Realizing Kasparovs KK-theory groups as the homotopy classes of maps of a Quillen Model Category

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ABSTRACT. In this article we build a model category structure on the category of sequentially complete $l.m.c.$-C^*-algebras such that the corresponding homotopy classes of maps $Ho(A, B)$ for separable C^*-algebras A and B coincide with the Kasparov group $KK(A, B)$.

Dedicated to the memory of Michel Matthey.

1. Introduction

The goal of the present article is to relate classical Kasparov KK-theory for C^*-algebras (only complex algebras will be considered throughout) with the concepts of Quillen's model category theory. The motivation for this project emerged from one of the fundamental theorems of KK-theory due to Cuntz ([**Cun83**, §5]).

THEOREM 1.1 (Cuntz). *Suppose A,B are C^*-algebras with A separable and B σ-unital. Then there is a natural isomorphism*

$$KK(A, B) \cong [qA, B \otimes \mathcal{K}],$$

*where qA denotes the kernel of the fold map $A*A \to A$, $\mathcal{K}$ stands for the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space, and where $[?, ?]$ stands for homotopy classes of *-homomorphisms.*

The initial intuition was that replacing A by qA and B by $B \otimes \mathcal{K}$ could be the cofibrant and fibrant approximation functors for a corresponding model category structure on the category of C^*-algebras which somehow realizes the KK-groups as homotopy classes of maps. However, one of the first observations when trying to build a model category of C^*-algebras is that the category of C^*-algebras is not appropriate. By the Gelfand-Naimark Theorem, commutative C^*-algebras all correspond to algebras of continuous functions on compact (which will always imply Hausdorff) spaces. However, looking for a model category structure on the category of compact spaces or the category of pointed compact spaces is not very reasonable since colimits in the category of compact spaces exist but do not descend

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to the (standard) homotopy category, as had been pointed out to the authors by Jesper Grodal. Instead one typically looks for model category structures on the category of compactly generated spaces, which can all be described as unions of their compact subspaces. In the algebra setting this step would correspond to considering inverse limits of C^*-algebras instead of C^*-algebras themselves. For technical reasons, we consider a slightly bigger class of algebras, namely the (complex) "locally multiplicatively convex C^*-algebras" or $l.m.c.$-C^*-algebras, of [**Mal86**]. By the Arens-Michael decomposition theorem (2.6 below) any $l.m.c.$-C^*-algebra can be embedded as a dense subalgebra into an inverse limit of C^*-algebras, and an $l.m.c.$-C^*-algebra is complete if and only if it is isomorphic to an inverse limit of C^*-algebras. To make sure that the notion of the tensor product in the category of C^*-algebras agrees with the tensor product in the category of algebras we are working with, we restrict to ν-sequentially complete $l.m.c.$-C^*-algebras for a fixed infinite ordinal ν.

Our main theorem now is

THEOREM 1.2 ($\equiv$ Theorem 9.6). *There is a model category structure on the category of ν-sequentially complete l.m.c.-C^*-algebras satisfying the following properties:*

(1) *for a separable C^*-algebra A and a σ-unital C^*-algebra B there is a natural isomorphism*
$$Ho(A, B) \cong KK(A, B);$$
(2) *a *-homomorphism $A \to A'$ of separable C^*-algebras is a weak equivalence for the model category structure if and only if it is a KK_*-equivalence;*
(3) *the model category structure is cofibrantly generated;*
(4) *each object is fibrant.*

While properties (1) and (2) were the designing criteria, we wanted (3) and (4) to hold as they are convenient from a model category theory point of view. Whenever a model category structure is cofibrantly generated, there are so-called basic cells and an inductive process for building cellular approximations. Moreover, one essentially needs only to understand the basic cells in sufficient detail in order to build the whole model structure using Quillen's Small Object Argument. By Property (4) the model category at hand is right proper, which implies that homotopy pullbacks (and general homotopy limits) behave well in our category. We do not know whether or not our model category is left proper (and hence proper altogether).

Since we wanted property (4) to hold, of course it shouldn't be necessecary to apply any fibrant approximation functor to an object B for determining a homotopy set $Ho(A, B)$. For that reason, we do not start building our model category from Cuntz's Theorem (1.1) but a variant of it, based on the existence of an adjoint to tensoring with the compact operators. For the category of inverse limits of C^*-algebras, such an adjoint was considered by Phillips, cf. [**Phi89**, Proposition 5.8 on page 1084]. For an inverse limit of C^*-algebras $A = \lim_\alpha A_\alpha$, let $W(A)$ denote the left adjoint of the functor which associates to an inverse limit of C^*-algebras $B = \lim_\beta B_\beta$ the inverse limit $\lim_\beta B_\beta \otimes \mathcal{K}$. We then have

THEOREM 1.3 (Cuntz, Phillips). *For a separable C^*-algebra A and a σ-unital C^*-algebra B there is a natural isomorphism*
$$KK(A, B) \cong [W(qA), B].$$

We now take this theorem as our starting point and build our model category structure with a weakened version of the standard lifting technique from the Serre structure on Top_*, the category of pointed topological spaces. A variation is necessary because the usual adjoint pair is replaced by a weaker condition, where the left adjoint is only defined on compact spaces rather than arbitrary spaces. However, this technical detail causes no serious added difficulty, but a detailed argument is provided in Section 8. The most difficult technical portion of our arguments arises from two other facts. First, we cannot work completely within the category of compactly generated spaces, as $l.m.c.$-C^*-algebras (other than C^*-algebras) are typically not compactly generated. For that reason, questions concerning the exponential law for mapping spaces have to be treated with a lot of care, as detailed in Section 3. The second technical difficulty is that we need to work with colimits in the category of $l.m.c.$-C^*-algebras and colimits in that category typically are not very easy to deal with. For that reason we have introduced what we call the seminorm extension property. By definition a *-homomorphism has the seminorm extension property if and only if each continuous C^*-seminorm on the domain is the pullback of some C^*-seminorm of the target. This condition is analogous to the notion of topological embedding which plays a crucial role when applying Quillen's Small Object Argument in building the Serre structure on Top_*. Sequential colimits over *-homomorphisms having the seminorm extension property behave much better than in the general case, and we then have to show that various relevant *-homomorphisms in fact do have this property in Section 7.

On a practical level, our main theorem (and Section 9 generally) provides a solution to an open question posed by Hovey, Problem 8.4 in his foundational book [**Hov99**]. The model category structure also provides a variety of new techniques which might be useful in studying KK-theory, along with a common language for discussing KK-theory with homotopy theorists familiar with those techniques. As we mentioned above, there will be well-behaved notions of homotopy inverse limits, which include homotopy fibers, in our structure. A variety of Spectral Sequence techniques can now be applied to computing the extended notion of KK-theory provided by the homotopy classes of maps in our model structure (cf. Remark 9.7). If one wanted to build a better model for some specific purpose, particularly with more familiar basic cells, the well-understood notion of a Quillen equivalence now makes that relatively straightforward. If one wanted to study how formally inverting a certain map might affect KK-theory, one could now apply the theory of localizing model categories, as introduced e.g. in [**Hir03**].

Now a word on the organization of the article. We begin with several technical results on the category of topological *-algebras in Section 2. Section 3 is a derivation from [**Ste67**] and [**tDKP70**] of some properties of mapping spaces, while Section 4 introduces a language for universal constructions and Section 5 is devoted to a construction which serves as a restricted sort of adjoint to a mapping space functor. Section 6 contains the construction of the functor which is left adjoint to tensoring with the C^*-algebra of compact operators $\mathcal{K}$. After that we proceed to analyze the seminorm extension property and operations which preserve this property in Section 7. Section 8 presents the slight alteration of the standard lifting argument necessary for constructing the model category structures we require. Finally, in Section 9 we give the definition of the model category structure (Definition 9.5) and state the main theorem (Theorem 9.6). In addition, Section 9 also contains the definition of a related model category structure in which the weak equivalences between C^*-algebras are the K-theory equivalences, which might be of independent interest.

Finally we would like to thank Ralf Meyer and Andreas Thom for various interesting and useful discussions while completing this piece of work.

2. Categorical Aspects of Topological *-Algebras

DEFINITION 2.1 (topological $*$–algebras and $l.m.c.$-C^*-algebras).

(1) A topological *-algebra is an algebra over $\mathbb{C}$ equipped with a Hausdorff vector space topology such that the multiplication $A \times A \to A$ is separately continuous, together with an involution $* : A \to A$. The involution is not required to be continuous. Let $\mathbb{T}$ denote the category of topological *-algebras together with the continuous *-homomorphisms.

