module spectra over a commutative symmetric ring spectrum then the corresponding theory becomes quite complicated. Moreover, a priori it is not clear how one should extend the definition of symmetric structure to the G-equivariant setting for an arbitrary compact Lie group G. There is an obvious generalization of the notion of a symmetric spectrum: one can replace the family of symmetric groups in the definition of a symmetric spectrum by the corresponding family of orthogonal groups. This resolves the two major disadvantages stated above: most of the difficulties one has with the model category structure on the category of symmetric spectra go away, and there is an obvious way for introducing an equivariant setting [14].

Let G be a compact Lie group. A nice way to define an equivariant orthogonal spectrum is provided by the so-called $\mathscr{I}$-FSPs, introduced in [14]. The letter $\mathscr{I}$ here stands for a certain symmetric monoidal category built out of finite dimensional real G-inner product spaces. The category $\mathscr{I}$ is a symmetric monoidal G-category as is the category $\mathscr{T}_*$ consisting of the well-pointed compactly generated weak Hausdorff G-spaces and all (not necessarily G-equivariant) maps between them.[1] Taking one point compactifications of finite dimensional real G-inner product spaces provides a strong symmetric monoidal functor $S : \mathscr{I} \to \mathscr{T}_*$, and an $\mathscr{I}$-FSP by definition is a pair $(\mathbb{X}, \eta)$ consisting of a lax symmetric monoidal functor $\mathbb{X} : \mathscr{I} \to \mathscr{T}_*$ and a monoidal natural transformation $\eta : S \to \mathbb{X}$ (see Section 2). An $\mathscr{I}$-FSP can be thought of as an G-equivariant FSP in the sense of Bökstedt which is just defined on the image $S(\mathscr{I}) \subset \mathscr{T}_*$. The concept of an $\mathscr{I}$-FSP is quite closely related to the old concept of E_∞-structure: given an $\mathscr{I}$-FSP one obtains a corresponding E_∞-ring spectrum, essentially given by a restriction of the underlying functor that defines the $\mathscr{I}$-FSP (see 2.2 and c.f. [12, VII, Proposition 2.4]).

In this paper we construct two $\mathscr{I}$-FSPs, $\mathbb{K}$ and $\mathbb{M}\mathrm{Spin}^c$, as well as a map of $\mathscr{I}$-FSPs $\alpha : \mathbb{M}\mathrm{Spin}^c \to \mathbb{K}$. The first $\mathscr{I}$-FSP $\mathbb{K}$ represents equivariant periodic (complex) K-theory as defined in [19], the second one represents equivariant $Spin^c$-cobordism and the map $\alpha : \mathbb{M}\mathrm{Spin}^c \to \mathbb{K}$ represents an equivariant version of the Atiyah-Bott-Shapiro orientation $MSpin^c \to K$. In particular this implies (using the old E_∞-terminology)

Theorem

(A) *For any compact Lie group equivariant periodic complex topological K-theory can be represented by an equivariant E_∞-ring spectrum.*

(B) *For any compact Lie group an equivariant version of the Atiyah-Bott-Shapiro orientation $MSpin^c \to K$ can be realized as a map of equivariant E_∞-ring spectra.*

Until recently it was not even known if the non-equivariant periodic K-theory spectrum can be represented by an E_∞-ring spectrum. It has been known for a long time though that non-equivariant connective K-theory can be represented by an E_∞-ring spectrum [16, VIII, Theorem 2.1], and that the periodic K-theory spectrum K can be obtained from the connective K-theory spectrum k by localizing with respect to the Bott class in $\pi_2(k)$. However

[1] Here we adopted the notation used in [14]; it is important in the set-up of the theory to carefully distinguish between the category $\mathscr{T}_*$ and the subcategory $\mathscr{T}_*^G$ consisting if all G-equivariant maps.

one did not know that one can perform localizations within the category of E_∞-ring spectra. The latter now can be shown using the theory of S-modules, so we can obtain an E_∞-model for K this way. A similar approach can be used to show that equivariant periodic K-theory can be represented through an equivariant E_∞-spectrum in case G is a finite group (see [20]), but the methods do not generalize to the case where G is an arbitrary compact Lie group. For $G = 1$ an explicit geometric construction of an E_∞-model for the periodic K-theory spectrum also can be obtained from [9], where we constructed a symmetric ring spectrum spectrum representing periodic real K-theory. Where the second statement of the theorem is concerned, here we are not even aware of any previously known explicit E_∞-model for the Atiyah-Bott-Shapiro orientation, even in the case where $G = 1$.

The construction of the $\mathscr{I}$-FSP $\mathbb{K}$ is in fact related to the definition of the symmetric ring spectrum KO representing non-equivariant KO-theory given in [9]. There we have used Clifford linear Fredholm operators to build the spaces $KO(n)$ for the symmetric ring spectrum KO. One technical difficulty in this approach was to introduce appropriate base points. In this paper we show that more recent concepts of operator theory can be used in a straightforward manner to construct an $\mathscr{I}$-FSP $\mathbb{K}$ which represents (periodic) equivariant K-theory. The appropriate concept is the so-called homomorphism picture for equivariant KK-theory. In the homomorphism picture ([21],[17]) the K-homology group of a G-C^*-algebra is identified with the set of path components of a space of $*$-homomorphism between certain C^*-algebras. It follows that all spaces which are involved in our definition of $\mathbb{K}$ will be spaces of $*$-homomorphisms, hence they all naturally come equipped with a base point, given by the trivial $*$-homomorphism which sends all elements to zero. Another advantage of this approach is that it is quite clear how one can modify the definition of the $\mathscr{I}$-FSP $\mathbb{K}$ in order to obtain E_∞-ring spectra $\mathbb{K}A$ representing the equivariant K-homology groups $K_*^G(A)$ for commutative G-C^*-algebras A.

In this paper we are working with equivariant periodic complex topological K-theory. However all constructions and proofs go through equally well for equivariant periodic real topological K-theory, with the obvious modifications which correspond to the different period of Bott periodicity in the real setting. It follows that our constructions also can be used to show that equivariant periodic real topological K-theory can be represented by an equivariant E_∞-spectrum, and that an equivariant version of the Atiyah-Bott-Shapiro orientation $MSpin \to KO$ can be modelled by a map of equivariant E_∞-ring spectra.

The paper is organized as follows. In Section 2 we recall the definition of an $\mathscr{I}$-FSP and we show how an $\mathscr{I}$-FSP $\mathbb{X}$ determines a multiplicative equivariant cohomology theory $\mathbb{X}^*$ on the category $\mathscr{T}_*^G$, which is the subcategory of $\mathscr{T}_*$ consisting of the equivariant maps. In the following two sections we construct the $\mathscr{I}$-*FSP* $\mathbb{K}$ which represents equivariant K-theory. By the latter we mean that there is a natural isomorphism $\Phi^* : \mathbb{K}^* \to \widetilde{K}^*_G$ of (reduced) multiplicative equivariant cohomology theories on $\mathscr{K}_*^G$, the full subcategory of $\mathscr{T}_*^G$ generated by the compact spaces in $\mathscr{T}_*^G$. In Section 5 we construct the corresponding natural isomorphism Φ^*, using well-known concepts of equivariant KK-theory. The construction of the $\mathscr{I}$-*FSP* $\mathbb{M}\mathrm{Spin}^c$ as well as the map $\alpha : \mathbb{M}\mathrm{Spin}^c \to \mathbb{K}$ are discussed in Section 6.

Convention: Throughout the paper we will work in the category of compactly generated weak Hausdorff spaces. If a certain construction (like taking mapping spaces) does not produce a compactly generated space at the first place we think of it as being hit by the Kelley functor k without notice.

2. Symmetric monoidal categories and *FSP*'s

A *symmetric monoidal topological G-category* is a topological G-category $\mathcal{C}$ together with a continuous G-functor $\oplus_{\mathscr{C}} : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ and a unit object $u_{\mathscr{C}}$ such that the functor $\oplus_{\mathscr{C}}$ is associative, commutative and unital up to coherent natural isomorphism.[2] Given two symmetric monoidal topological G-categorgies $\mathscr{C}$ and $\mathscr{D}$, a *lax symmetric monoidal G-functor* $\mathscr{C} \to \mathscr{D}$ consists of a continuous G-functor $\mathbb{X} : \mathscr{C} \to \mathscr{D}$ together with a natural transformation of functors from $\mathscr{C} \times \mathscr{C}$ to $\mathscr{D}$ consisting of equivariant maps

$$\mu = \mu_{a_1,a_2} : \mathbb{X}(a_1)\oplus_{\mathscr{D}}\mathbb{X}(a_2) \to \mathbb{X}(a_1\oplus_{\mathscr{C}}a_2), \quad a_1, a_2 \in \mathscr{C}$$

and an equivariant map $\lambda : u_{\mathscr{D}} \to \mathbb{X}(u_{\mathscr{C}})$ such that all coherence diagrams relating associativity, commutativity and unit isomorphisms of $\mathscr{C}$ and $\mathscr{D}$ are commutative. A lax symmetric monoidal G-functor $\mathbb{X} = (\mathbb{X}, \mu, \lambda)$ is called a *strong symmetric monoidal G-functor* if μ and λ are isomorphisms. A natural transformation of lax symmetric monoidal G-functors $\alpha : \mathbb{X} \to \mathbb{Y}$ is called *monoidal* if the following diagrams commute

$$\begin{tikzcd} \mathbb{X}(a_1)\oplus_{\mathscr{D}}\mathbb{X}(a_2) \arrow[r, "\alpha\oplus_{\mathscr{D}}\alpha"] \arrow[d, "\mu_{\mathbb{X}}"'] & \mathbb{Y}(a_1)\oplus_{\mathscr{D}}\mathbb{Y}(a_2) \arrow[d, "\mu_{\mathbb{Y}}"'] \\ \mathbb{X}(a_1\oplus_{\mathscr{C}}a_2) \arrow[r, "\alpha"] & \mathbb{Y}(a_1\oplus_{\mathscr{C}}a_2) \end{tikzcd} \qquad \begin{tikzcd} & u_{\mathscr{D}} \arrow[dl, "\lambda_{\mathbb{X}}"'] \arrow[dr, "\lambda_{\mathbb{Y}}"] & \\ \mathbb{X}(u_{\mathscr{C}}) \arrow[rr, "\alpha"] & & \mathbb{Y}(u_{\mathscr{C}}). \end{tikzcd}$$

Note that if $\mathbb{X}_1, \mathbb{X}_2 : \mathscr{C} \to \mathscr{D}$ are lax symmetric monoidal G-functors, then we can define the product of the two functors $\mathbb{X}_1\oplus_{\mathscr{D}}\mathbb{X}_2 : \mathscr{C} \to \mathscr{D}$, given by $(\mathbb{X}_1\oplus_{\mathscr{D}}\mathbb{X}_2)(a) = \mathbb{X}_1(a)\oplus_{\mathscr{D}}\mathbb{X}_2(a)$, which naturally is a lax symmetric monoidal G-functor.

Example 2.1. Let $\mathscr{C} = (\mathscr{C}, \oplus_{\mathscr{C}}, u_{\mathscr{C}})$ and $\mathscr{D} = (\mathscr{D}, \oplus_{\mathscr{D}}, u_{\mathscr{D}})$ be symmetric monoidal topological G-categories. Then there always is the *trivial* strong

[2] See [13, VII §1, §7] for a precise definition of the latter.

symmetric monoidal G-functor called $u_{\mathscr{D}} : \mathscr{C} \to \mathscr{D}$, denoted as the unit of $\mathscr{D}$. It associates to any object $a \in \mathscr{C}$ the unit $u_{\mathscr{D}}$ and to any morphism $a \to b$ in $\mathscr{C}$ the identity on $u_{\mathscr{D}}$. The corresponding natural transformation $\mathscr{C}\times\mathscr{C} \to \mathscr{D}$ for any pair of objects $(a_1, a_2) \in \mathscr{C}^2$ is given through the canonical map $u_{\mathscr{D}} \oplus_{\mathscr{D}} u_{\mathscr{D}} \to u_{\mathscr{D}}$, and the corresponding morphism λ is the identity on $u_{\mathscr{D}}$. Note also, given any lax symmetric monoidal G-functor $\mathbb{X} : \mathscr{C} \to \mathscr{D}$ then there is a canonical monoidal natural isomorphism of lax symmetric monoidal G-functors $\mathbb{X} \to \mathbb{X} \oplus_{\mathscr{D}} u_{\mathscr{D}}$. This terminology will be used in the following sections.