(2) Suppose S is a set of C^*-seminorms on an (algebraic) *-algebra A. Then there exists a coarsest topology τ_S on A for which each of the seminorms in S is continuous (compare [**Mal86**, I.3.1]). We write A_S for the corresponding Hausdorff quotient of A, which then is an object in $\mathbb{T}$. If a topology on A is given which coincides with τ_S for some set of C^*-seminorms, it is called an $l.m.c.$-C^*-topology. Note that we do not require an $l.m.c.$-C^*-topology to be Hausdorff.

(3) If A is a topological *-algebra and $A \cong A_S$ for some set of C^*-seminorms S then A is called an $l.m.c.$-C^*-algebra. Equivalently, one could say that an $l.m.c.$-C^*-algebra is a topological *-algebra whose topology is given by a (Hausdorff) $l.m.c.$-C^*-topology. As in [**Dix77**, 1.3.4 on page 9], it follows that the involution of an $l.m.c.$-C^*-algebra is continuous. Let $\mathbb{L}$ denote the full subcategory of $l.m.c.$-C^*-algebras in $\mathbb{T}$.

(4) For any $A \in \mathbb{T}$, let $S(A)$ denote the set of all continuous C^*-seminorms on A. Thus, for any $A \in \mathbb{L}$ we have a canonical isomorphism $A \cong A_{S(A)}$.

We follow Mallios (cf. [**Mal86**, page 484]) in the sense that we do not require an $l.m.c.$-C^*-algebra to be complete. However, we would like to require some weak form of completeness for the objects that we are going to work with. This will be convenient at some point, but also allows our notion of tensor product to generalize the minimal tensor product of C^*-algebras in a natural way.

DEFINITION 2.2 (ν-complete algebras and the category $\mathbb{A}_\nu$).

(1) Let ν be an ordinal, whose cardinality is greater than or equal to the cardinality of the integers. A ν-sequence in a topological $*$-algebra simply is a map $x : \nu \to A$ of underlying sets. A ν-sequence is called a Cauchy sequence if, for any open neighborhood U of $0 \in A$, there is an element $\gamma \in \nu$ such that for any two $\alpha, \beta \in \nu$ satisfying $\alpha, \beta \geq \gamma$ one has $x(\alpha) - x(\beta) \in U$. An algebra $A \in \mathbb{T}$ is called ν-complete if each Cauchy ν-sequence converges to a point in A. We define the category $\mathbb{A}_\nu$ to be the full subcategory of $\mathbb{T}$ consisiting of the ν-complete $l.m.c.$-C^*-algebras. Moreover, let C^*-alg denote the full subcategory of $\mathbb{T}$ consisting of the C^*-algebras. Then we have inclusions of full subcategories

$$C^*\text{-}alg \subset \mathbb{A}_\nu \subset \mathbb{L} \subset \mathbb{T}.$$

(2) For any $A \in \mathbb{T}$ one can define its ν-completion as the expected quotient of the space of Cauchy ν-sequences, which we denote $\overline{(A)}^{\nu}$. As usual $\overline{(A)}^{\nu}$ comes equipped with a canonical dense inclusion $A \subset \overline{(A)}^{\nu}$, which yields a functor $\overline{(?)}^{\nu} : \mathbb{L} \to \mathbb{A}_{\nu}$. By contrast, the (honest) completion of an algebra $A \in \mathbb{T}$ will be denoted $\overline{A}$ and we will use the symbol $\approx$ to differentiate isomorphisms in the category $\mathbb{A}_{\nu}$ from ordinary bijections (denoted $\cong$).

For ease of reference later, we state the following consequence of the idempotence of the ν-completion operation.

Lemma 2.3. *The functor $\overline{(?)}^{\nu} : \mathbb{L} \to \mathbb{A}_{\nu},\ A \to \overline{(A)}^{\nu}$ is left adjoint to the inclusion $\mathbb{A}_{\nu} \subset \mathbb{L}$ of the ν-complete l.m.c.-C^*-algebras into the category of all l.m.c.-C^*-algebras, i.e. for any $B \in \mathbb{A}_{\nu}$ we have a natural bijection*

$$Hom(\overline{(A)}^{\nu}, B) \cong Hom(A, B).$$

Complete $l.m.c.$-C^*-algebras may be identified with inverse limits of C^*-algebras by the Arens-Michael decomposition theorem (see Theorem 2.6 below). Inverse limits of C^*-algebras, sometimes called pro-C^*-algebras, have been studied by many authors, most notably by Phillips (cf. [**Phi88a**], [**Phi88b**]). For technical reasons that will become clear below (Remark 7.6), we are forced to work with the weaker notion of ν-completeness. Nevertheless, we can choose the ordinal ν to be arbitrarily large.

Lemma 2.4. *The category $\mathbb{A}_{\nu}$ has all set-indexed limits.*

Proof. Limits in $\mathbb{T}$ are formed as in topological spaces, with the algebraic operations defined coordinatewise (compare [**Mal86**, III, Lemma 2.1]). Moreover, a limit in $\mathbb{T}$ over a system of $l.m.c.$-C^*-algebras is an $l.m.c.$-C^*-algebra (compare [**Mal86**, III, (2.8)]). The limit in $\mathbb{T}$ over a system of ν-complete $l.m.c.$-C^*-algebras is also a ν-complete $l.m.c.$-C^*-algebra, since a Cauchy ν-sequence in a product or limit is simply a ν-sequence that is Cauchy in each factor as in [**KN63**, 7.2 on page 57]. □

Lemma 2.5. *The category $\mathbb{A}_{\nu}$ has all set-indexed colimits.*

Proof. First note that it suffices to construct colimits in $\mathbb{L}$. Colimits in $\mathbb{A}_{\nu}$ are then obtained by applying the ν-completion to the corresponding colimit in $\mathbb{L}$ (by 2.3). Also, as commented above Theorem IX.1.1 in [**ML98**], it suffices to construct coproducts, coequalizers (also called difference cokernels) and directed colimits in $\mathbb{L}$.

The existence of directed colimits can be shown as in the proof of Lemma 2.2 in Section IV of [**Mal86**].

Given two morphisms $u, v : A \to B$ in $\mathbb{L}$, let $S(u,v) \subset S(B)$ denote the set of continuous C^*-seminorms on B which vanish on the ideal generated by elements of the form $u(a) - v(a)$ for $a \in A$. Then the desired coequalizer is given by the canonical map $B \to B_{S(u,v)}$.

Suppose $\{A_i\}$ denotes a set of $l.m.c.$-C^*-algebras indexed on the set I. We define a topology on the algebraic free product $*_i A_i$ by considering the collection of all *-homomorphisms from the free product to $l.m.c.$-C^*-algebras B such that the canonical algebraic compositions $A_i \to *_i A_i \to B$ are continuous for each $i \in I$. Given such a *-homomorphism we pull back the defining C^*-seminorms of B to the free product $*_i A_i$. As there is only a set of possible C^*-seminorms on $*_i A_i$, this procedure yields a set S of seminorms on $*_i A_i$ (even though the collection of *-homomorphisms we are considering might not form a set). It is now easy to check that $(*_i A_i)_S$ is the coproduct of the A_i in $\mathbb{L}$. (For a second approach see 4.3 (3)). □

Let A be a topological *-algebra. For any $p \in S(A)$, let A_p denote the Hausdorff quotient $A_{\{p\}}$. Then A_p is a pre-C^*-algebra in the sense that its completion $\overline{A_p}$ is a C^*-algebra (here ν-completion agrees with completion since the topology is first countable). Notice also that the set $S(A)$ is partially ordered by $p \leq q$ when $p(a) \leq q(a)$ for each $a \in A$, so there is a canonical map $A \to \lim A_p$, where the inverse limit is taken over the directed system $S(A)$. This map is shown to be an isomorphism for any complete $l.m.c.$-C^*-algebra by the remarkable Arens-Michael decomposition theorem.

Theorem 2.6 ([**Are52**],[**Are58**],[**Mic52**], compare [**Mal86**, III.Theorem 3.1 on page 88]). *For an l.m.c.-C^*-algebra A there is a string of natural inclusions*

$$A \subset \lim A_p \subset \lim \overline{A_p} \approx \overline{A}.$$

In particular, for any complete l.m.c.-C^*-algebra there is a natural isomorphism

$$A \approx \lim A_p.$$

The Arens-Michael decomposition theorem is quite helpful in understanding the nature of $l.m.c.$-C^*-algebras in general. This already becomes apparent when we are considering the tensor products of $l.m.c.$-C^*-algebras, which we now introduce.