Recall that a *G-universe* U is a sum of countably many copies of each real G-inner product space in some set of irreducible representations of G that includes the trivial representation $\mathbb{R}$. U is called *complete* if it contains all irreducible representations of G up to isomorphism. A finite dimensional G-invariant linear subspace of U is called an *indexing space*. The subspace relation gives the set of indexing spaces the structure of a directed set, which we will denote $\mathscr{D} = \mathscr{D}(U)$ in the sequel. Let $\mathscr{I} = \mathscr{I}(U)$ be the topological G-category whose objects are all real G-inner product spaces that are equivariantly isomorphic to indexing spaces in U, and whose morphisms are the linear isometric isomorphisms equipped with the G-action given by conjugation. $\mathscr{I}$ becomes a symmetric monoidal topological G-category under taking direct sums of inner product spaces in $\mathscr{I}$. From this point of view we want to regard the direct sum on $\mathscr{I}$ as a functor $\oplus : \mathscr{I} \times \mathscr{I} \to \mathscr{I}$, and without loss of generality we will assume that the unit is the trivial subspace $0 \subset U$.

Next let us consider the topological G-category $\mathscr{T}_*$ of well-pointed compactly generated weak Hausdorff G-spaces and all (not necessarily G-equivariant) continuous maps between them. The topology on the morphism sets shall be given by the compact-open topology. The category has the structure of a symmetric monoidal topological G-category given through the smash product of pointed G-spaces with unit the space S^0 equipped with the trivial G-action. Let $S : \mathscr{I} \to \mathscr{T}_*$ be the strong symmetric monoidal G-functor that maps an inner product space $V \in \mathscr{I}$ to its one point compactification S^V.

Definition 2.2. A commutative *$\mathscr{I}$-FSP* is a lax symmetric monoidal G-functor $\mathbb{X} : \mathscr{I} \to \mathscr{T}_*$ together with a monoidal natural transformation $\eta : S \to \mathbb{X}$ of lax symmetric monoidal G-functors.

Let $\eta : S \to \mathbb{X}$ be a commutative $\mathscr{I}$-FSP. For any pair of indexing spaces V_1, V_2 with $V_1 \perp V_2$ and any pair of inner product spaces $W, Z \in \mathscr{I}$ the natural transformation η provides maps

$$\begin{aligned} (2.3) \qquad \sigma : S^{V_1 \oplus W} \wedge \mathbb{X}(V_2 \oplus Z) &\xrightarrow{\eta_{V_1 \oplus W} \wedge Id} \mathbb{X}(V_1 \oplus W) \wedge \mathbb{X}(V_2 \oplus Z) \\ &\xrightarrow{\mu_{\mathbb{X}}} \mathbb{X}(V_1 \oplus W \oplus V_2 \oplus Z) \xrightarrow{\cong} \mathbb{X}(V_1 + V_2 \oplus W \oplus Z). \end{aligned}$$

Here, the last map[3] in the composition is given by applying the functor $\mathbb{X}$ to the canonical map

$$V_1 \oplus W \oplus V_2 \oplus Z \to V_1 + V_2 \oplus W \oplus Z.$$

The maps σ define a coordinate-free prespectrum in the sense of Lewis-May-Steinberger (cf. [12]).

As usual, a prespectrum defines a corresponding equivariant cohomology theory on $\mathscr{T}_*^G$, the subcategory of $\mathscr{T}_*$ consisting of the G-equivariant maps. Given inner product spaces $W, Z \in \mathscr{I}$ we obtain groups $\mathbb{X}^{W,Z}$ for pointed spaces $X \in \mathscr{T}_*$ by defining

$$\mathbb{X}^{W,Z}(X) = \operatorname*{colim}_{V \in \mathscr{D}} [S^{V \oplus W} \wedge X,\ \mathbb{X}(V \oplus Z)]_G. \tag{2.4}$$

Here, $[\ ,\ ]_G$ stands for pointed G-homotopy classes of pointed G-maps, and the colimites are taken over the indexing spaces $V \in \mathscr{D}$ using the suspension maps (2.3). The bigraded groups (2.4) can be used (using standard techniques) to define a cohomology theory $\mathbb{X}^*$ whose grading is over the real representation ring $RO(G)$.

Given inner product spaces $W_1, Z_1, W_2, Z_2 \in \mathscr{I}$ and pointed spaces $X_1, X_2 \in \mathscr{T}_*$ the multiplicative structure of $\mathbb{X}$ provides a natural pairing

$$\cup\ :\ \mathbb{X}^{W_1,Z_1}(X_1) \times \mathbb{X}^{W_2,Z_2}(X_2) \longrightarrow \mathbb{X}^{W_1 \oplus W_2, Z_1 \oplus Z_2}(X_1 \wedge X_2). \tag{2.5}$$

More explicitly, if G-equivariant pointed maps

$$\begin{aligned} \varphi_1 : S^{V_1 \oplus W_1} \wedge X_1 &\ \to\ \mathbb{X}(V_1 \oplus Z_1), \\ \varphi_2 : S^{V_2 \oplus W_2} \wedge X_2 &\ \to\ \mathbb{X}(V_2 \oplus Z_2) \end{aligned}$$

represent classes $x_1 \in \mathbb{X}^{W_1,Z_1}(X_1)$ and $x_2 \in \mathbb{X}^{W_2,Z_2}(X_2)$, and $V_1 \perp V_2$ in U, then the product $x_1 \cup x_2 \in \mathbb{X}^{W_1 \oplus W_2, Z_1 \oplus Z_2}(X_1 \wedge X_2)$ is represented by the map

$$\begin{aligned} S^{V_1+V_2 \oplus W_1 \oplus W_2} \wedge X_1 \wedge X_2 &\longrightarrow \big(S^{V_1 \oplus W_1} \wedge X_1\big) \wedge \big(S^{V_2 \oplus W_2} \wedge X_2\big) \\ &\xrightarrow{\varphi_1 \wedge \varphi_2} \mathbb{X}(V_1 \oplus Z_1) \wedge \mathbb{X}(V_2 \oplus Z_2) \xrightarrow{\mu_{\mathbb{X}}} \mathbb{X}(V_1 \oplus Z_1 \oplus V_2 \oplus Z_2) \\ &\qquad \to \mathbb{X}(V_1 + V_2 \oplus Z_1 \oplus Z_2), \end{aligned}$$

where the first map is induced by the various natural isomorphisms and the last one is obtained by applying $\mathbb{X}$ to the canonical map $V_1 \oplus Z_1 \oplus V_2 \oplus Z_2 \to V_1 + V_2 \oplus Z_1 \oplus Z_2$. As every pair of equivalence classes $x_1 \in \mathbb{X}^{W_1,Z_1}, x_2 \in \mathbb{X}^{W_2,Z_2}$ can be represented by maps φ_1 and φ_2 as above (i.e. with $V_1 \perp V_2$) this can actually be used as the definition of the pairing (2.5). The natural pairing (2.5) induces a corresponding natural pairing for the $RO(G)$-graded

[3]The appearance of the last isomorphism in the composition (2.3) is motivated by the fact that $V_1 + V_2$ is an indexing space while $V_1 \oplus V_2$ is not. The issue will become relevant when we define $\mathbb{X}^{W,Z}(X)$ in (2.4).

cohomology groups $\mathbb{X}^*$ which equips the $RO(G)$-graded cohomology theory $\mathbb{X}^*$ with a multiplicative structure.

3. Clifford-algebras and compact operators

Recall that a $\mathbb{Z}/2$-graded G-C^*-algebra A is a $\mathbb{Z}/2$-graded C^*-algebra A with a grading preserving covariant representation of G on A (see [5, 10.1]), i.e. in particular, A regarded as a topological space carries a continuous G-action. Let us denote by $\mathscr{A}$ the topological G-category of $\mathbb{Z}/2$-graded G-C^*-algebras with morphisms the grading preserving $*$-homomorphisms. The topology on its morphism sets $Hom(A, B)$ shall be the (compactly generated) compact-open topology. The $\mathbb{Z}/2$-graded tensor product $\otimes : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ with unit $\mathbb{C}$ gives $\mathscr{A}$ the structure of a symmetric monoidal topological G-category.[4] The unit for $\otimes$ is the C^*-algebra $\mathbb{C}$ regarded as a $\mathbb{Z}/2$-graded C^*-algebra using the trivial $\mathbb{Z}/2$-grading and the trivial G-action.

Definition 3.1. Given an inner product space $V \in \mathscr{I}$ we define the Clifford algebra $\mathbb{C}l_V$ to be the quotient of the complex tensor algebra on V modulo the ideal generated by the elements $v \otimes v - \langle v, v\rangle \cdot 1,\ v \in V$. The Clifford algebras naturally come with a $\mathbb{Z}/2$-grading $\mathbb{C}l_V = \mathbb{C}l_V^{even} + \mathbb{C}l_V^{odd}$ induced from the $\mathbb{N}_0$-grading of the complex tensor algebra generated by V. Moreover there are canonical isomorphisms $\mathbb{C}l_V \otimes \mathbb{C}l_W \cong \mathbb{C}l_{V\oplus W}$, and we have the obvious isomorphism $\mathbb{C} \to \mathbb{C}l_0$ sending the unit in $\mathbb{C}$ to the unit in $\mathbb{C}l_0$. Finally $\mathbb{C}l_V$ is canonically isomorphic as a vector space to the exterior algebra $\Lambda^*(V \otimes \mathbb{C})$. Using this isomorphism left multiplication on $\mathbb{C}l_V$ defines a $*$-algebra monomorphism $\mathbb{C}l_V \to End(\Lambda^*(V \otimes \mathbb{C}))$. Using the standard scalar product product on $\Lambda^*V \otimes \mathbb{C}$ we interpret $End(\Lambda^*(V \otimes \mathbb{C}))$ as the C^*-algebra of bounded operators of a (finite dimensional) Hilbert space. The inclusion $\mathbb{C}l_V \subset End(\Lambda^*(V \otimes \mathbb{C}))$ then induces a C^*-norm on $\mathbb{C}l_V$, which gives $\mathbb{C}l_V$ the structure of a $\mathbb{Z}/2$-graded G-C^*-algebra. Altogether we get a strong symmetric monoidal G-functor $\mathbb{C}l : \mathscr{I} \to \mathscr{A}$ by sending V to $\mathbb{C}l_V$.

Definition 3.2. Given an inner product space $V \in \mathscr{I}$ let $L^2(V)$ denote the L^2-completion of the pre-Hilbert space of functions of compact support on V with values in $\mathbb{C}$. The space $L^2(V)$ comes equipped with a natural $\mathbb{Z}/2$-grading which corresponds to the decomposition into even and odd functions on V. For two inner product spaces $V, W \in \mathscr{I}$ there is the canonical G-invariant isomorphism

$$L^2(V) \otimes L^2(W) \xrightarrow{\cong} L^2(V \oplus W),\ f \otimes g \mapsto f \cdot g \tag{3.3}$$

[4] On the algebraic tensor product of two C^*-algebras there is typically more than one meaningful definition of a C^*-norm, and the algebraic tensor product typically is not complete with respect to any of these C^*-norms. Thus, in order to define a tensor product which is a C^*-algebra one specifies a corresponding choice of a C^*-norm and defines the tensor product to be the completion of the algebraic tensor product with respect to it. The C^*-norm that we want to use is the so-called spatial C^*-norm, see [22, App. T] for details.

where the tensor product here is the graded tensor product of Hilbert spaces.

We now associate to V the $\mathbb{Z}/2$-graded G-C^*-algebra of compact operators on $L^2(V)$

$$\mathcal{K}_V = \mathcal{K}(L^2(V)).$$

The canonical maps (3.3) induce corresponding canonical isomorphisms

$$\mathcal{K}_V \otimes \mathcal{K}_W \xrightarrow{\cong} \mathcal{K}_{V\oplus W}.$$

Moreover we have the obvious map $\mathbb{C} \to \mathcal{K}_0$ sending the unit $1 \in \mathbb{C}$ to the unit in $\mathcal{K}_0$. This gives $\mathcal{K}$ the structure of a strong symmetric monoidal G-functor $\mathscr{I} \to \mathscr{A}$.

Now consider the $\mathbb{Z}/2$-graded G-C^*-algebra $\mathcal{S}$ of continuous functions on $\mathbb{R}$ vanishing at infinity, equipped with the grading given by the even and odd functions therein and the trivial G-action. Note that there is one and only one non trivial grading preserving $*$-homomorphism $ev_0 : \mathcal{S} \to \mathbb{C}$. It is given by evaluating a function $u \in \mathcal{S}$ at $0 \in \mathbb{R}$, i.e. $ev_0(u) = u(0)$.