As in the category of C^*-algebras, there are various possibilities to define a reasonable tensor product in $\mathbb{A}_\nu$. Recall that there is the notion of a maximal or a minimal tensor product in the category of C^*-algebras, and both definitions agree if one factor is nuclear. By the tensor product of C^*-algebras, we will always mean the minimal tensor product.

Definition 2.7. Let $A, B \in \mathbb{A}_\nu$. Given $p \in S(A)$ and $q \in S(B)$ let pq denote the C^*-seminorm on the algebraic tensor product $A \otimes_{alg} B$ obtained by pulling back the C^*-norm on $\overline{A_p} \otimes \overline{B_q}$ along the canonical homomorphism $A \otimes_{alg} B \to \overline{A_p} \otimes \overline{B_q}$. Let $S(AB)$ be the set of C^*-seminorms given by $\{pq \mid (p,q) \in S(A) \times S(B)\}$. We define the tensor product $A \otimes B$ in $\mathbb{A}_\nu$ to be the ν-completion of the $l.m.c.$-C^*-algebra $(A \otimes_{alg} B)_{S(AB)}$.

By the Arens-Michael decomposition theorem, $(A \otimes_{alg} B)_{S(AB)}$ can also be identified with the Hausdorff quotient of the *-algebra $A \otimes_{alg} B$ equipped with the topology induced by the canonical homomorphism

$$A \otimes_{alg} B \to \prod_{pq \in S(AB)} \overline{A_p} \otimes \overline{B_q}.$$

Remark 2.8. The tensor product described above gives $\mathbb{A}_\nu$ a symmetric monoidal structure. The fact that we restrict to ν-complete algebras ensures that the category of C^*-algebras forms a symmetric monoidal subcategory of $\mathbb{A}_\nu$.

3. On Mapping Spaces

For any pair of Hausdorff spaces X and Y, let $\mathcal{C}(X,Y)$ denote the space of continuous maps with the compact-open topology, which is then a Hausdorff space as well. Similary, for a pair of pointed Hausdorff spaces X and Y, let $\mathcal{C}_*(X,Y)$ denote the subspace of $\mathcal{C}(X,Y)$ consisiting of the basepoint preserving maps. Let Cpt denote the category of compact spaces and let Cpt_* denote the category of pointed compact spaces. Note that adding a disjoint basepoint yields an inclusion of categories $\mathrm{Cpt} \subset \mathrm{Cpt}_*$. Moreover, given a pointed Hausdorff space Y and a compact space X we have an obvious homeomorphism $\mathcal{C}(X,Y) \cong \mathcal{C}_*(X_+,Y)$, where X_+ denotes adding a disjoint basepoint to X.

Now notice that for A and $B \in \mathbb{A}_\nu$, one can consider the set of continuous *-homomorphisms $\mathrm{Hom}(A,B)$ as a topological space with the compact-open topology and think of 0 as the basepoint in each algebra. Thus, $\mathrm{Hom}(A,B)$ is a subspace of $\mathcal{C}_*(A,B)$, hence is a (pointed) Hausdorff space as well. We will use the symbol $\mathcal{K}$ to denote the C^*-algebra of compact operators on a separable Hilbert space.

Given $X \in \mathrm{Cpt}$ and $B \in \mathbb{T}$, it should be clear that $\mathcal{C}(X,B) \in \mathbb{T}$, where each algebraic operation is defined pointwise, and similarly for the pointed mapping space $\mathcal{C}_*(X,B)$ for $X \in \mathrm{Cpt}_*$. However, we will need the following stronger result.

Lemma 3.1. *Suppose $X \in Cpt$ and $B \in \mathbb{A}_\nu$, then $\mathcal{C}(X,B) \in \mathbb{A}_\nu$. Similarly, if $X \in Cpt_*$, then $\mathcal{C}_*(X,B) \in \mathbb{A}_\nu$.*

Proof. First notice that the compact-open topology on $\mathcal{C}(X,B)$ is the same as the topology of uniform convergence, since X is compact. Given $p \in S(B)$, one has an associated C^*-seminorm p_X on $\mathcal{C}(X,B)$ given by

$$p_X(f) = \sup_{x\in X} p(f(x)).$$

In fact, the collection of these C^*-seminorms p_X on $\mathcal{C}(X,B)$ determines the topology of uniform convergence. To see this, notice the sets

$$U_{p,r,f} = \{g \in \mathcal{C}(X,B) : \forall x \in X, p(f(x) - g(x)) < r\}$$

with $p \in S(B)$, $r \in \mathbb{R}_{>0}$ and $f \in \mathcal{C}(X,B)$ form a subbasis for the topology in both cases. Thus, $\mathcal{C}(X,B)$ is an $l.m.c.$-C^*-algebra. Since $B \in \mathbb{A}_\nu$ is ν-complete, so is $\mathcal{C}(X,B)$ with this topology, thus $\mathcal{C}(X,B) \in \mathbb{A}_\nu$.

For $X \in \mathrm{Cpt}_*$, the algebra $\mathcal{C}_*(X,B)$ is a closed ν-complete subalgebra of $\mathcal{C}(X,B)$, hence we also have $\mathcal{C}_*(B,X) \in \mathbb{A}_\nu$. □

The following lemma is provided here mainly for ease of reference later.

Lemma 3.2. *If $X \in Cpt$ and $B \in \mathbb{A}_\nu$, then there is an isomorphism in $\mathbb{A}_\nu$, $\mathcal{C}(X,B) \otimes \mathcal{K} \approx \mathcal{C}(X, B \otimes \mathcal{K})$. If $X \in Cpt_*$ and $B \in \mathbb{A}_\nu$, we have a corresponding isomorphism $\mathcal{C}_*(X,B) \otimes \mathcal{K} \approx \mathcal{C}_*(X, B \otimes \mathcal{K})$.*

Proof. Since by the previous lemma $\mathcal{C}(X, B\otimes\mathcal{K})$ is ν-complete, the canonical injection $\mathcal{C}(X,B) \otimes_{alg} \mathcal{K} \to \mathcal{C}(X, B\otimes\mathcal{K})$ yields an inclusion $\mathcal{C}(X,B)\otimes\mathcal{K} \subset \mathcal{C}(X, B\otimes \mathcal{K})$ (compare [**Phi88a**, Proposition 3.4]). Thus, we have to show that the map is surjective.

For pairs of natural numbers (i,j), let $x_{ij} : \mathcal{K} \to \mathbb{C}, k \mapsto k_{ij}$ denote the family of linear maps which are defined through the equation $k(e_i) = \sum_{i,j} k_{ij} e_j$ for some fixed orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of the underlying separable Hilbert space that $\mathcal{K}$ is acting on. Moreover, abusing notation, let x^B_{ij} also stand for the corresponding linear map $x_{ij}^B : B \otimes \mathcal{K} \to B \otimes \mathbb{C} \approx B$ that is obtained by tensoring the map $x_{ij} : \mathcal{K} \to \mathbb{C}$ with $id : B \to B$. In addition, let $e_{ij} \in \mathcal{K}$ denote the compact operator which maps a general vector $\sum_k h_k e_k$ (with the $h_k \in \mathbb{C}$) to $h_j e_i$. Each element $b \in B \otimes \mathcal{K}$ then has a unique presentation $b = \sum_{ij} x_{ij}^B(b) \otimes e_{ij}$.

Let $f : X \to B \otimes \mathcal{K}$ be a pointed continuous function and define corresponding functions $f_{ij} \in \mathcal{C}(X, B)$ by $f_{ij}(x) = x_{ij}^B(f(x))$. The sum $\sum_{ij} f_{ij} \otimes e_{ij} \in \mathcal{C}(X, B) \otimes \mathcal{K}$ then is a pre-image for $f \in \mathcal{C}(X, B \otimes \mathcal{K})$.