There is a comultiplication map $\Delta : \mathcal{S} \longrightarrow \mathcal{S}\otimes\mathcal{S}$. To present it we consider the isomorphism of linear spaces $\mathcal{S} \otimes \mathcal{S} \cong C_0(\mathbb{R}) \otimes C_0(\mathbb{R}) \cong C_0(\mathbb{R}^2)$, where $C_0(\mathbb{R})$ and $C_0(\mathbb{R}^2)$ denote the C^*-algebras of continuous functions vanishing at infinity on $\mathbb{R}$ and $\mathbb{R}^2$ respectively. We also use the fact that the C^*-algebra $\mathcal{S}$ is generated by the functions $u(t) = (1 + t^2)^{-1}$ and $v(t) = t(1 + t^2)^{-1}$. The map Δ then is given by

$$\Delta(u)(x, y) = \frac{1}{1 + x^2 + y^2} \quad \text{and} \quad \Delta(v)(x, y) = \frac{x + y}{1 + x^2 + y^2}.$$

One easily checks that Δ is coassociative and cocommutative. The following lemma is an immediate consequence of the coassociativity and the cocommutativity of the comultiplication map.

Below we want to consider the spaces $Hom(\mathcal{S}, A)$ for $\mathbb{Z}/2$-graded G-C^*-algebras A. Without further assumptions on A one maybe should not expect this homomorphism space to be well-pointed. Thus we temporarily will work with the category $\mathscr{T}'_*$ of pointed compactly generated weak Hausdorff spaces and all continuous maps between them.

Lemma 3.4. *The functor $Hom(\mathcal{S}, ?) : \mathscr{A} \to \mathscr{T}'_*$ sending a $\mathbb{Z}/2$-graded G-C^*-algebra A to the pointed G-space $Hom(\mathcal{S}, A)$ together with the natural transformation $* : \mathscr{A} \times \mathscr{A} \to \mathscr{T}'_*$ given by the family of maps*

$$* : Hom(\mathcal{S}, A) \wedge Hom(\mathcal{S}, B) \longrightarrow Hom(\mathcal{S} \otimes \mathcal{S}, A \otimes B) \xrightarrow{\Delta^*} Hom(\mathcal{S}, A \otimes B)$$

and the pointed isomorphism $S^0 \cong Hom(\mathcal{S}, \mathbb{C})$ is a lax symmetric monoidal G-functor.

(The unlabeled map in the composition is given by taking the tensor product of $*$-homomorphisms.)

Corollary 3.5. *Let $\mathscr{C}$ be a symmetric monoidal topological G-category.*

(1) *Let $\mathbb{X} = (\mathbb{X}, \mu, \lambda) : \mathscr{C} \to \mathscr{A}$ be a lax symmetric monoidal G-functor. Then the G-functor $\widehat{\mathbb{X}} : \mathscr{C} \to \mathscr{T}'_*$, defined by*

$$\widehat{\mathbb{X}} : a \longmapsto Hom(\mathcal{S}, \mathbb{X}(a)), \quad a \in \mathscr{C},$$

together with the natural transformation of G-functors $\mathscr{C} \times \mathscr{C} \to \mathscr{T}'_*$

$$Hom(\mathcal{S}, \mathbb{X}(a)) \wedge Hom(\mathcal{S}, \mathbb{X}(b)) \xrightarrow{*} Hom(\mathcal{S}, \mathbb{X}(a) \otimes \mathbb{X}(b)) \xrightarrow{\mu_{a,b*}} Hom(\mathcal{S}, \mathbb{X}(a \oplus b))$$

and the canonical map $S^0 \xrightarrow{\cong} Hom(\mathcal{S}, \mathbb{C}) \xrightarrow{\lambda_*} Hom(\mathcal{S}, \mathbb{X}(u_{\mathscr{C}}))$ is a lax symmetric monoidal G-functor $\widehat{\mathbb{X}}$ from $\mathscr{C}$ to $\mathscr{T}'_*$.

(2) *Let $\mathbb{X}_1, \mathbb{X}_2 : \mathscr{C} \to \mathscr{A}$ be lax symmetric monoidal G-functors, then the maps*

$$* : Hom(\mathcal{S}, \mathbb{X}_1(a)) \wedge Hom(\mathcal{S}, \mathbb{X}_2(a)) \longrightarrow Hom(\mathcal{S}, \mathbb{X}_1(a) \otimes \mathbb{X}_2(a))$$

define a monoidal natural transformation $* : \widehat{\mathbb{X}_1} \wedge \widehat{\mathbb{X}_2} \to \widehat{\mathbb{X}_1 \otimes \mathbb{X}_2}$.

Definition 3.6. Let $\mathbb{K}$ be the lax symmetric monoidal G-functor $\mathbb{K} = \widehat{\mathbb{C}l \otimes \mathcal{K}}$, where $\widehat{\mathbb{C}l \otimes \mathcal{K}}$ is defined as in Corollary 3.5. In particular we have

$$\mathbb{K}(V) = Hom(\mathcal{S}, \mathbb{C}l_V \otimes \mathcal{K}_V) \quad \text{for all } V \in \mathscr{I}.$$

One can show that these homomorphism spaces are in fact all well-pointed; thus $\mathbb{K}$ is a lax symmetric monoidal G-functor from $\mathscr{I}$ to $\mathscr{T}_*$.

4. The K-theory spectrum

One goal of this section is to complete the definition of of the $\mathscr{I}$-FSP $\mathbb{K}$, i.e. in this section we will present the relevant monoidal natural transformation $\eta : S \to \mathbb{K}$. Moreover we will state the first main theorem and give some comments on the proof.

As already mentioned earlier the C^*-algebra $\mathcal{S}$ is generated by the two functions $u(t) = (1 + t^2)^{-1}$ and $v(t) = t(1 + t^2)^{-1}$. Thus a $*$-homomorphism out of $\mathcal{S}$ is specified by saying where these two elements go to. Now let W be a finite dimensional G-inner product space and let $w \in W$. The element w then can be regarded as an element of the Clifford algebra $\mathbb{C}l_W$ by means of the natural inclusion $W \subset \mathbb{C}l_W$. We can specify a $*$-homomorphism $fc(w) : \mathcal{S} \to \mathbb{C}l_W$ by requiring that $fc(w)(u) = (1+|w|^2)^{-1}$ and $fc(w)(v) = w \cdot (1+|w|^2)^{-1}$. Here the letters fc stand for "functional calculus". We shall think of $fc(w)$ as been given by substituting the variable t by w. If w becomes big in norm then the norm of $fc(w)$ tends to zero.

Definition 4.1. Let V be an inner product space in $\mathscr{I}$. Associating to $v \in V$ the $*$-homomorphism $fc(v)$ induces a continuous map $fc_V : V \to Hom(\mathcal{S}, \mathbb{C}l_V)$. Since the norm of $fc(v)$ tends to zero when v goes to infinity it follows that the map fc_V has a continuous pointed extension

$$\beta_V : S^V \longrightarrow Hom(\mathcal{S}, \mathbb{C}l_V).$$

For inner product spaces $V, W \in \mathscr{I}$ and $v \in V, w \in W$ the element $v + w \in \mathbb{C}l_{V\oplus W}$ is the image of $v \otimes 1 + 1 \otimes w \in \mathbb{C}l_V \otimes \mathbb{C}l_W$ under the canonical isomorphism $\mathbb{C}l_V \otimes \mathbb{C}l_W \cong \mathbb{C}l_{V\oplus W}$. A direct calculation shows that $fc(v+w)$ coincides with the composition

$$\mathcal{S} \xrightarrow{\Delta} \mathcal{S} \otimes \mathcal{S} \xrightarrow{fc(v)\otimes fc(w)} \mathbb{C}l_V \otimes \mathbb{C}l_W \cong \mathbb{C}l_{V\oplus W}.$$

We define a monoidal natural transformation $\beta : S \to \widehat{\mathbb{C}l}$ through the maps $\beta_V : S^V \to \mathit{Hom}(\mathcal{S}, \mathbb{C}l_V)$ which send a vector $v \in V \subset S^V$ to $\beta_V(v) = fc(v)$.

Definition 4.2. Given an inner product space $V \in \mathscr{I}$, let $P_V \in \mathcal{K}_V$ denote the projection onto $\mathbb{C}i_V$, the linear subspace in $L^2(V)$ generated by the function $i_V \in L^2(V)$ which sends a vector v to $\exp(-\langle v, v\rangle)$. We then define the element $p_V^0 \in \mathit{Hom}(\mathcal{S}, \mathcal{K}_V)$ to be the composition of the non-trivial $*$-homomorphism $ev_0 : \mathcal{S} \to \mathbb{C}$ with the $*$-homomorphism $\mathbb{C} \to \mathcal{K}_V$ which sends the unit in $\mathbb{C}$ to the projection P_V. The family $(p_V^0)_{\{V\in\mathscr{I}\}}$ is multiplicative, since $i_{V\oplus W} = i_V \cdot i_W$ for inner product spaces $V, W \in \mathscr{I}$. Now recall from the second section that the unit S^0 of the symmetric monoidal category $\mathscr{T}_*$ gives rise to a lax monoidal functor $S^0 : \mathscr{I} \to \mathscr{T}_*$, which sends any inner product space $V \in \mathscr{I}$ to $S^0_V = S^0$. The family $(p_V^0)_{\{V\in\mathscr{I}\}}$ then defines a lax symmetric monoidal natural transformation $p : S^0 \to \widehat{\mathcal{K}}$. More explicitly this transformation is given by the collection of pointed maps $p_V : S^0_V \to \widehat{\mathcal{K}_V} = \mathit{Hom}(\mathcal{S}, \mathcal{K}_V)$ which send the point different from the basepoint in S^0_V to the $*$-homomorphism $p_V^0 \in \mathit{Hom}(\mathcal{S}, \mathcal{K}_V)$.

Definition 4.3. We define $\eta : S \to \mathcal{K}$ to be the monoidal natural transformation given by the composition

$$\eta : S \xrightarrow{\cong} S \wedge S^0 \xrightarrow{\beta\wedge p} \widehat{\mathbb{C}l} \wedge \widehat{\mathcal{K}} \xrightarrow{*} \widehat{\mathbb{C}l \otimes \mathcal{K}} = \mathbb{K},$$

i.e. for an inner product space $V \in \mathscr{I}$ the map η_V sends $v \in V \subset S^V$ to the $*$-homomorphism $fc(v) * p_V^0 : \mathcal{S} \to \mathbb{C}l_V \otimes \mathcal{K}_V$.

Recall from the second section that a monoidal natural transformation as $\eta : S \to \mathbb{K}$ gives rise to an equivariant spectrum in the sense of Lewis-May-Steinberger [12]. Hence η defines a multiplicative G-equivariant cohomology theory on $\mathscr{T}_*^G$, the category of well-pointed weak Hausdorff G-spaces and equivariant pointed maps between them.

Let $\mathscr{K}^G$ denote the category of (unpointed) compact G-spaces and equivariant maps. In the following $K_G^*(X)$ shall stand for unreduced equivariant K-cohomology for spaces $X \in \mathscr{K}^G$. That is, $K_G^0(X)$ is the Grothendieck group of isomorphism classes of G-vector bundles over X. For any pair of inner product spaces $V, W \in \mathscr{I}$ one has the groups

$$K_G^{V,W}(X) = \widetilde{K}_G^0(S^V \wedge S^W \wedge X_+),$$

where X_+ as usual denotes the one point compactification of X. The groups can be used (using standard techniques) to define an $RO(G)$-graded cohomology theory (c.f. Section 2); the relevant suspension isomorphisms

$$\sigma : K^0(X) \longrightarrow \widetilde{K}_G^{0,V}(S^V \wedge X_+) = \widetilde{K}_G^0(S^{V\oplus V} \wedge X_+)$$

are given by multiplication with the so-called Bott element $b_V^{top} \in \widetilde{K}_G^0(S^{V\oplus V})$. For the definition of the Bott class we refer to (6.8) below.

Theorem 4.4. *Let U be a complete universe. Then the $\mathscr{I}$-FSP $(\mathbb{K}, \eta)$, defined by 3.6 and 4.3, represents equivariant K-theory, i.e. there is a natural isomorphism of unreduced $RO(G)$-graded multiplicative (equivariant) cohomology theories on $\mathscr{K}^G$*

$$\Phi^*(X) : \mathbb{K}^*(X_+) \longrightarrow K_G^*(X).$$

Note that the natural isomorphism $\Phi^*(X)$ of the previous theorem induces a corresponding natural isomorphism $\widetilde{\Phi}^*(X) : \mathbb{K}^*(X) \to \widetilde{K}_G^*(X)$ of reduced $RO(G)$-graded multiplicative (equivariant) cohomology theories on the category of well-pointed compact spaces $\mathscr{K}_*^G$. The definition of the natural isomorphism Φ^* as well as a rigorous proof of the theorem will be given in the following section using elements of Kasparovs KK-theory. In the remaining part of this section we comment a little on some ideas that go into the proof so that the statement becomes plausible even without knowing too much about Kasparov K-theory and operator theory respectively.

Assume ℓ is an evenly graded G-Hilbert space and P is a G-fixed compact operator which also is a projection. Then $P(\ell)$ is a finite dimensional G-represention. Conversely, if $V \subset \ell$ is a finite-dimensional G-invariant linear subspace, then we have an isomorphism $\ell \cong V \oplus V^\perp$, and we can define a compact projection P_V by

$$P_V : \ell \to \ell, \quad P_V(h) = \begin{cases} h & \text{for } h \in V \\ 0 & \text{for } h \in V^\perp \end{cases}.$$

The space[5] of all G-invariant projections in $\mathcal{K}(\ell)$ is homeomorphic to the space of $*$-homomorphisms $Hom(\mathbb{C}, \mathcal{K}(\ell))$: the homeomorphism is given by mapping a G-invariant compact projection P to the unique $*$-homomorphism f_P which maps the unit $1 \in \mathbb{C}$ to P. Given a compact G-space X an equivariant continuous map $f : X \to Hom(\mathbb{C}, \mathcal{K}(\ell))$ defines a corresponding family of compact projections $\{P_{f(x)}\}_{x\in X}$, which defines a G-vector bundle E_f over X with fiber $E_{fx} = P_{f(x)}(\ell)$ over a point $x \in X$. By construction the vector bundle E_f comes equipped with a G-equivariant linear embedding $E_f \subset X \times \ell$ which is a map over X. Conversely, given a G-vector bundle E and a G-equivariant linear embedding $j : E \subset X \times \ell$ over X one defines a corresponding G-equivariant map $X \to Hom(\mathbb{C}, \mathcal{K}(\ell))$ by mapping a point $x \in X$ to the

[5]For the topology we take the norm topology.

$*$-homomorphism $f_{P(x)}$ where $P(x)$ is the compact projection which projects to the embedded fiber $j(E_x) \subset \ell$.

If ℓ is universal in the sense that any G-vector bundle over any compact space X can be G-equivariantly embedded into $X \times \ell$ (as a map over X) then $Hom(\mathbb{C}, \mathcal{K}(\ell))$ represents the functor $X \mapsto Vect_G(X)$, i.e. we have a natural isomorphism

$$Vect_G(X) \cong [X_+, Hom(\mathbb{C}, \mathcal{K}(\ell))]_G,$$

where $[\ ,\ ]_G$ stands for pointed G-homotopy classes of pointed continuous G-equivariant maps.

Now let H be the $\mathbb{Z}/2$-graded G-vectorspace with $H^{even} = H^{odd} = \ell$ for a universal ungraded G-Hilbert space ℓ.[6] Let $\iota : \mathcal{K}(\ell) \to \mathcal{K}(H)$ be the inclusion which is induced by the inclusion $\ell = H^{even} \subset H$, and let $ev_0 : \mathcal{S} \to \mathbb{C}$ be the $*$-homomorphism which evaluates a function $u \in \mathcal{S}$ at the point $0 \in \mathbb{R}$. Then one has a commutative diagram as follows

$$\begin{tikzcd}
Vect_G(X) \arrow[r, "\cong"] \arrow[dd] & {[X_+, Hom(\mathbb{C}, \mathcal{K}(\ell))]_G} \arrow[d, "\iota_*"] \\
 & {[X_+, Hom(\mathbb{C}, \mathcal{K}(H))]_G} \arrow[d, "ev_0^*"] \\
K_G(X) \arrow[r, "\cong"] & {[X_+, Hom(\mathcal{S}, \mathcal{K}(H))]_G,}
\end{tikzcd}$$

where the left vertical arrow is group completion. The fact that the space $Hom(\mathcal{S}, \mathcal{K}(H))$ represents K-theory is a well-known result in operator theory. Roughly speaking an element $f \in Hom(\mathcal{S}, \mathcal{K}(H))$ can be used to define[7] an odd self-adjoint G-equivariant Fredholm operator $\mathcal{F}_f$ on a sub-G-Hilbert space $\mathcal{E}_f \subset H$. The corresponding element in $K^0_G(\{pt\})$ which is represented by $[f] \in \pi_0(Hom(\mathcal{S}, \mathcal{K}(H)))$ then is given by the $\mathbb{Z}/2$-graded index of $\mathcal{F}_f$. More generally one can define an index map $[X_+, Hom(\mathcal{S}, \mathcal{K}(H))]_G \to K^0_G(X)$ which then is an inverse to the bottom isomorphism in the above diagram. This index map is similiar in spirit to the one given in [2, Appendix].

If V is a finite dimensional G-inner product space there is an analogous interpretation for the space $Hom(\mathcal{S}, \mathbb{C}l_V \otimes \mathcal{K}(H))$. Elements in $\mathbb{C}l_V \otimes \mathcal{K}(H)$ correspond to $\mathbb{C}l_V$-right linear compact operators on the G-Hilbert space $\mathbb{C}l_V \otimes H$ which we regard as a right $\mathbb{C}l_V$-module by means of right multiplication on the first factor. An element $f \in Hom(\mathcal{S}, \mathbb{C}l_V \otimes \mathcal{K}(H))$ now can be used to define an odd self-adjoint $\mathbb{C}l_V$-right linear G-equivariant Fredholm operator $\mathcal{F}_f$ on a sub-G-Hilbertspace $\mathcal{E}_f \subset \mathbb{C}l_V \otimes H$, so one can think of $Hom(\mathcal{S}, \mathbb{C}l_V \otimes \mathcal{K}(H))$ as a space of $\mathbb{C}l_V$-linear Fredholm operators. Both

[6] Then H is universal in the sense defined on page 102.

[7] To see how this works see 5.2: take $A = \mathbb{C}$ and use Theorem 5.2 to identify $KK^0_G(\{pt\})$ with $K^0_G(\{pt\})$; moreover use the fact that a Hilbert-$\mathbb{C}$-module is nothing but a $\mathbb{Z}/2$-graded G-Hilbert space.

spaces, $Hom(\mathcal{S}, \mathbb{C}l_V \otimes \mathcal{K}(H))$ as well as the honest space of odd self-adjoint $\mathbb{C}l_V$-right linear G-equivariant Fredholm operators on $\mathbb{C}l_V \otimes H$ are V-th deloopings of a space representing equivariant K-theory.[8] The statement for the space of odd self-adjoint $\mathbb{C}l_V$-right linear G-equivariant Fredholm operator is more classical and essentially due to Atiyah and Singer who proved the statement in the non-equivariant setting [4, Theorem A(k), Prop. 5.3], but see [9, Theorem 3.3] for a formulation which fits better with the present context.

Given a $\mathbb{Z}/2$-graded G-Hilbert space we call it universal in the following, if it is isomorphic to H above. It follows from Lemma 5.5 below that there are indexing spaces W for which $L^2(W)$ is universal, and that these indexing spaces form a cofinal diagram in the diagram of all indexing spaces.

We now want to connect the above to Theorem 4.4. Recall that by definition we have

$$\mathbb{K}^0(X) = \operatorname*{colim}_{V \in \mathscr{D}} [S^V \wedge X_+, Hom(\mathcal{S}, \mathbb{C}l_V \otimes \mathcal{K}_V)]_G.$$

Computing this colimit involves homomorphisms which fit as bottom rows in commutative diagrams of the following form (with the remaining maps defined as below)

$$\begin{tikzcd}
{[X_+, Hom(\mathcal{S}, \mathcal{K}_V)]_G} \arrow[r] \arrow[d] & {[X_+, Hom(\mathcal{S}, \mathcal{K}_{V+W})]_G} \arrow[d] \\
{[S^V \wedge X_+, Hom(\mathcal{S}, \mathbb{C}l_V \otimes \mathcal{K}_V)]_G} \arrow[r, bend right] & {[S^{V+W} \wedge X_+, Hom(\mathcal{S}, \mathbb{C}l_{V+W} \otimes \mathcal{K}_{V+W})]_G.}
\end{tikzcd}$$

We may assume that $L^2(V)$ is universal, and since $L^2(V) \subset L^2(V+W)$ then so is $L^2(V+W)$. Therefore the two groups on the top of the above diagram are both canonically isomorphic to $K_G^0(X)$. The map between them is the one which is induced by the natural inclusion $\mathcal{K}(j) : \mathcal{K}_V \subset \mathcal{K}_{V+W}$ and induces an isomorphism (in fact the identity)[9] on $K_G^0(X)$. The left vertical arrow in the diagram is induced by the map which maps a pointed function $f : X_+ \to Hom(\mathcal{S}, \mathcal{K}_V)$ to the function $\beta_V * f : S^V \wedge X_+ \to Hom(\mathcal{S}, \mathbb{C}l_V \otimes \mathcal{K}_V)$ given by $(\beta_V * f)(v, x) = \beta_V(v) * f(x)$; while the vertical right arrow is induced by the corresponding map which maps a pointed function $g : X_+ \to Hom(\mathcal{S}, \mathcal{K}_{V+W})$ to the function $\beta_{V+W} * g$. The statement of Theorem 4.4 then follows from the fact that the vertical arrows are isomorphisms, which follows from the

[8] The topology on the spaces of Fredholm operators considered here is the norm topology. We would like to emphasize that the natural G-action on the set of Fredholm operators in general is not compatible with the norm topology, i.e. the G-action does not give the spaces of Fredholm operators the structure of a G-space.

[9] To see that assume that an element in $[X_+, Hom(\mathcal{S}, \mathcal{K}_V)]_G$ is represented by $ev_0 \circ f$ for some map $f : X \to Hom(\mathbb{C}, \mathcal{K}(L^2(W)))$. As explained above, the corresponding element in $K_G^0(X)$ then is represented by the G-vector bundle E_f. The element in $K_G^0(X)$ which is represented by $j \circ ev_0 \circ f$ then is $E_{j \circ f}$ which is isomorphic to E_f since j is an embedding. Hence $[E_f] = [E_{j \circ f}] \in K_G^0(X)$.

fact that the vertical arrows are the operator-theoretic descriptions of Bott periodicity isomorphisms within the theory of Clifford algebras.

5. Elements of equivariant KK-theory

In this section we will define the natural homorphism $\Phi^*(X) : \mathbb{K}^*(X_+) \to K^*_G(X)$ of Theorem 4.4 and see that it is an isomorphism which then proves the theorem. For doing this we need to recall some facts about equivariant Kasparov K-theory first.

5.1. **The KK-homology groups.** Given a $\mathbb{Z}/2$-graded G-algebra A there is the notion of a *Hilbert-A-module* (see [5, 13.1 & 20.1.1]). In particular, a Hilbert-A-module is a $\mathbb{Z}/2$-graded Banach space with a right A-action, an A-sesquilinear A-valued scalar product and a continuous representation of G on it which is compatible with the $\mathbb{Z}/2$-grading, the A-action and the A-valued scalar product. The concept of a Hilbert-A-module generalizes the notion of a Hilbert space in the sense that a Hilbert-$\mathbb{C}$-module is nothing but a $\mathbb{Z}/2$-graded Hilbert space with a unitary representation of G on it, which preserves the grading. Given a Hilbert-A-module $\mathcal{E}$ one defines the corresponding $\mathbb{Z}/2$-graded C^*-algebras $\mathbb{B}(\mathcal{E})$, consisting of the so-called *(adjointable) bounded operators* on $\mathcal{E}$, and $\mathbb{K}(\mathcal{E})$, consisting of the *(adjointable) compact operators* on $\mathcal{E}$ ([5, 13.2.1 & 20.1.1]). By definition, $\mathbb{B}(\mathcal{E})$ is contained in the subspace of A-linear bounded linear operators $\mathcal{B}(\mathcal{E})_A$, and $\mathbb{K}(\mathcal{E})$ is a subspace of the A-linear compact operators $\mathcal{K}(\mathcal{E})_A$. If A is finite dimensional then we have equality, i.e. $\mathbb{B}(\mathcal{E}) = \mathcal{B}(\mathcal{E})_A$ and $\mathbb{K}(\mathcal{E}) = \mathcal{K}(\mathcal{E})_A$. Note that $\mathbb{K}(\mathcal{E})$ is a G-C^*-algebra while $\mathbb{B}(\mathcal{E})$ in general is not (cf. [5, 20.1]).

For the following let A be σ-unital. Moreover let us regard $[0,1]$ as a G-space with trivial G-action, and let $A[0,1]$ be the G-C^*-algebra consisting of the functions on $[0,1]$ with values in A. If $\mathcal{E}$ is a Hilbert-A-module let $\mathcal{E}[0,1]$ denote the Hilbert-$A[0,1]$-module consisting of the functions from the unit interval into $\mathcal{E}$. Moreover, for any $t \in [0,1]$ we have the evaluation $*$-homomorphism $ev_t : A[0,1] \to A$, $ev_t(f) = f(t)$, and given a Hilbert-$A[0,1]$-module $\mathcal{E}$ we use the evaluation $*$-homomorphisms to define the *evaluations* of $\mathcal{E}$ at t by $\mathcal{E}_t = \mathcal{E} \otimes_{A[0,1]} A$. For bounded operators $F \in \mathbb{B}(\mathcal{E})$ the *evaluation* at t is given by $F_t = F \otimes_{A[0,1]} Id \in \mathbb{B}(\mathcal{E}_t)$.