To obtain the pointed statement observe that evaluation at the basepoint induces a natural transformation of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathcal{C}_*(X,B) \otimes \mathcal{K} \arrow[r] \arrow[d] & \mathcal{C}(X,B) \otimes \mathcal{K} \arrow[r, "ev \otimes id"] \arrow[d, "\approx"] & B \otimes \mathcal{K} \arrow[r] \arrow[d, "\approx"] & 0 \arrow[d] \\
0 \arrow[r] & \mathcal{C}_*(X, B \otimes \mathcal{K}) \arrow[r] & \mathcal{C}(X, B \otimes \mathcal{K}) \arrow[r, "ev"] & B \otimes \mathcal{K} \arrow[r] & 0
\end{tikzcd}$$

The pointed statement therefore follows from the unpointed statement. ☐

Notice, in the previous proof we have used the ν-completion with ν at least countable to see that the infinite sum still lies in $\mathcal{C}(X, B) \otimes \mathcal{K}$.

In order to prove our key result on mapping spaces, we will first require another technical result which comes from [**Ste67**] and [**tDKP70**].

LEMMA 3.3. *Suppose X is compact, while Y and Z are Hausdorff. Then there is a natural topological inclusion*

$$\mathcal{C}(Y, \mathcal{C}(X, Z)) \to \mathcal{C}(X, \mathcal{C}(Y, Z))$$

which is a homeomorphism if Y is compactly generated.

PROOF. Define topological inclusions $\iota_X : \mathcal{C}(X \times Y, Z) \to \mathcal{C}(X, \mathcal{C}(Y, Z))$ by

$$(\iota_X(x))(y) = f(x, y)$$

and similarly $\iota_Y : \mathcal{C}(X \times Y, Z) \to \mathcal{C}(Y, \mathcal{C}(X, Z))$ by

$$(\iota_Y(y))(x) = f(x, y).$$

These maps are inclusions as both X and Y are Hausdorff spaces (e.g. see [**tDKP70**, Satz (4.9) on page 88]). In general, these inclusions are not surjective. However, ι_Y is surjective since X is compact (see [**Ste67**, 3.4.7]) while ι_X is onto if Y, hence $X \times Y$, is compactly generated (see [**Ste67**, 3.4.9]). Thus, in any case $\iota_X \circ (\iota_Y)^{-1}$ is a topological inclusion and it becomes a homeomorphism if Y is compactly generated. ☐

We now have our key property of mapping spaces.

PROPOSITION 3.4. *For A,$B \in \mathbb{A}_\nu$ and $X \in Cpt$, there is a natural topological inclusion*

$$\mathrm{Hom}(A, \mathcal{C}(X, B)) \to \mathcal{C}(X, \mathrm{Hom}(A, B)) \tag{3.1}$$

which is a homeomorphism if A is compactly generated as a topological space. In particular, the inclusion is a homeomorphism if $A \in C^*$-alg.

In case $X \in Cpt_*$, the same is true for the corresponding map of pointed mapping spaces

$$\mathrm{Hom}(A, \mathcal{C}_*(X, B)) \to \mathcal{C}_*(X, \mathrm{Hom}(A, B)). \tag{3.2}$$

PROOF. The statement for C^*-algebras follows from the general one because $A \in C^*\text{-}alg$ would imply A is compactly generated as a normed, hence first countable, space (see [**Ste67**]).

We use the notation of the previous proof and consider the case $X \in \mathrm{Cpt}$ first. On the set-theoretic level, one easily checks that the restriction of $\iota_X \circ (\iota_A)^{-1}$ to the subspace

$$\mathrm{Hom}(A, \mathcal{C}(X, B)) \subset \mathcal{C}(A, \mathcal{C}(X, B))$$

factors through the subspace

$$\mathcal{C}(X, \mathrm{Hom}(A, B)) \subset \mathcal{C}(X, \mathcal{C}(A, B)).$$

Hence we obtain a topological inclusion

$$\mathrm{Hom}(A, \mathcal{C}(X, B)) \to \mathcal{C}(X, \mathrm{Hom}(A, B)).$$

On the other hand, if $\iota_X \circ (\iota_A)^{-1}$ is surjective, it is also straightforward to check that the pre-image of $\mathcal{C}(X, \mathrm{Hom}(A, B))$ is precisely $\mathrm{Hom}(A, \mathcal{C}(X, B))$, giving the expected homeomorphism.

Now assume X is pointed. Notice that $\mathcal{C}_*(X, B) \subset \mathcal{C}(X, B)$ consists of those maps which vanish on the basepoint $x_0 \in X$. Thus, a *-homomorphism $\psi : A \to \mathcal{C}(X, B)$ factors through $\mathcal{C}_*(X, B)$ precisely when $(\psi(a))(x_0) = 0$ for each $a \in A$. However, this is equivalent to saying that the adjoint map $X \to \mathrm{Hom}(A, B)$ sends the basepoint x_0 to the constant map on zero, which is the basepoint of $\mathrm{Hom}(A, B)$. Thus we obtain the pointed version from the unpointed statement simply by restriction. □

4. Universal objects defined through generators and relations

From the Arens-Michael decomposition theorem, we see that $l.m.c.$-C^*-algebras (resp. ν-complete $l.m.c.$-C^*-algebras) are very closely related to inverse limits of C^*-algebras, or pro-C^*-algebras. In [**Phi88b**, 1.3] Phillips has introduced the concept of a weakly admissible set of generators and relations and showed that a weakly admissible set of generators and relations can be used to define a universal associated inverse limit of C^*-algebras. The corresponding concept works equally well for the category $\mathbb{A}_\nu$. A pair (G, R) of generators and relations in this context then simply consists of a set G and a collection R of statements about the elements in G which make sense for elements of a ν-complete $l.m.c.$-C^*-algebra. A representation of (G, R) in a ν-complete $l.m.c.$-C^*-algebra A then consists of a set-valued map $\varrho : G \to A$ such that the elements $\varrho(g)$ for $g \in G$ satisfy the relations R in A.

DEFINITION 4.1 ([**Phi88b**, Definition 1.3.4]). A set (G, R) of generators and relations is called weakly admissible if

(1) The zero map from G to the zero C^*-algebra is a representation of (G, R).
(2) If ϱ is a representation of (G, R) in a C^*-algebra A, and B is a C^*-subalgebra of A containing $\varrho(G)$, then ϱ is a representation of (G, R) in B.
(3) If ϱ is a representation of (G, R) into a ν-complete $l.m.c.$-C^*-algebra A, and $\varphi : A \to B$ is a continuous surjective *-homomorphism onto a C^*-algebra B, then $\varphi \circ \varrho$ is a representation of (G, R) in B.
(4) If A is a ν-complete $l.m.c.$-C^*-algebra, and $\varrho : G \to A$ is a function such that, for every $p \in S(A)$ the composition of ϱ with the *-homomorphism

$A \to \overline{A_p}$ (notation of Theorem 2.6) is a representation of (G, R) in $\overline{A_p}$, then ϱ is a representation of (G, R) in A.

(5) If $\varrho_1, ..., \varrho_n$ are representations of (G, R) in C^*-algebras $A_1, ..., A_n$, then $g \mapsto (\varrho_1(g), ..., \varrho_n(g))$ is a representation of (G, R) in $A_1 \times ... \times A_n$.

Proposition 4.2 ([**Phi88b**, Proposition 1.3.6]). *Let (G, R) be a weakly admissible set of generators and relations. Then there exists a ν-complete l.m.c.-C^*-algebra $\mathbb{A}_\nu(G, R)$, equipped with a representation $\varrho : G \to \mathbb{A}_\nu(G, R)$ of (G, R), such that, for any representation ς of (G, R) into a ν-complete l.m.c.-C^*-algebra B, there is a unique continuous *-homomorphism $\varphi : \mathbb{A}_\nu(G, R) \to B$ satisfying $\varsigma = \varphi \circ \varrho$.*

Proof. The proof carries over verbatim from the one given in [**Phi88b**], except that the *l.m.c.*-C^*-algebra $\mathbb{A}_\nu(G, R)$ has to be defined as the Hausdorff ν-completion $F(G)_D$ of the free associative complex $*$-algebra $F(G)$ generated by G with respect to the set of seminorms D (rather than being its Hausdorff completion). $\square$

Example 4.3.