Definition 5.1. Let A be a $\mathbb{Z}/2$-graded σ-unital G-C^*-algebra. Then we define a *Kasparov-A-cycle* to be a pair $(\mathcal{E}, \mathcal{F})$ consisting of a countably generated Hilbert-A-module $\mathcal{E}$ and a G-invariant self-adjoint bounded operator $\mathcal{F} \in \mathbb{B}(\mathcal{E})$ satisfying $\mathcal{F}^2 - Id \in \mathbb{K}(\mathcal{E})$ and $||\mathcal{F}|| \leq 1$. Two Kasparov-A-cycles $(\mathcal{E}_0, \mathcal{F}_0), (\mathcal{E}_1, \mathcal{F}_1)$ are called *isomorphic (or unitary equivalent)* if there is an isomorphism (i.e. a G-equivariant grading preserving unitary) $U : \mathcal{E}_0 \to \mathcal{E}_1$ such that $\mathcal{F}_1 = U\mathcal{F}_0U^*$. The two cycles are called *homotopic* if there is a Kasparov-$A[0,1]$-cycle $(\mathcal{E}, \mathcal{F})$ whose evaluations at 0 and 1 are $(\mathcal{E}_0, \mathcal{F}_0)$ and $(\mathcal{E}_1, \mathcal{F}_1)$. The corresponding Kasparov cycle $(\mathcal{E}, \mathcal{F})$ then is called a *Kasparov* *homotopy* between $(\mathcal{E}_0, \mathcal{F}_0)$ and $(\mathcal{E}_1, \mathcal{F}_1)$. Homotopy induces an equivalence relation on the set of isomorphism classes of Kasparov-A-cycles, and the direct sum of Kasparov-A-cycles induces a group structure on this set of homotopy classes. Our notation for this group is $KK_0^G(A)$.

Equivariant KK-theory is related to equivariant K-theory by means of the equivariant Swan isomorphism.

Theorem 5.2 (Equivariant Swan Theorem, (cf. [18, 2.3]))**.** *Let X be a compact G-space and let $C(X)$ denote the trivially graded G-C^*-algebra of continuous functions on X with values in $\mathbb{C}$. Associating to a G-vector bundle with a G-invariant inner product over X the corresponding Hilbert-$C(X)$-module of continuous sections $\Gamma(E)$ induces an isomorphism*

$$\psi_X : K_G^0(X) \cong KK_0^G(C(X)). \tag{5.3}$$

5.2. **The homomorphism picture.** Given a countably generated Hilbert-A-module $\mathcal{E}$ we want to think about the G-fixpoint set $Hom(\mathcal{S}, \mathbb{K}(\mathcal{E}))^G$ as a space of Kasparov-A-cycles as follows. Let us be given an equivariant $*$-homomorphism $f : \mathcal{S} \to \mathbb{K}(\mathcal{E})$. Then we define the Hilbert-A-module $\mathcal{E}_f$ to be the completion

$$\mathcal{E}_f = \overline{span\ f(\mathcal{S})\mathcal{E}}.$$

As all operators $K \in f(\mathcal{S})$ are G-invariant $\mathcal{E}_f$ is G-invariant, and $\mathcal{E}_f$ canonically has the structure of a Hilbert-A-module. Moreover we have $\mathbb{K}(\mathcal{E}_f) = \overline{f(\mathcal{S})\mathbb{K}(\mathcal{E})f(\mathcal{S})}$. Thus we obtain a $*$-homomorphism $f_r : \mathcal{S} \to \mathbb{K}(\mathcal{E}_f)$ by restricting the range of the $*$-homomorphism $f : \mathcal{S} \to \mathbb{K}(\mathcal{E})$. The $*$-homomorphism f_r uniquely extends to a unital $*$-homomorphism of the multiplier algebras $f_r : C_b(\mathbb{R}) \to \mathbb{B}(\mathcal{E}_f)$ (cf. [11, Proposition 2.1]). The G-invariant operator $\mathcal{F}_f$ then is given by

$$\mathcal{F}_f = f_r(q) \in \mathbb{B}(\mathcal{E}_f),$$

with $q \in C_b(\mathbb{R})$ given by $q(s) = s/\sqrt{1+s^2}$. It follows from this construction that $(\mathcal{E}_f, \mathcal{F}_f)$ is a Kasparov-A-cycle. Note that the pair $(\mathcal{E}_f, \mathcal{F}_f)$ is a function of f. We can associate to $(\mathcal{E}_f, \mathcal{F}_f)$ its equivalence class in $KK_0^G(A)$, which leads to a map $kc : Hom(\mathcal{S}, \mathbb{K}(\mathcal{E}))^G \longrightarrow KK_0^G(A)$ which factors through taking path components on the left side, i.e. we have a corresponding map

$$kc_* : [Hom(\mathcal{S}, \mathbb{K}(\mathcal{E}))]_G \longrightarrow KK_0^G(A),$$

where $[\ ,\ ]_G$ denotes pointed G-homotopy classes of pointed G-maps. Using the description above one sees that an inclusion $i : \mathcal{E} \subset \mathcal{E}'$ yields a commutative diagram

$$\begin{tikzcd}
{[Hom(\mathcal{S}, \mathbb{K}(\mathcal{E}))]_G} \arrow[rr, "\mathbb{K}(i)_*"] \arrow[dr, "kc_*"'] & & {[Hom(\mathcal{S}, \mathbb{K}(\mathcal{E}'))]_G} \arrow[dl, "kc_*"] \\
& KK_0^G(A) &
\end{tikzcd}$$

In special cases kc_* is an isomorphism as we shall recall next. For the following let $\widehat{H}$ be a fixed choice of a separable $\mathbb{Z}/2$-graded Hilbert space satisfying $\dim \widehat{H}^{even} = \dim \widehat{H}^{odd} = \infty$. The standard $\mathbb{Z}/2$-graded G Hilbert space then is defined to be $L^2(G) \otimes \widehat{H}$. A separable $\mathbb{Z}/2$-graded G-Hilbert space H is called *universal* if it is isomorphic to $L^2(G) \otimes \widehat{H}$.

Theorem 5.4. *If H is a universal $\mathbb{Z}/2$-graded G-Hilbert space then for any σ-unital G-C^*-algebra A the map*

$$kc_* : [Hom(\mathcal{S}, \mathbb{K}(A \otimes H))]_G \longrightarrow KK_0^G(A)$$

is an isomorphism.

Proof. The theorem is folklore in operator theory; although we are not aware of any written account for the equivariant setting. The non-equivariant statement is proved in [21]. □

Lemma 5.5. *Let W be a G-representation equipped with a G-invariant inner product. Assume that there is a G-equivariant smooth embedding of G into W. Then the $\mathbb{Z}/2$-graded G-Hilbert space $L^2(W \oplus \mathbb{R})$ is universal.*

Proof. By the equivariant Kasparov stabilization theorem [10, §1, 12] it suffices to show that there is an embedding of the standard $\mathbb{Z}/2$-graded G-Hilbert space $L^2(G) \otimes \widehat{H}$ as a direct summand into $L^2(W \oplus \mathbb{R})$. We have $L^2(W \oplus \mathbb{R}) \cong L^2(W) \otimes L^2(\mathbb{R})$ and $\widehat{H} \cong L^2(\mathbb{R})$. Thus it suffices to show that $L^2(G)$ is contained in $L^2(W)$ as a direct summand (ignoring the $\mathbb{Z}/2$-gradings). A corresponding direct summand can be constructed from a smooth embedding of G into W and a choice of a G-invariant tubular neighborhood of the embedding. □

In the following we will say that a G-representation W is *full* if $L^2(W)$ is universal. For later reference note that if the universe U is complete then the full indexing spaces are cofinal in the diagram of all indexing spaces.

5.3. **The exterior Kasparov product and Bott periodicity.** If $\mathcal{E}_1$ and $\mathcal{E}_2$ are countably generated Hilbert-modules over G-C^*-algebras A_1 and A_2 respectively let $\mathcal{E}_1 \otimes \mathcal{E}_2$ denote the Hilbert module tensor product (cf. [5, 13.5]). In particular, $\mathcal{E}_1 \otimes \mathcal{E}_2$ is a Hilbert-$A_1 \otimes A_2$-module. The canonical map $\mathbb{K}(\mathcal{E}_1) \otimes \mathbb{K}(\mathcal{E}_2) \to \mathbb{K}(\mathcal{E}_1 \otimes \mathcal{E}_2)$ then is an isomorphism, which we will use in the sequel without notice. If we are given Kasparov-A_i-cycles $(\mathcal{E}_i, \mathcal{F}_i)$ for $i = 1, 2$ then one can define a corresponding Kasparov-$A_1 \otimes A_2$-cycle $(\mathcal{E}_1 \otimes \mathcal{E}_2, \mathcal{F}_1 \# \mathcal{F}_2)$. The definition of $\mathcal{F}_1 \# \mathcal{F}_2$ typically involves various choices (c.f. [5, 18.2]), however the resulting operators for any two sets of choices just differ by a Kasparov homotopy and one obtains a lax symmetric monoidal pairing, the exterior Kasparov product

$$KK_0^G(A_1) \times KK_0^G(A_2) \xrightarrow{\otimes} KK_0^G(A_1 \otimes A_2).$$

Theorem 5.6 (Exterior Kasparov product, e.g. see [17, p. 221]). *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be countably generated Hilbert modules over $\mathbb{Z}/2$-graded σ-unital G-C^*-algebras A_1 and A_2. Then the $*$-homomorphism $\mathbb{K}(\mathcal{E}_1)\otimes\mathbb{K}(\mathcal{E}_2)\to\mathbb{K}(\mathcal{E}_1\otimes\mathcal{E}_2)$ induces a commutative diagram*

$$\begin{tikzcd} Hom(\mathcal{S},\mathbb{K}(\mathcal{E}_1))^G\times Hom(\mathcal{S},\mathbb{K}(\mathcal{E}_2))^G \arrow[r, "*"] \arrow[d, "kc\times kc"] & Hom(\mathcal{S},\mathbb{K}(\mathcal{E}_1\otimes\mathcal{E}_2))^G \arrow[d, "kc"] \\ KK_0^G(A_1)\times KK_0^G(A_2) \arrow[r, "\otimes"] & KK_0^G(A_1\otimes A_2). \end{tikzcd}$$

For more details concerning the Kasparov product we refer the reader to [5, VIII.18].

Theorem 5.7 (Bott periodicity in KK-theory, c.f. [10, §5]). *Let V be a G-inner product space and let A be a σ-unital $\mathbb{Z}/2$-graded G-C^*-algebra. Then there is a natural (in A) isomorphism*

$$KK_0^G(A)\cong KK_0^G(C_0(S^V)\otimes\mathbb{C}l_V\otimes A),$$

where $C_0(S^V)$ denotes the trivially graded G-C^*-algebra consisting of continuous functions on S^V vanishing at infinity. The isomorphism is given by exterior multiplication with a specific element, the so-called Bott element $b_V\in KK_0^G(C_0(S^V)\otimes\mathbb{C}l_V)$.

5.4. **The exponential law.** For spaces $X,Y\in\mathscr{T}_*$ let $C_0(X,Y)\in\mathscr{T}_*$ denote the pointed space of pointed maps from X to Y equipped with the compact-open topology. Given a G-C^*-algebra we also regard it as an object in $\mathscr{T}_*$, the basepoint being $0\in A$. In case $X\in\mathscr{K}_*$, the full subcategory of $\mathscr{T}_*$ whose objects are the pointed compact G-spaces, the corresponding space $C_0(X,A)$ naturally has the structure of a G-C^*-algebra. The C^*-algebra $C_0(X,A)$ can also be identified with the tensor product of $C_0(X)=C_0(X,\mathbb{C})$ and A since the canonical map $C_0(X)\otimes_{alg}A\to C_0(X,A)$, $f\otimes a\mapsto f\cdot a$ becomes an isomorphism after completing the algebraic tensor product $C_0(X)\otimes_{alg}A$.

Lemma 5.8. *Given a G-space $X\in\mathscr{K}$ and a G-C^*-algebra A then there is a pointed G-homeomorphism*

$$\nu: C_0(X,Hom(\mathcal{S},A))\ \cong\ Hom(\mathcal{S},C_0(X)\otimes A). \tag{5.9}$$

The G-homeomorphism is natural in X and A.