(1) Let G be a set of generators and let $F(G)$ denote the free associative complex *-algebra on the set G. The requirement that a set map $X \to F(G)$ from a topological space X into $F(G)$ is continuous defines a set of weakly admissible relations. (cf. [**Phi88b**, Example 1.3.5 (4)])

(2) Given a ν-complete *l.m.c.*-C^*-algebra A, define G to be the set A and define the set of relations to be the set of algebraic relations between the elements of A together with the relations that imply the obvious composite map $A \to G \subset F(G)$ is continuous. Then (G, R) is weakly admissible and $\mathbb{A}_\nu(G, R) \approx A$. (cf. [**Phi88b**, 1.3.7])

(3) More generally, given a set $\{A_i\}_{i \in I}$ of ν-complete *l.m.c.*-C^*-algebras take the disjoint union of the elements in the A_i as a set of generators G and let R be the set of all the algebraic relations in the individual A_i together with the relations which imply that each of the maps $A_i \to G$ is continuous. This yields a set of weakly admissible relations and $\mathbb{A}_\nu(G, R)$ is a model for the coproduct of the A_i in $\mathbb{A}_\nu$.

(4) Given a ν-complete *l.m.c.*-C^*-algebra A, a topological space X and a set map $X \to A$ one can define a weakly admissible set of relations with $G = A$ and the set of relations R given by all the algebraic relations of A together with the relations which imply the two maps $A \to G \subset F(G)$ and $X \to G \subset F(G)$ to be continuous. (This construction will be used in 5.2.)

5. Partial Adjoint Pairs

Ideally we would like to have a simple and easy to work with construction for a left adjoint to the functor

$$\mathbb{R}_A = \mathrm{Hom}(A, ?) : \mathbb{A}_\nu \to \mathrm{Top}_*$$

associated to any $A \in \mathbb{A}_\nu$. However, technically it is easier to deal with functors with the following much weaker property, which suffices for our purposes.

Definition 5.1. Suppose $\mathbb{L}_A : \mathrm{Cpt}_* \to \mathbb{A}_\nu$ is a functor together with a bijection

$$\mathrm{Hom}(\mathbb{L}_A(X), B) \to \mathcal{C}_*\left(X, \mathbb{R}_A(B)\right)$$

natural in both $X \in \mathrm{Cpt}_*$ and $B \in \mathbb{A}_\nu$. Then $\mathbb{L}_A$ will be referred to as a partial left adjoint to $\mathbb{R}_A$.

We now have the following construction, which is closely related to that of Phillips in [**Phi88b**, on page 180].

CONSTRUCTION 5.2. Suppose $A \in \mathbb{A}_\nu$ and $X \in \mathrm{Cpt}$. For $x \in X$, let A_x denote a copy of A. By 4.3 (3) we can obtain a model for the free product $*_{x\in X} A_x$ in $\mathbb{A}_\nu$ in terms of generators and relations. And by 4.3 (4) we can put further relations on the free product which imply the natural inclusion $A \times X \to *_{x\in X} A_x$, given by the isomorphisms $A \times \{x\} \to A_x$, is continuous. Let us denote the resulting object by $A \otimes X$.

By construction, $A \otimes X$ comes equipped with a natural continuous map

$$i : A \times X \to A \otimes X.$$

In particular, for each $a \in A$ we obtain a continuous map $i_a : X \to A \otimes X$ by

$$X \cong \{a\} \times X \to A \times X \to A \otimes X$$

and similarly we obtain a continuous map $i_x : A \to A \otimes X$ for each $x \in X$.

If $x_0 \in X$ is a basepoint for X, then we can modify the previous construction to yield an *l.m.c.-C^**-algebra $A \oslash X \in \mathbb{A}_\nu$ and a corresponding continuous map

$$j : A \wedge X \to A \oslash X$$

simply by removing x_0 from the indexing set of the free product and requiring the canonical map $A \wedge X \to *_{x\in X\setminus\{x_0\}} A_x$ to be continuous. As above, we then obtain continuous maps $j_a : X \to A \oslash X$ for each $a \in A$ and $j_x : A \to A \oslash X$ for each $x \in X$, with the maps j_0 and j_{x_0} mapping constantly to $0 \in A \oslash X$.

PROPOSITION 5.3. *Let A,$B \in \mathbb{A}_\nu$ and $X \in Cpt$. Then there is a continuous bijection*

$$\mathrm{Hom}(A \otimes X, B) \to \mathrm{Hom}(A, \mathcal{C}(X, B)) \tag{5.1}$$

natural in A,B and X.

PROOF. There is a continuous composition of

$$i^* : \mathrm{Hom}(A \otimes X, B) \to \mathcal{C}(A \times X, B)$$

and the map

$$\iota_A : \mathcal{C}(A \times X, B) \to \mathcal{C}(A, \mathcal{C}(X, B))$$

which restricts to give the continuous map

$$\mathrm{Hom}(A \otimes X, B) \to \mathrm{Hom}(A, \mathcal{C}(X, B))$$

in the statement. Clearly, this map is natural in all variables, so it suffices to verify this is a bijection.

In order to establish surjectivity, let ψ be an arbitrary *-homomorphism in $\mathrm{Hom}(A, \mathcal{C}(X, B))$. As in the proof of Proposition 3.4, the map ι_A is a homeomorphism since X is compact. Let f denote the continuous map $f : A \times X \to B$ with $\iota_A(f) = \psi$ so $(\psi(a))(x) = f(a, x)$. Then f is a representation of the set of generators and relations defining $A \otimes X$. So, by Proposition 4.2, we get a *-homomorphism $g : A \otimes X \to B$ which then maps to ψ by construction.

In order to establish injectivity, note that two *-homomorphisms $g_1, g_2 : A \otimes X \to B$ mapping to a *-homomorphism $\psi : A \to \mathcal{C}(X, B)$ restrict to the same representation $f : A \times X \to B$, so we obtain $g_1 = g_2$ by the uniqueness assertion of Proposition 4.2. □

PROPOSITION 5.4. *Let A, $B \in \mathbb{A}_\nu$ and $X \in Cpt_*$. Then there is a continuous bijection*

$$\operatorname{Hom}(A \circledS X, B) \to \operatorname{Hom}(A, \mathcal{C}_*(X, B)) \tag{5.2}$$

natural in A,B and X.

PROOF. There is a continuous composition of

$$j^* : \operatorname{Hom}(A \circledS X, B) \to \mathcal{C}_*(A \wedge X, B)$$

and the map

$$\iota_A : \mathcal{C}_*(A \wedge X, B) \to \mathcal{C}_*(A, \mathcal{C}_*(X, B))$$

which restricts to give the continuous map

$$\operatorname{Hom}(A \circledS X, B) \to \operatorname{Hom}(A, \mathcal{C}_*(X, B))$$

in the statement. Clearly, this map is natural in all variables, so it suffices to verify this is a bijection, which follows as in the proof of Proposition 5.3. □

In the next section, we will have a mild extension of the following result which produces partial adjoint pairs.

PROPOSITION 5.5. *For a compactly generated $A \in \mathbb{A}_\nu$, the functor*

$$A \circledS ? : Cpt_* \to \mathbb{A}_\nu$$

is a partial left adjoint to the functor

$$\operatorname{Hom}(A, ?) : \mathbb{A}_\nu \to Top_*.$$

PROOF. The statement is equivalent to the existence of a bijection

$$\operatorname{Hom}(A \circledS X, B) \to \mathcal{C}_*(X, \operatorname{Hom}(A, B))$$

natural in $X \in \mathrm{Cpt}_*$ and $B \in \mathbb{A}_\nu$. However, Proposition 5.4 yields a (continuous) bijection

$$\operatorname{Hom}(A \circledS X, B) \to \operatorname{Hom}(A, \mathcal{C}_*(X, B))$$

natural in X and B. For compactly generated A, Proposition 3.4 yields a bijection (actually, a homeomorphism) natural in X and B

$$\operatorname{Hom}(A, \mathcal{C}_*(X, B)) \to \mathcal{C}_*(X, \operatorname{Hom}(A, B)).$$

□

6. The adjoint of tensoring with the compact operators

From the work of Phillips ([**Phi89**, Proposition 5.8 on page 1084]), we know that in the category of inverse limits of C^*-algebras there is an adjoint to tensoring an object with the compact operators $\mathcal{K}$. The corresponding statement also holds in the category $\mathbb{A}_\nu$. As with Phillips' construction, the adjoint can be obtained by an application of Freyd's Adjoint Functor Theorem, but we instead give a constructive proof.

Before we present the construction, we first have the following lemma which verifies a necessary condition for the existence of an adjoint to tensoring with $\mathcal{K}$.

LEMMA 6.1. *Taking the tensor product with the compact operators in the category $\mathbb{A}_\nu$ commutes with inverse limits $B = \lim_\beta B_\beta$ in $\mathbb{A}_\nu$, i.e. the canonical map*

$$(\lim_\beta B_\beta) \otimes \mathcal{K} \xrightarrow{\approx} \lim_\beta (B_\beta \otimes \mathcal{K})$$

is an isomorphism in $\mathbb{A}_\nu$.