Proof. Using the canonical isomorphism $C_0(X)\otimes A\cong C_0(X,A)$ this is a consequence of the fact that the exponential map $C_0(X\wedge Y,Z)\to C_0(X,C_0(Y,Z))$ is a pointed G-homeomorphism for spaces $X,Y,Z\in\mathscr{T}_*$. □

Any G-C^*-algebra A can be regarded as a Hilbert module over itself, and there is the canonical isomorphism $\iota: A\to\mathbb{K}(A)$, which sends an element $a\in A$ to the bounded operator that multiplies with a from the left. In case

B is a G-C^*-algebra and $\mathcal{E}$ a Hilbert-B-module we thus have a canonical isomorphism $\iota : A \otimes \mathbb{K}(\mathcal{E}) \cong \mathbb{K}(A) \otimes \mathbb{K}(\mathcal{E}) \cong \mathbb{K}(A \otimes \mathcal{E})$. We use this isomorphism for $A = C_0(X)$ to define the composition

$$ad = \iota_* \circ \nu : C_0(X, Hom(\mathcal{S}, \mathbb{K}(\mathcal{E}))) \to Hom(\mathcal{S}, C_0(X) \otimes \mathbb{K}(\mathcal{E}))$$

$$\to Hom(\mathcal{S}, \mathbb{K}(C_0(X) \otimes \mathcal{E})).$$

Let A be a $\mathbb{Z}/2$-graded G-C^*-algebra, let $\mathcal{E}$ be a countably generated Hilbert-A-module, and let $X \in \mathscr{K}_*$ such that $C_0(X)$ is σ-unital. Given an equivariant map $f : X \to Hom(\mathcal{S}, \mathbb{K}(\mathcal{E}))$ we associate to it the Kasparov-$C_0(X) \otimes A$-cycle $(\mathcal{E}_f, \mathcal{F}_f) = (\mathcal{E}_{ad(f)}, \mathcal{F}_{ad(f)})$. This allows us to extend the definition of kc above to a natural map

$$kc : C_0(X, Hom(\mathcal{S}, \mathbb{K}(\mathcal{E})))^G \longrightarrow KK_0^G(C_0(X) \otimes A)$$

by defining $kc(f) = [(\mathcal{E}_f, \mathcal{F}_f)]$. If we are given a path $w : [0,1] \to C_0(X, Hom(\mathcal{S}, \mathbb{K}(\mathcal{E})))^G$ this corresponds to an element $h \in C_0(X \wedge [0,1]_+, Hom(\mathcal{S}, \mathbb{K}(\mathcal{E})))^G$, and $(\mathcal{E}_h, \mathcal{F}_h)$ is a Kasparov homotopy between $\mathcal{E}_{w(0)}$ and $\mathcal{E}_{w(1)}$. Hence kc factors through taking homotopy classes, i.e. we obtain an induced map

(5.10)

$$kc_* : [X, Hom(\mathcal{S}, \mathbb{K}(\mathcal{E}))]_G = [X, Hom(\mathcal{S}, \mathbb{K}(\mathcal{E}))]_G \quad \to \quad KK_0^G(C_0(X) \otimes A).$$

5.5. **Proof of Theorem 4.4.** We now come back to our special situation. For any inner product space $V \in \mathscr{I}$ we can identify $\mathbb{C}l_V$ with the adjointable compact operators $\mathbb{K}(\mathbb{C}l_V)$, if we regard $\mathbb{C}l_V$ as a Hilbert-$\mathbb{C}l_V$-module using the action of $\mathbb{C}l_V$ given by right multiplication. The identification $\mathbb{C}l_V \cong \mathbb{K}(\mathbb{C}l_V)$ is given by identifying an element $v \in \mathbb{C}l_V$ with the operator which multiplies with v from the left. Moreover, if we regard $\mathbb{C}l_V \otimes L^2(V)$ as a Hilbert-$\mathbb{C}l_V$-module with $\mathbb{C}l_V$ acting by right muliplication on the first factor and trivially on $L^2(V)$, then we have a canonical isomorphism $\mathbb{K}(\mathbb{C}l_V \otimes L^2(V)) \cong \mathbb{C}l_V \otimes \mathcal{K}_V$.

Recall from Section 3 that we constructed natural transformations

$$p : S^0 \to \widehat{\mathcal{K}} \quad \text{and} \quad \beta : S \to \widehat{\mathbb{C}l} \quad \text{and} \quad \eta = \beta * p : S \to \mathbb{K} = \widehat{\mathbb{C}l \otimes \mathcal{K}}.$$

By construction, if we apply kc to the map $p_V : S^0_V \to Hom(\mathcal{S}, \mathcal{K}_V)$ we obtain $kc(p_V) = [\mathbb{C}i_V, 0] = [\mathbb{C}, 0]$, which represents the unit $1 \in KK_0^G(\mathbb{C})$. If we apply kc to the map $\beta_V : S^V \to Hom(\mathcal{S}, \mathbb{C}l_V)$ introduced in Definition 4.2 then we obtain the Bott elements

$$b_V = kc(\beta_V) \in KK_0^G(C_0(S^V) \otimes \mathbb{C}l_V).$$

It follows from Theorem 5.6 that

$$kc(\eta_V) = kc(\beta_V) \otimes kc(p_V) = b_V \otimes 1 = b_V.$$

Let $X \in \mathscr{K}$, let $V \in \mathscr{D}$ be an indexing space. Using the canonical isomorphisms $C_0(S^V) \otimes C_0(X_+) \cong C_0(S^V \wedge X_+)$ we obtain commutative diagrams

$$\begin{tikzcd}
{[X_+, Hom(\mathcal{S}, \mathcal{K}_V)]_G} \arrow[r, "kc_*"] \arrow[dd, "(\beta_V *)_*"'] & KK_0^G(C_0(X_+)) \arrow[d, "b_V \otimes KK_0^G(\tau)"'] \\
& KK_0^G(C_0(S^V) \otimes \mathbb{C}l_V \otimes C_0(X_+)) \arrow[d, "KK_0^G(Id\otimes\tau)"', "\cong"] \\
{[S^V \wedge X_+, Hom(\mathcal{S}, \mathbb{C}l_V \otimes \mathcal{K}_V)]_G} \arrow[r, "kc_*"] & KK_0^G(C_0(S^V) \otimes C_0(X_+) \otimes \mathbb{C}l_V).
\end{tikzcd}$$

Taking colimits over the indexing spaces V we obtain commutative diagrams

$$\begin{tikzcd}
& KK_0^G(C_0(X_+)) \arrow[dd] \\
\operatorname*{colim}_{V\in\mathscr{D}} [X_+, Hom(\mathcal{S}, \mathcal{K}_V)]_G \arrow[ur] \arrow[d] & \\
\operatorname*{colim}_{V\in\mathscr{D}} [S^V \wedge X_+, Hom(\mathcal{S}, \mathbb{C}l_V \otimes \mathcal{K}_V)]_G \arrow[uur, dotted, "\phi_X"] \arrow[r] & \operatorname*{colim}_{V\in\mathscr{D}} KK_0^G(C_0(S^V) \otimes C_0(X_+) \otimes \mathbb{C}l_V).
\end{tikzcd}$$

It follows from Bott periodicity that each of the right vertical arrows in the two diagrams above is an isomorphism. In particular this allows us to define the homomorphism ϕ_X as indicated in the diagram. Since the full[10] indexing spaces are cofinal in the diagram of all indexing spaces (cf. Lemma 5.5) it follows from Lemma 5.8 and Theorem 5.4 that ϕ_X is an isomorphism. Note that the lower left corner, by definition, is $\mathbb{K}^0(X_+)$.

We now define $\Phi^0(X) : \mathbb{K}^0(X) \to K_G^0(X)$ as the composition

(5.11)

$$\Phi^0(X) : \mathbb{K}^0(X_+) \xrightarrow{\phi_X} KK_0^G(C_0(X_+)) \;\cong\; KK_0^G(C(X)) \xrightarrow{\psi_X^{-1}} K_G^0(X).$$

The definition of $\Phi^0(X)$ has a unique extension to a family of isomorphisms

$$\Phi^{V,W}(X) : \mathbb{K}^{V,W}(X_+) \to K_G^{V,W}(X) \tag{5.12}$$

[10]For the definition of full see page 102

for inner product spaces $V, W \in \mathscr{I}$ such that $\Phi^{V,0}(X) = \widetilde{\Phi}^0(S^V \wedge X_+)$ for all V and such that the diagrams

$$\begin{tikzcd}
\mathbb{K}^0(X_+) \arrow[r, "\Phi^0(X)"] \arrow[d, "\sigma"] & K_G^0(X) \arrow[d, "\otimes b_W^{top}"] \\
\mathbb{K}^{0,W}(S^W \wedge X_+) \arrow[r, "\widetilde{\Phi}^{0,W}(S^W \wedge X_+)"] & \widetilde{K}_G^{0,W}(S^W \wedge X_+)
\end{tikzcd}$$

commute for all W. It follows that the natural isomorphisms $\Phi^{*,*}$ can be used to define a corresponding natural isomorphism of $RO(G)$-graded cohomology theories $\Phi^*(X) : \mathbb{K}^*(X_+) \to K_G^*(X)$. It also is not hard to see that Φ^* is compatible with the multiplicative structures of the two cohomology theories. This completes the proof of Theorem 4.4.

Remark 5.13. One can extend the definition of the KK-homology groups to a setting where the corresponding groups are labeled by pairs of inner product spaces $V, W \in \mathscr{I}$ by defining

$$KK^G_{V,W}(A) = KK^G_0(C_0(S^V) \otimes \mathbb{C}l_W \otimes A).$$

The methods above also provide natural isomorphisms $\phi_X^{V,W} : \mathbb{K}^{V,W}(X_+) \cong KK^G_{V,W}(C_0(X_+))$. The isomorphisms $\Phi^{V,W}$ of (5.12) also can be described using the isomorphisms $\phi_X^{V,W}$.

6. The Atiyah-Bott-Shapiro orientation

In this section we construct an $\mathscr{I}$-FSP $\mathbb{M}\mathrm{Spin}^c$ whose underlying prespectrum is a model for $MSpin^c$. We also construct a map $\mathbb{M}\mathrm{Spin}^c \to \mathbb{K}$ and show that it provides a model for an equivariant version of the Atiyah-Bott-Shapiro orientation.

Given an inner product space $V \in \mathscr{I}$ we consider the group $Pin^c_V \subset \mathbb{C}l_V$ which is generated by the elements in the unit sphere $S(V) \subset V \subset \mathbb{C}l_V$ and the elements of the unit circle $S^1 \subset \mathbb{C} \cdot 1 \subset \mathbb{C}l_V$. Note that Pin^c_V is contained in the units of $\mathbb{C}l_V$ since for any $v \in S(V)$ we have $v^2 = 1 \in \mathbb{C}l_V$, and obviously any $\lambda \in S^1$ is invertible. The $\mathbb{Z}/2$-grading of $\mathbb{C}l_V$ induces a homomorphism $\xi : Pin^c_V \to \mathbb{Z}/2$. The homomorphism ξ is called the *grading homomorphism*. We now consider the twisted conjugation action of Pin^c_V on $\mathbb{C}l_V$; for elements $v \in Pin^c_V$ and $w \in \mathbb{C}l_V$ it is given by $v : w \mapsto \xi(v)vwv^{-1}$. One easily checks that the twisted conjugation action leaves $V \subset \mathbb{C}l_V$ invariant. For each $e \in Pin^c_V$ the restricted action $c_e : V \to V$ is an isometry. It is well-known that associating c_e to e defines a surjective group homomorphism $\varrho_V : Pin^c_V \to O_V$ with kernel S^1. On the other hand any isometry $S : V \to V$ uniquely extends to an algebra automorphism $\mathbb{C}l_S : \mathbb{C}l_V \to \mathbb{C}l_V$. Hence the twisted conjugation action of e on $\mathbb{C}l_V$ coincides with $\mathbb{C}l_{\varrho_V(e)}$.

Given an inner product space $V \in \mathscr{I}$ let us write $B(\mathbb{C}l_V \otimes L^2(V))$ for the space consisting of the bounded operators on $\mathbb{C}l_V \otimes L^2(V)$, equipped with the strong $*$-topology, and let $\mathbb{U}_V \subset B(\mathbb{C}l_V \otimes L^2(V))$ be the group generated by the odd unitary operators and the scalars. By construction $\mathbb{U}_V$ comes equipped with a corresponding grading homomorphism $\xi : \mathbb{U}_V \to \mathbb{Z}/2$. The group $\mathbb{U}_V$ acts on $B(\mathbb{C}l_V \otimes L^2(V))$ by twisted conjugation, and this action factors through an effective action of the projective group $\mathbb{PU}_V = \mathbb{U}_V/S^1$. More precisely, given a unitary operator $U \in \mathbb{U}_V$ and a bounded operator $F \in B(\mathbb{C}l_V \otimes L^2(V))$ we define $\overline{U} \in \mathbb{PU}_V$ to be the equivalence class of U in $\mathbb{PU}_V$, and $\overline{U}F = \xi(U)UFU^*$. Note that the restriction of the action to the subset $\mathbb{K}(\mathbb{C}l_V \otimes L^2(V))$ consisting of the compact operators in $B(\mathbb{C}l_V \otimes L^2(V))$ is compatible with the norm topology in the sense that $\mathbb{PU}_V$ is in fact acting on $\mathcal{K}_V = \mathbb{K}(\mathbb{C}l_V \otimes L^2(V))$.

Given an element $e \in Pin^c_V$ we can associate to it the unitary operator $U_e \in B(\mathbb{C}l_V \otimes L^2(V))$ given by

$$U_e(v \otimes f) = ev \otimes (f \circ \varrho_V(e)^{-1}), \quad v \in \mathbb{C}l_V, f \in L^2(V).$$

This induces a homomorphism $\iota_V : Pin^c_V \to \mathbb{U}_V$. Since the kernel of ϱ_V maps onto $S^1 \subset \mathbb{U}_V$ the homomorphism ι_V induces a homomorphism $\jmath_V : O_V \to \mathbb{PU}_V$. We use $\jmath_V$ to regard $\mathbb{K}(\mathbb{C}l_V \otimes L^2(V))$ as an O_V-space. Using the canoncial isomorphism $\mathbb{K}(\mathbb{C}l_V \otimes L^2(V)) \cong \mathbb{C}l_V \otimes \mathcal{K}_V$ the action of an isometry $S \in O_V$ is given by $S(v \otimes k) = \mathbb{C}l_S v \otimes \mathcal{K}_S k$ for $v \otimes k \in \mathbb{C}l_V \otimes \mathcal{K}_V$.

Lemma 6.1. *The map $\eta_V : S^V \to Hom(\mathcal{S}, \mathbb{K}(\mathbb{C}l_V \otimes L^2(V)))$, mentioned in Section 4, is O_V-equivariant.*

Proof. Note that it follows directly from the definition of β_V (see Definition 4.1) that for any orthogonal transformation $S \in O_V$ we have $S(\beta_V(v)) = \beta_V(Sv)$ for all $v \in V$. The same applies to the map $p_V : S^0_V \to Hom(\mathcal{S}, \mathbb{K}(L^2(V)))$. Hence $\eta_V = \beta * p_V$ is O_V-equivariant. □

It follows that the maps $\widetilde{\alpha_V} : (\mathbb{U}_V)_+ \wedge S^V \longrightarrow Hom(\mathcal{S}, \mathbb{K}(\mathbb{C}l_V \otimes L^2(V)))$ given by $\widetilde{\alpha_V}(U, v) = \xi(U)U\eta_V(v)U^*$ induce corresponding maps

$$\alpha_V : \mathbb{PU}_{V+} \wedge_{O_V} S^V \longrightarrow Hom(\mathcal{S}, \mathbb{K}(\mathbb{C}l_V \otimes L^2(V))). \tag{6.2}$$

Definition 6.3. Let $\mathbb{M}\mathrm{Spin}^c : \mathscr{I} \to \mathscr{T}_*$ be the G-functor given by

$$\mathbb{M}\mathrm{Spin}^c_V = \mathbb{PU}_{V+} \wedge_{O_V} S^V. \tag{6.4}$$

Let $\mu : \mathbb{M}\mathrm{Spin}^c \wedge \mathbb{M}\mathrm{Spin}^c \to \mathbb{M}\mathrm{Spin}^c$ be the natural transformation of G-functors $\mathscr{I} \times \mathscr{I} \to \mathscr{T}_*$ given by the maps $\mu_{V,W} : \mathbb{M}\mathrm{Spin}^c_V \wedge \mathbb{M}\mathrm{Spin}^c_W \to \mathbb{M}\mathrm{Spin}^c_{V\oplus W}$ which are induced by the obvious natural maps $\mathbb{U}_{V+} \wedge \mathbb{U}_{W+} \to \mathbb{U}_{V\oplus W+}$ and $S^V \wedge S^W \to S^{V\oplus W}$ respectively. Finally let $\lambda : S^0 \to \mathbb{M}\mathrm{Spin}^c_0$ be the unique pointed homeomorphism from S^0 to $\mathbb{M}\mathrm{Spin}^c_0 = \{\overline{Id_{L^2(0)}}\}_+ \wedge S^0 \cong S^0$. Then the triple $(\mathbb{M}\mathrm{Spin}^c, \mu, \lambda)$ defines a lax symmetric monoidal G-functor, and the natural transformation of lax monoidal G-functors $\eta : S \to$

$\mathbb{M}\mathrm{Spin}^c$ given by the maps

$$(6.5)\quad \eta_V : S^V \cong O_{V+} \wedge_{O_V} S^V \xrightarrow{\jmath_{V+} \wedge_{O(V)} Id} \mathbb{P}\mathbb{U}_{V+} \wedge_{O_V} S^V = \mathbb{M}\mathrm{Spin}^c_V$$

gives $\mathbb{M}\mathrm{Spin}^c$ the structure of an $\mathscr{I}$-FSP. Moreover the maps α_V introduced in (6.2) define a natural transformation $\alpha : \mathbb{M}\mathrm{Spin}^c \to \mathbb{K}$ of lax symmetric monoidal G-functors.

Theorem 6.6. *The $\mathscr{I}$-FSP $(\mathbb{M}\mathrm{Spin}^c, \eta)$ given by (6.4) and (6.5) provides a model for the G-equivariant $Spin^c$-bordism spectrum.*

Proof. Let $V \in \mathscr{I}$ be an inner product space. We have the following diagram of groups with exact rows

$$\begin{tikzcd}
1 \arrow[r] & S^1 \arrow[r] \arrow[d, "\cong"] & Pin^c_V \arrow[r, "\varrho_V"] \arrow[d, "\iota_V"] & O_V \arrow[r] \arrow[d, "\jmath_V"] & 1 \\
1 \arrow[r] & S^1 \arrow[r] & \mathbb{U}_V \arrow[r] & \mathbb{P}\mathbb{U}_V \arrow[r] & 1.
\end{tikzcd}$$

It follows that we have a homeomorphism $\mathbb{P}\mathbb{U}_V/O_V \cong \mathbb{U}_V/Pin^c_V$.

Now let us write $\mathbb{U}_V^{even}$ for the even part of $\mathbb{U}_V$ and recall that $Spin^c_V$, by definition, is the even part of Pin^c_V. Assume that $V \neq 0$, so that $\xi : Pin^c_V \to \mathbb{Z}/2$ is onto. We then have a commutative diagram of groups with exact rows

$$\begin{tikzcd}
1 \arrow[r] & Spin^c_V \arrow[r] \arrow[d] & Pin^c_V \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & \mathbb{U}_V^{even} \arrow[r] & \mathbb{U}_V \arrow[r] & \mathbb{Z}/2 \arrow[r] & 1.
\end{tikzcd}$$

It follows that we have a G-homeomorphism

$$\mathbb{P}\mathbb{U}_V/O_V \cong \mathbb{U}_V/Pin^c_V \cong \mathbb{U}_V^{even}/Spin^c_V.$$

If V in addition has the property that $L^2(V)^{even}$ is stable in the sense that $L^2(V)^{even} \cong L^2(V)^{even} \otimes \ell$ with ℓ an infinite dimensional separable Hilbert space with trivial G-action, then $\mathbb{U}_V^{even}$ is weakly G-contractible. Hence the space

$$\mathbb{P}\mathbb{U}_V/O_V \cong \mathbb{U}_V^{even}/Spin^c_V$$

is a model for $BSpin^c_V$. Note that the two assumptions on V are satisfied if V contains a copy of the trivial representation, and indexing spaces with this property are cofinal in the diagram of all indexing spaces.

Given two such inner product spaces V_1, V_2 the homomorphism

$$\mu_{V_1,V_2} : Spin^c_{V_1} \times Spin^c_{V_2} \to Spin^c_{V_1 \oplus V_2}$$

induces a commutative diagram

$$\begin{tikzcd}
\mathbb{PU}_{V_1}/O_{V_1} \times \mathbb{PU}_{V_2}/O_{V_2} \arrow[r] \arrow[d, "\cong"] & \mathbb{PU}_{W_1\oplus V_2}/O_{V_1\oplus V_2} \arrow[d, "\cong"] \\
\mathbb{U}_{V_1}^{even}/Spin^c_{V_1} \times \mathbb{U}_{V_2}^{even}/Spin^c_{V_2} \arrow[r] & \mathbb{U}_{V_1\oplus V_2}^{even}/Spin^c_{V_1\oplus V_2}.
\end{tikzcd}$$

It follows that the natural maps

$$\nu_{V_1,V_2} : \mathbb{PU}_{V_1}/O_{V_1} \times \mathbb{PU}_{V_2}/O_{V_2} \to \mathbb{PU}_{W_1\oplus V_2}/O_{V_1\oplus V_2}$$

provide a model for $B\mu_{V_1,V_2}$.

Finally for any V satisfying the above two properties the natural map of G-vector bundles

$$\mathbb{U}_V^{even} \times_{Spin^c_V} V \longrightarrow \mathbb{PU}_V \times_{O_V} V = E_V,$$

where $Spin^c_V$ acts on V in the standard way (i.e. via ϱ_V), is an isomorphism. Hence $\mathbb{M}\mathrm{Spin}^c_V = \mathbb{PU}_{V+} \wedge_{O_V} S^V$, which is homeomorphic to the Thom space of E_V, is a model for $MSpin^c_V$.

The G-vector bundles E_V also fit together in the sense that $E_{V_1} \oplus E_{V_2}$ is a pullback of $E_{V_1\oplus V_2}$ along ν_{V_1,V_2}. Hence the maps $\mu_{V_1,V_2} : \mathbb{M}\mathrm{Spin}^c_{V_1} \wedge \mathbb{M}\mathrm{Spin}^c_{V_2} \to \mathbb{M}\mathrm{Spin}^c_{V_1\oplus V_2}$ provide a model for the map

$$MSpin^c(V_1) \wedge MSpin^c(V_2) \to MSpin^c(V_1 \oplus V_2).$$

Since the indexing spaces V with the two required properties are cofinal in the diagram of all indexing spaces it follows that $\mathbb{M}\mathrm{Spin}^c$ provides a model for $MSpin^c$. $\square$

Definition 6.7. Motivated by the results just obtained we define for an inner product space $V \in \mathscr{I}$

$$\begin{aligned}
\mathbb{E}\mathrm{Spin}^c_V &= \mathbb{U}_V^{even} \\
\mathbb{B}\mathrm{Spin}^c_V &= \mathbb{U}_V^{even}/Spin^c_V \\
\gamma_{Spin^c_V} &= \mathbb{E}\mathrm{Spin}^c_V \times_{Spin^c_V} V.
\end{aligned}$$

Note that $\mathbb{E}\mathrm{Spin}^c_V$ and $\mathbb{B}\mathrm{Spin}^c_V$ are in general not models for the spaces $ESpin^c_V$ and $BSpin^c_V$ respectively, unless Pin^c_V has a G-invariant odd element and $L^2(V)^{even}$ is stable in the sense considered in the proof of the previous theorem.

The remaining part of this section is devoted to prove that $\alpha : \mathbb{M}\mathrm{Spin}^c \to \mathbb{K}$ represents an equivariant version of the classical Atiyah-Bott-Shapiro orientation. In order to give the more precise statement (Theorem 6.9 below) we need to recall the definition of the latter.

Let $W = V \oplus V$ for some G-inner product space V. There is a standard right $\mathbb{C}l_W$-action on the exterior algebra $\Lambda^*(V \otimes \mathbb{C})$. The action is given as follows. Let us consider the isomorphism $W = V \oplus V \to V \otimes \mathbb{C}$ which sends $w = (v_0', v_1') \in W$ to $v' = v_0' + iv_1'$. For $z \in \Lambda^*(V \otimes \mathbb{C})$ let

$$z \cdot w = z \wedge v' + z \lrcorner\, v',$$

where $z \lrcorner\, v'$ for an element $z = v_1 \wedge \cdots \wedge v_p \in \Lambda^p(V \otimes \mathbb{C})$ with $v_i \in V \otimes \mathbb{C}$ is given by the formula

$$z \lrcorner\, v' = \sum_{i=1}^{p} (-1)^{p-i} \langle v', v_i \rangle v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_p.$$

One easily checks that the action of w is $\mathbb{C}$-linear (i.e. $(\lambda z) \cdot w = \lambda(z \cdot w)$, $\lambda \in \mathbb{C}$) and that for any $z \in \Lambda^*(V \otimes \mathbb{C})$ one has $z \cdot w \cdot w = z \cdot \langle w, w \rangle$. On the other hand $\Lambda^*(V \otimes \mathbb{C})$ is a complex vector space. Hence the action of the elements $w \in W$ induces a right action of $\mathbb{C}l_W$ on $\Lambda^*(V \otimes \mathbb{C})$. In the following let us write $\mathbb{S}_W$ for the right $\mathbb{C}l_W$-module $\Lambda^*(V \otimes \mathbb{C})$, and let us write $\mathbb{S}_W^t$ for the left $\mathbb{C}l_W$-module, which coincides with $\mathbb{S}_W$, but where the elements of $\mathbb{C}l_W$ act by their adjoints. We then have a $\mathbb{C}l_W$-bimodule isomorphism

$$\mathbb{C}l_W \cong \mathbb{S}_W^t \otimes \mathbb{S}_W.$$

Since $\dim_{\mathbb{C}} \mathbb{S}_W = 2^{\dim_{\mathbb{R}} V}$ it follows that $\mathbb{S}_W$ is an irreducible $\mathbb{Z}/2$-graded right $\mathbb{C}l_W$-module, and that $\mathbb{S}_W^t$ is an irreducible $\mathbb{Z}/2$-graded left $\mathbb{C}l_W$-module.