PROOF. Using the Arens-Michael Decomposition Theorem 2.6 and the definition of the tensor product we obtain a commutative diagram

$$\begin{tikzcd}
(\lim_\beta B_\beta) \otimes \mathcal{K} \arrow[r] \arrow[d] & \lim_\beta (B_\beta \otimes \mathcal{K}) \arrow[d] \\
\prod_{q \in S(\lim_\beta B_\beta)} \overline{(\lim_\beta B_\beta)_q} \otimes \mathcal{K} \arrow[r] & \prod_\beta \prod_{p \in S(B_\beta)} \overline{(B_\beta)_p} \otimes \mathcal{K},
\end{tikzcd}$$

where the vertical maps are inclusions. Now let $pr_\alpha : \lim_\beta B_\beta \to B_\alpha$ for an index α denote the canonical projection. The set $S = \bigcup_\beta \{pr_\beta^* p \mid p \in S(B_\beta)\}$ of C^*-seminorms on $\lim_\beta B_\beta$ then determines the topology of $\lim_\beta B_\beta$, which implies that the bottom horizontal arrow is an inclusion. Consequently so is the horizontal arrow on the top, which is the map of the lemma. Thus, it remains to see that the *-homomorphism is surjective.

For pairs of natural numbers (i, j), let $x_{ij}^B : B \otimes \mathcal{K} \to B$ be defined as in the proof of Lemma 3.2. The *-homomorphism of the lemma then fits into a commutative diagram

$$\begin{tikzcd}
(\lim_\beta B_\beta) \otimes \mathcal{K} \arrow[rr] \arrow[dr, "\prod x_{ij}^{\lim_\beta B_\beta}"'] & & \lim_\beta (B_\beta \otimes \mathcal{K}) \arrow[dl, "\prod \lim_\beta x_{ij}^{B_\beta}"] \\
& \prod_{i,j} \lim_\beta B_\beta &
\end{tikzcd}$$

Given any element $b = (b_\beta)_\beta \in \lim_\beta (B_\beta \otimes \mathcal{K})$, the element $\sum_{i,j} x_{ij}^{B_\beta}(b_\beta) \otimes e_{ij}$ is contained in $(\lim_\beta B_\beta) \otimes \mathcal{K}$ and provides a pre-image of b (establishing surjectivity). Note that we have used that $(\lim_\beta B_\beta) \otimes \mathcal{K}$ is sequentially complete, since the infinite sum above is not contained in the algebraic tensor product. ☐

CONSTRUCTION 6.2. Suppose $A \in \mathbb{A}_\nu$. For any $l.m.c.$-C^*-algebra Z and any *-homomorphism $f : A \to Z \otimes \mathcal{K}$, define $Z_f \subset Z$ to be the topological sub-*-algebra generated by the set $\{f(a)_{ij} \mid i, j \in \mathbb{N}, a \in A\}$, whose elements are given by $f(a)_{ij} = x_{ij}^Z(f(a))$ (notation from the previous proof). Let D be the set of all isomorphism classes of pairs (Z, f) where Z is an $l.m.c.$-C^*-algebra and f is a *-homomorphism $f : A \to Z \otimes \mathcal{K}$ such that $Z = Z_f$. Notice that D is in fact a set, as there is only a set of isomorphism classes of $l.m.c.$-C^*-algebras that can be generated by the set $\mathbb{N} \times \mathbb{N} \times A$. On the set D there is a partial ordering defined as follows. Pick representatives $d = (Z_d, f_d)$ and define $d \geq e$ if there exits a *-homomorphism $\varphi_e^d : Z_d \to Z_e$ such that $f_e = (\varphi_e^d \otimes id_{\mathcal{K}}) \circ f_d$. The partially ordered set D is directed, as for $d, e \in D$ one can define $f : A \to (Z_d \times Z_e) \otimes \mathcal{K} \cong (Z_d \otimes \mathcal{K}) \times (Z_e \otimes \mathcal{K})$ by $f(a) = (f_d(a), f_e(a))$ and define $Z = Z_f$. The topological *-algebra Z then is an $l.m.c.$-C^*-algebra and the *-homomorphism f can be regarded as a *-homomorphism into $Z \otimes \mathcal{K}$, with (Z, f) bigger than d and e respectively by construction. Now define $A \boxtimes \mathcal{K}$ to be the ν-completion of the inverse limit $\lim_{d \in D} Z_d$.

Proposition 6.3. *For $A, B \in \mathbb{A}_\nu$ there is a natural continuous bijection*

$$Hom(A \boxtimes \mathcal{K}, B) \to Hom(A, B \otimes \mathcal{K}) \tag{6.1}$$

which induces an isomorphism on homotopy classes

$$[A \boxtimes \mathcal{K}, B] \to [A, B \otimes \mathcal{K}] \tag{6.2}$$

Proof. First notice the statement concerning homotopy classes follows by simply considering $\mathcal{C}(I, B)$ as well as B, exploiting the naturality.

By the previous lemma, taking the tensor product with the compact operators $\mathcal{K}$ commutes with inverse limits. Moreover, taking the tensor product with $\mathcal{K}$ induces a natural continuous map $Hom(A, B) \to Hom(A \otimes \mathcal{K}, B \otimes \mathcal{K})$. Thus, there is a corresponding continuous map $Hom(\lim_d Z_d, B) \to Hom(\lim_d Z_d \otimes \mathcal{K}, B \otimes \mathcal{K})$. On the other hand, there is the canonical continuous *-homomorphism $A \to \lim_d Z_d \otimes \mathcal{K}$, whose composition with the projection to the factor $Z_d \otimes \mathcal{K}$ corresponding to the index $d = (Z_d, f_d : A \to Z_d \otimes \mathcal{K})$ is the *-homomorphism f_d. The map (6.1) then is defined as the composition of the continuous maps

$$Hom(\lim_d Z_d, B) \to Hom(\lim_d Z_d \otimes \mathcal{K}, B \otimes \mathcal{K}) \to Hom(A, B \otimes \mathcal{K}),$$

where the second map is induced by precomposing a *-homomorphism with the canonical *-homomorphism $A \to \lim_d Z_d \otimes \mathcal{K}$. In particular, (6.1) is continuous by construction.

That the map described yields a bijection essentially corresponds to an application of Freyd's Adjoint Functor Theorem ([**Bor94**, Theorem 3.3.3 on page 109]). Instead of reducing to the abstract result, we give a direct proof.

To see surjectivity, let $f : A \to B \otimes \mathcal{K}$ be a continuous *-homomorphism. Let Z be the topological sub-*-algebra of B which is generated by the set of elements $\bigcup_{i,j \in \mathbb{N}, a \in A} x_{ij}^B(f(a))$. Then f can be factored through $Z \otimes \mathcal{K}$, so that (Z, f) can be regarded as an index in D. The *-homomorphism $f : A \to B \otimes \mathcal{K}$ then is the image of the composite *-homomorphism

$$\lim_d Z_d \to Z \to B$$

which is given by the projection corresponding to the index (Z, f) followed by the inclusion $Z \subset B$.