Let E' be a G-vector bundle over a compact space X with a $Spin_W^c$-structure. Using the $Spin_W^c$-structure bundle $Spin_{E'}^c$ of E' we have a canonical isomorphism

$$Spin_{E'}^c \times_{Spin_W^c} W \cong E'.$$

Let

$$\begin{aligned} \mathbb{C}l_{E'} &= Spin_{E'}^c \times_{Spin_W^c} \mathbb{C}l_W, \\ \mathbb{S}_{E'}^t &= Spin_{E'}^c \times_{Spin_W^c} \mathbb{S}_W, \\ \mathbb{S}_{E'}^{\mathbb{C}l} &= Spin_{E'}^c \times_{Spin_W^c} \mathbb{C}l_W. \end{aligned}$$

Here the $Spin_W^c$ action on the fiber in the first case is given by conjugation, in the second and the third case it is given by left multiplication. The bundle $\mathbb{C}l_{E'}$ is denoted the *Clifford bundle* associated to E'. The other two bundles are called the *spinor bundle* of E' and the *Clifford spinor bundle* of E' respectively. The latter two are module bundles over $\mathbb{C}l_{E'}$ by means of the action given by $[s, w] \cdot [s, z] = [s, wz]$ for $s \in Spin_{E'}^c$, $w \in \mathbb{C}l_W$, and $z \in \mathbb{S}_W^t$ or $z \in \mathbb{C}l_W$ respectively. Note that E' itself is canonically contained in $\mathbb{C}l_{E'}$. Hence an element e' in the fiber E_x' over some $x \in X$ gives rise to endomorphisms $e' \cdot : (\mathbb{S}_{E'}^t)_x \to (\mathbb{S}_{E'}^t)_x$ and $e' \cdot : (\mathbb{S}_{E'}^{\mathbb{C}l})_x \to (\mathbb{S}_{E'}^{\mathbb{C}l})_x$. In case e is of norm 1, the endomorphism is a unitary isomorphism.

Now let E be a G-vector bundle equipped with a $Spin_V^c$-structure for some inner product space $V \in \mathscr{I}$. Let $E' = V \oplus E$. Let $D(E')$ and $S(E')$ denote the disk and the sphere bundle of E' respectively. The *Atiyah-Bott-Shapiro* *orientation* of E is a cohomology class $u_E \in \widetilde{K}_G^{0,V}(Th(E))$. We define it by saying what its image is under the sequence of isomorphisms

$$\widetilde{K}_G^V(Th(E)) = \widetilde{K}_G^0(S^V \wedge Th(E)) \cong \widetilde{K}_G^0(D(E')/S(E')) \cong K_G^0(D(E'), S(E')).$$

Recall that elements in relative K-theory for a pair of compact spaces (X, A) can be described by pairs $(\widetilde{E}, \widetilde{\varphi})$ consisting of a $\mathbb{Z}/2$-graded G-vector bundle $\widetilde{E} = \widetilde{E}^{even} \oplus \widetilde{E}^{odd}$ over X and an isomorphism $\widetilde{\varphi} : \widetilde{E}|_A^{even} \to \widetilde{E}|_A^{odd}$. The image of the Atiyah-Bott-Shapiro orientation u_E by definition is represented by the pair $[\mathscr{S}_{E'}^t, \varphi']$, where $\mathscr{S}_{E'}^t = \pi^* \mathbb{S}_{E'}^t$ is the pullback of $\mathbb{S}_{E'}^t$ along $\pi : D(E') \to X$, and where $\varphi' : \mathscr{S}_{E'}^t|_{S(E')} \to \mathscr{S}_{E'}^t|_{S(E')}$ is the isomorphism which in a fiber over a point $e' \in S(E')$ is given by multiplication with e'. For the case where E just is an inner product space $E = V \in \mathscr{I}$ the Atiyah-Bott-Shapiro orientation is the Bott periodicity element introduced in Section 4

$$b_V^{top} = u_V \in \widetilde{K}_G^{0,V}(S^V) = \widetilde{K}_G^0(S^V \wedge S^V). \tag{6.8}$$

Theorem 6.9. *The natural transformation of $\mathscr{I}$-FSPs $\alpha : \mathbb{M}\mathrm{Spin}^c \to \mathbb{K}$, defined in Definition 6.3, represents an equivariant version of the classical Atiyah-Bott-Shapiro orientation, i.e. in particular: if G is the trivial group then the map $\alpha : \mathbb{M}\mathrm{Spin}^c \to \mathbb{K}$ induces the classical Atiyah-Bott-Shapiro orientation homomorphism $MSpin_*^c \to K_*$ on the level of homotopy groups.*

Taking for granted Theorem 6.6 a technically precise statement is given as follows. If E is a G-vector bundle with structure group $Spin_V^c$ over some compact space X for some inner product space $V \in \mathscr{I}$, with a classifying map $c : E \to \gamma_{Spin_V^c}$, then the isomorphism

$$\widetilde{\Phi}^{0,V}(Th(E)) : \mathbb{K}^{0,V}(Th(E)) \to \widetilde{K}_G^{0,V}(Th(E)),$$

which is the reduced version of the natural isomorphism $\Phi^{0,V}(Th(E))$ introduced in (5.12), maps the class $\{f_E\} \in \mathbb{K}^{0,V}(Th(E))$ which is represented by the composition

$$f_E = \alpha_V \circ Th(c) : Th(E) \to Th(\gamma_{Spin_V^c}) = \mathbb{M}\mathrm{Spin}_V^c \xrightarrow{\alpha_V} \mathbb{K}_V \tag{6.10}$$

to the Atiyah-Bott-Shapiro orientation $u_E \in \widetilde{K}_G^{0,V}(Th(E))$.

It is easy to see that the natural transformation $\alpha : \mathbb{M}\mathrm{Spin}^c \to \mathbb{K}$ defines an orientation in the sense of stable homotopy theory. That it is a model for the Atiyah-Bott-Shapiro orientation essentially follows from Theorem 6.6 and the fact that the definition of the natural transformation α is modelled on an operator-theoretic description of the classical Atiyah-Bott-Shapiro.

Sketch of Proof. Let $W = V \oplus V$ and regard the irreducible $\mathbb{Z}/2$-graded $\mathbb{C}l_W$-module $\mathbb{S}_W$ as a Hilbert-$\mathbb{C}l_W$-module. Let $(\mathbb{S}_W, 0)$ be the Kasparov-$\mathbb{C}l_W$-cycle consisting of the irreducible $\mathbb{Z}/2$-graded $\mathbb{C}l_W$-module $\mathbb{S}_W$ and the zero operator on it. The Kasparov cycle represents a generator $[\mathbb{S}_W, 0] \in$

$KK_0^G(\mathbb{C}l_W) \cong \mathbb{Z}$. Performing the exterior product with this generator induces an isomorphism

$$\otimes[\mathbb{S}_W, 0] : KK_0^G(A) \longrightarrow KK_0^G(A \otimes \mathbb{C}l_W)$$

for any σ-unital $\mathbb{Z}/2$-graded G-C^*-algebra A. Given a Kasparov-A-cycle $(\mathcal{E}, \mathcal{F})$ the image in $KK_0^G(A \otimes \mathbb{C}l_W)$ is represented by the Kasparov cycle $(\mathcal{E} \otimes \mathbb{S}_W, \mathcal{F} \otimes Id)$.

On the other hand there is a version of the Swan isomorphism (5.3) which applies to pointed compact spaces Y

$$\widetilde{\psi}_Y : \widetilde{K}_G^0(Y) \to KK_0^G(C_0(Y)).$$

Similarly there is a pointed version $\widetilde{\phi}_Y^{0,V} : \mathbb{K}^{0,V}(Y) \to KK_0^G(C_0(Y) \otimes \mathbb{C}l_V)$ of the isomorphism $\phi_Y^{0,V}$ that we introduced in Remark 5.13. The isomorphism $\widetilde{\Phi}^{0,V}(Y) : \mathbb{K}^{0,V}(Y) \to \widetilde{K}_G^{0,V}(Y)$, which by definition is the reduced version of the natural isomorphism $\Phi^{0,V}(Y)$ introduced in (5.12), then fits into the commutative diagram

$$\begin{tikzcd}
\mathbb{K}^{0,V}(Y) \arrow[rr, "\widetilde{\Phi}^{0,V}(Y)"] \arrow[d, "\widetilde{\phi}_Y^{0,V}"] & & \widetilde{K}_G^{0,V}(Y) \arrow[dd, "\widetilde{\psi}_{S^V \wedge Y}"] \\
KK_0^G(C_0(Y) \otimes \mathbb{C}l_V) \arrow[d, "\otimes b_V"] & & \\
KK_0^G(C_0(Y) \otimes \mathbb{C}l_V \otimes C_0(S^V) \otimes \mathbb{C}l_V) \arrow[d, "KK_0^G(Id \otimes \tau \otimes Id)"] & & KK_0^G(C_0(S^V) \otimes C_0(Y)) \arrow[d, "\otimes[\mathbb{S}_W, 0]"] \\
KK_0^G(C_0(Y) \otimes C_0(S^V) \otimes \mathbb{C}l_V \otimes \mathbb{C}l_V) \arrow[rr, bend right, "KK_0^G(\tau \otimes Id \otimes Id)"'] & & KK_0^G(C_0(S^V) \otimes C_0(Y) \otimes \mathbb{C}l_W)
\end{tikzcd}$$

We now compare

$$a = \widetilde{\psi}_{S^V \wedge Th(E)}(u_E) \in KK_0^G(C_0(S^V) \otimes C_0(Th(E)))$$

with the element

$$b \in KK_0^G(C_0(S^V) \otimes C_0(Th(E)) \otimes \mathbb{C}l_W)$$

which we define to be the image of $\{f_E\} \in \mathbb{K}^{0,V}(Th(E))$ under the composition in the above diagram (for $Y = Th(E)$) which starts in the upper left corner, passes through the lower left corner and ends in the lower right corner.

Therefore consider the diagram of isomorphisms

$$\begin{tikzcd} a \in KK_0^G(C_0(S^V)\otimes C_0(Th(E))) \arrow[r] \arrow[d, "\otimes[\mathbb{S}_W,0]"] & KK_0^G(C_0(D(E')/S(E'))) \arrow[d, "\otimes[\mathbb{S}_W,0]"] \\ b \in KK_0^G(C_0(S^V)\otimes C_0(Th(E))\otimes \mathbb{C}l_W) \arrow[r] & KK_0^G(C_0(D(E')/S(E'))\otimes \mathbb{C}l_W) \end{tikzcd}$$

The image of a in $KK_0^G(C_0(D(E')/S(E')))$ is represented by the Kasparov cycle

$$(\mathcal{E}^t,\mathcal{F}^t)=(\Gamma_0(\mathscr{S}^t_{E'}),L),\ \text{with}\ \ L:\sigma\longmapsto(L(\sigma):e'\mapsto e'\sigma(e')),$$

while the image of b in $KK_0^G(C_0(D(E')/S(E'))\otimes\mathbb{C}l_W)$ is represented by

$$(\mathcal{E}^{\mathbb{C}l},\mathcal{F}^{\mathbb{C}l})=(\Gamma_0(\mathscr{S}^{\mathbb{C}l}_{E'}),L),\ \text{with}\ \ L:\sigma\longmapsto(L(\sigma):e'\mapsto e'\sigma(e')).$$

Hence $(\mathcal{E}^t\otimes\mathbb{S}_W,\mathcal{F}^t\otimes Id)\cong(\mathcal{E}^{\mathbb{C}l},\mathcal{F}^{\mathbb{C}l})$. Therefore $a\otimes[\mathbb{S}_W,0]=b$, hence $\widetilde{\Phi}^{0,V}(\{f_E\})=u_E$. $\square$

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