To see injectivity, we first show that any *-homomorphism $\lim_d Z_d \to B$ is given by the composition of some projection $\lim_d Z_d \to Z_d$ followed by an inclusion $Z_d \subset B$. Given any *-homomorphism $\lim_d Z_d \to B$, let Z' be its topological image and apply $? \otimes \mathcal{K}$. The corresponding *-homomorphism $\lim_d Z_d \otimes \mathcal{K} \to B \otimes \mathcal{K}$ then factors as the composition

$$\lim_d Z_d \otimes \mathcal{K} \to Z' \otimes \mathcal{K} \to B \otimes \mathcal{K}.$$

When we precompose this *-homomorphism with the canonical *-homomorphism $A \to \lim_d Z_d \otimes \mathcal{K}$ we obtain a *-homomorphism $f : A \to B \otimes \mathcal{K}$. Define the index $d = (Z, f)$, where Z is the topological *-algebra generated by the set $\bigcup_{i,j \in \mathbb{N}, a \in A} x_{ij}^{Z'} f(a)_{ij}$ where (abusing notation) the range of f is restricted to $Z' \otimes \mathcal{K}$. It follows that the *-homomorphism $\lim_d Z_d \otimes \mathcal{K} \to Z' \otimes \mathcal{K}$ can be factored as

$$\lim_d Z_d \otimes \mathcal{K} \to Z \otimes \mathcal{K} \to Z' \otimes \mathcal{K},$$

where the first *-homomorphism is induced by the projection $\lim_d Z_d \to Z$. Since the composition $\lim_d Z_d \to Z'$ is surjective and $Z \to Z'$ is injective, it follows that

$Z \cong Z'$, so $\lim_d Z_d \to B$ indeed is given by the composition of a projection followed by an inclusion. Moreover, the index d as well as the corresponding inclusion $Z \subset B$ are uniquely determined by the *-homomorphism $f : A \to B \otimes \mathcal{K}$. Thus, the map (6.1) is injective. $\square$

PROPOSITION 6.4. *Let $A \in C^*\text{-}alg$, $B \in \mathbb{A}_\nu$, $X \in Cpt$ and $Y \in Cpt_*$. Then there are natural homeomorphisms*

$$Hom(A \boxtimes \mathcal{K}, \mathcal{C}(X, B)) \to \mathcal{C}(X, Hom(A \boxtimes \mathcal{K}, B)), \tag{6.3}$$

$$Hom(A \boxtimes \mathcal{K}, \mathcal{C}_*(Y, B)) \to \mathcal{C}_*(Y, Hom(A \boxtimes \mathcal{K}, B)), \tag{6.4}$$

and the following two maps, given by composing a map with the map (6.1)*,*

$$\mathcal{C}(X, Hom(A \boxtimes \mathcal{K}, B)) \to \mathcal{C}(X, Hom(A, B \otimes \mathcal{K})), \tag{6.5}$$

$$\mathcal{C}_*(Y, Hom(A \boxtimes \mathcal{K}, B)) \to \mathcal{C}_*(Y, Hom(A, B \otimes \mathcal{K})), \tag{6.6}$$

are continuous bijections.

PROOF. First, notice the statements for the pointed setting follow from the corresponding unpointed version, as in the proof of Proposition 3.4. Also by Proposition 3.4, the map (6.3) is a topological inclusion, while the map (6.5) certainly is continuous. Thus, we need to show that the two maps are bijections.

Consider the following commutative diagram, where the unlabeled vertical arrows are induced by the continuous bijections (6.1) and where the horizontal arrows are induced by the maps (3.1) of Proposition 3.4

$$\begin{tikzcd}
Hom(A \boxtimes \mathcal{K}, \mathcal{C}(X, B)) \arrow[r] \arrow[d] & \mathcal{C}(X, Hom(A \boxtimes \mathcal{K}, B)) \arrow[dd] \\
Hom(A, \mathcal{C}(X, B) \otimes \mathcal{K}) \arrow[d, "\overset{3.2}{\cong}"] & \\
Hom(A, \mathcal{C}(X, B \otimes \mathcal{K})) \arrow[r] & \mathcal{C}(X, Hom(A, B \otimes \mathcal{K})).
\end{tikzcd}$$

From Proposition 6.3, it follows that the first left vertical arrow is a bijection. Therefore, the composition of the two left vertical arrows is also a bijection by Lemma 3.2. Proposition 6.3 in addition implies that the right vertical arrow is an injective map, and the lower horizontal arrow is a bijection by Proposition 3.4. It follows that the right vertical arrow is a bijection, therefore the upper horizontal arrow must also be a bijection. $\square$

PROPOSITION 6.5. *For $A \in C^*\text{-}alg$ and $B \in \mathbb{A}_\nu$ the canonical bijection* (6.1)

$$Hom(A \boxtimes \mathcal{K}, B) \to Hom(A, B \otimes \mathcal{K})$$

is a weak homotopy equivalence.

PROOF. Given a choice of a basepoint for $Hom(A \boxtimes \mathcal{K}, B)$ we regard $Hom(A \boxtimes \mathcal{K}, B)$ as a pointed space, and (6.1) as a pointed map. We need to show that the maps

$$\pi_n(Hom(A \boxtimes \mathcal{K}, B)) \to \pi_n(Hom(A, B \otimes \mathcal{K}))$$

are isomorphisms for any choice of a basepoint in $Hom(A \boxtimes \mathcal{K}, B)$.

This immediately follows from the previous proposition. To see surjectivity let $f' \in \mathcal{C}_*(S^n, Hom(A, B \otimes \mathcal{K}))$ be a pointed map. Consider the map (6.5) for $X = S^n$. The pre-image $f \in \mathcal{C}(S^n, Hom(A \boxtimes \mathcal{K}, B))$ of $f' \in \mathcal{C}(S^n, Hom(A, B \otimes \mathcal{K}))$ then is a pointed map representing a pre-image $[f] \in \pi_n(Hom(A \boxtimes \mathcal{K}, B))$ of $[f'] \in \pi_n(Hom(A, B \otimes \mathcal{K}))$.

Injectivity follows similarly. Let $f, g : S^n \to Hom(A \boxtimes \mathcal{K}, B)$ be pointed maps and assume that their images $f', g' : S^n \to Hom(A, B \otimes \mathcal{K})$ are pointed homotopic. Let $H' : S^n \times I \to Hom(A, B \otimes \mathcal{K})$ be a pointed homotopy between f' and g'. Now consider the map (6.5) for $X = S^n \times I$. The pre-image of H' then yields a pointed homotopy $H : S^n \times I \to Hom(A \boxtimes \mathcal{K}, B)$ between f and g. □

We now show that for any $A \in C^*\text{-}alg$ the functor

$$\mathbb{L}_{A\boxtimes\mathcal{K}} = (A \boxtimes \mathcal{K}) \circledS ? : \mathrm{Cpt}_* \to \mathbb{A}_\nu$$

is a partial left adjoint to the functor

$$\mathbb{R}_{A\boxtimes\mathcal{K}} = \mathrm{Hom}(A \boxtimes \mathcal{K}, ?) : \mathbb{A}_\nu \to \mathrm{Top}_*.$$

PROPOSITION 6.6. *For $X \in Cpt_*$, $A \in C^*\text{-}alg$ and $B \in \mathbb{A}_\nu$, there is a natural continuous bijection*

$$Hom((A \boxtimes \mathcal{K}) \circledS X, B) \cong \mathcal{C}_*(X, Hom(A \boxtimes \mathcal{K}, B))\,.$$

PROOF. The bijection is obtained by composing the continuous bijection

$$Hom((A \boxtimes \mathcal{K}) \circledS X, B) \to Hom(A \boxtimes \mathcal{K}, \mathcal{C}_*(X, B))$$

provided by Proposition 5.4 with the homeomorphism (6.4). □

The statements of the following discussion (in particular those of Proposition 6.11 and Proposition 6.12) will be needed in order to identify the homotopy classes of maps of the model categories we are going to define in Section 9.

NOTATION 6.7. Let $p : \mathbb{C} \to \mathcal{K}$ be a *-homomorphism which maps the unit $1 \in \mathbb{C}$ to a rank one projection. For any $C \in \mathbb{A}_\nu$, let $p_C : C \approx C \otimes \mathbb{C} \to C \otimes \mathcal{K}$ be the map induced by tensoring id_C with p, and let $q_C : C \boxtimes \mathcal{K} \to C$ be its adjoint.

PROPOSITION 6.8. *For any $A, B \in \mathbb{A}_\nu$ composing with q_B yields a natural isomorphism*

$$[A \boxtimes \mathcal{K}, B \boxtimes \mathcal{K}] \to [A \boxtimes \mathcal{K}, B].$$

We need a few technical facts before we proceed to prove the proposition.

LEMMA 6.9.

1. *The map $p_\mathcal{K} : \mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$ is homotopic to an isomorphism.*
2. *Any two isomorphisms $\mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$ are homotopic.*
3. *For $B, B' \in \mathbb{A}_\nu$, precomposing with $p_B : B \to B \otimes \mathcal{K}$ induces a natural isomorphism $[B \otimes \mathcal{K}, B' \otimes \mathcal{K}] \to [B, B' \otimes \mathcal{K}]$.*

PROOF. All three statements are well-known. Statements (1) and (2) follow for example from [**Mey00**, Lemma 4.3]. For the third statement, choose any isomorphism $\phi : \mathcal{K} \otimes \mathcal{K} \to \mathcal{K}$. Using (1) and (2), one can show that sending a *-homomorphism $f : B \to B' \otimes \mathcal{K}$ to the *-homomorphism $(id_{B'} \otimes \phi) \circ (f \otimes id_\mathcal{K}) : B \otimes \mathcal{K} \to B' \otimes \mathcal{K} \otimes \mathcal{K} \to B' \otimes \mathcal{K}$ induces an inverse to the map considered in (3). □

PROOF OF PROPOSITION 6.8. Use Lemma 6.9 and Proposition 6.3 to see that there is a natural sequence of isomorphisms

$$\begin{aligned}[A \boxtimes \mathcal{K}, B] &\overset{(6.2)}{\cong} [A, B \otimes \mathcal{K}] \\ &\overset{(p_{B\otimes\mathcal{K}})_*}{\cong} [A, B \otimes \mathcal{K} \otimes \mathcal{K}] \\ &\overset{(6.2)}{\cong} [A \boxtimes \mathcal{K}, B \otimes \mathcal{K}] \\ &\overset{6.9(3)}{\cong} [(A \boxtimes \mathcal{K}) \otimes \mathcal{K}, B \otimes \mathcal{K}].\end{aligned}$$

Thus, in order to prove Proposition 6.8 it suffices to show that the corresponding map

$$[(A \boxtimes \mathcal{K}) \otimes \mathcal{K}, (B \boxtimes \mathcal{K}) \otimes \mathcal{K}] \to [(A \boxtimes \mathcal{K}) \otimes \mathcal{K}, B \otimes \mathcal{K}]$$

is an isomorphism.

To see this, consider the map

$$[A' \otimes \mathcal{K}, B' \otimes \mathcal{K}] \to [(A' \boxtimes \mathcal{K}) \otimes \mathcal{K}, B' \otimes \mathcal{K}] \tag{6.7}$$

induced by precomposing with $q_{A'} \otimes id_{\mathcal{K}} : (A' \boxtimes \mathcal{K}) \otimes \mathcal{K} \to A' \otimes \mathcal{K}$. It is a natural isomorphism since, as above, it is given through a composition of isomorphisms

$$\begin{aligned}[A' \otimes \mathcal{K}, B' \otimes \mathcal{K}] &\cong [A', B' \otimes \mathcal{K}] \\ &\cong [A', B' \otimes \mathcal{K} \otimes \mathcal{K}] \\ &\cong [A' \boxtimes \mathcal{K}, B' \otimes \mathcal{K}] \\ &\cong [(A' \boxtimes \mathcal{K}) \otimes \mathcal{K}, B' \otimes \mathcal{K}].\end{aligned}$$

If we take $A' = B' = B$ then, by construction, (6.7) maps the homotopy class $[id_{B\otimes\mathcal{K}}]$ to $[q_B \otimes id_{\mathcal{K}}]$. On the other hand, if we take $A' = B$ and $B' = B \boxtimes \mathcal{K}$ the isomorphism guarantees that we can pick a map $g : B \otimes \mathcal{K} \to (B \boxtimes \mathcal{K}) \otimes \mathcal{K}$ such that the composition $g \circ f$ with $f = q_B \otimes id_{\mathcal{K}}$ is homotopic to $id_{(B\boxtimes\mathcal{K})\otimes\mathcal{K}}$. The map g then is a homotopy inverse to f, since the map (6.7) for $A' = B' = B$ maps $[f \circ g]$ to $[f \circ g \circ f] = [f \circ id_{(B\boxtimes\mathcal{K})\otimes\mathcal{K}}] = [f]$, and therefore $[f \circ g] = [id_{B\otimes\mathcal{K}}]$, using the injectivity of (6.7). It follows that composing with g induces an isomorphism $[A', B \otimes \mathcal{K}] \to [A', (B \boxtimes \mathcal{K}) \otimes \mathcal{K}]$ for any $A' \in \mathbb{A}_\nu$, in particular for $A' = (A \boxtimes \mathcal{K}) \otimes \mathcal{K}$. □

DEFINITION 6.10. For $A \in \mathbb{A}_\nu$, let qA be the kernel of the fold map $A * A \to A$.

In order to prove the following two propositions below we first need to recall the well-known fact that the space $Hom(qA, B\otimes\mathcal{K})$ is a group-like H-space, for any choice of $B \in \mathbb{A}_\nu$. The H-space structure is induced by the H-space structure of $\mathcal{K}$; and the structure is group-like with homotopy inverse given by the automorphism τ on qA induced by switching the two factors of $A * A$ ([**Cun87**, Proposition 1.4]). To be more precise, let $\psi : M_2 \otimes \mathcal{K} \cong \mathcal{K}$ be an isomorphism. It induces a natural isomorphism $\psi_* : M_2(B \otimes \mathcal{K}) \cong B \otimes \mathcal{K}$. Given *-homomorphisms $f, g : qA \to B \otimes \mathcal{K}$ then the sum of their homotopy classes is defined as the homotopy class of $\psi_*(diag(f, g))$. To see that τ induces a homotopy inverse first note that the map representing $[f] + [f \circ \tau]$ can be decomposed as follows

$$qA \overset{diag(id,\tau)}{\longrightarrow} M_2(qA) \overset{M_2(f)}{\longrightarrow} M_2(B \otimes \mathcal{K}) \overset{\psi_*}{\cong} B \otimes \mathcal{K},$$

so that the assertion follows from $diag(id, \tau)$ being null homotopic. An explicit null homotopy is obtained by restricting $(u, v) : A * A \to \mathcal{C}([0,1], M_2(A * A))$ given by the maps $u_t, v_t : A \to M_2(A * A),\ t \in [0,1]$,

$$u_t(a) = \begin{pmatrix} i_1(a) & 0 \\ 0 & i_2(a) \end{pmatrix}, \quad v_t(a) = A(t) \begin{pmatrix} i_1(a) & 0 \\ 0 & i_2(a) \end{pmatrix} A(t)^*$$

$$\text{with } A(t) = \begin{pmatrix} \cos(\pi t/2) & \sin(\pi t/2) \\ -\sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix},$$

where the *-homomorphisms $i_1, i_2 : A \to A * A$ refer to the inclusion of A into the first and into the second factor respectively of the free product.

Recall that a map $X \to X'$ of group-like H-spaces is a weak homotopy equivalence if and only if it is a π_*-isomorphism (with the existing basepoint).

PROPOSITION 6.11. *Let $A \in C^*$-alg, and let $B \to B'$ a morphism in $\mathbb{A}_\nu$. Then the induced map*

$$Hom(qA \boxtimes \mathcal{K}, B) \to Hom(qA \boxtimes \mathcal{K}, B') \tag{6.8}$$

is a weak equivalence if and only if the map $Hom(qA, B \otimes \mathcal{K}) \to Hom(qA, B' \otimes \mathcal{K})$ is a π_*-isomorphism.

In case A is separable, and B and B' are σ-unital C^*-algebras, the two equivalent assertions above are equivalent to the induced map $KK_*(A, B) \to KK_*(A, B')$ being an isomorphism.

The last assertion we have included here for later reference.

PROOF. The equivalence of the first two statements follows from Proposition 6.5 and the fact that the map $Hom(qA, B \otimes \mathcal{K}) \to Hom(qA, B' \otimes \mathcal{K})$ is a map of group-like H-spaces.

The last statement follows from the existence of a natural isomorphism of groups (which essentially is Cuntz' theorem)

$$\pi_* Hom(qA, B \otimes \mathcal{K}) \cong KK_*(A, B).$$

This arises as a string of natural isomorphisms

$$\begin{aligned} \pi_n Hom(qA, B \otimes \mathcal{K}) &\cong \pi_0 Hom(qA, \mathcal{C}_* (S^n, B \otimes \mathcal{K})) \\ &\cong \pi_0 \mathcal{C}_* (Hom(qA, \mathcal{C}_* (S^n, B) \otimes \mathcal{K})) \\ &\cong KK_0(A, \mathcal{C}_* (S^n, B)) \\ &\cong KK_n(A, B). \end{aligned}$$

The first isomorphism results from Proposition 3.4, the second isomorphism is induced by the isomorphism $\mathcal{C}_* (S^n, B) \otimes \mathcal{K} \cong \mathcal{C}_* (S^n, B \otimes \mathcal{K})$ of Lemma 3.2, the third one is Cuntz's Theorem 1.1, and the last one is standard.

□

PROPOSITION 6.12. *For $A \in C^*$-alg, $B \in \mathbb{A}_\nu$ the map induced by composing with $q_B : B \boxtimes \mathcal{K} \to B$ (defined in 6.7)*

$$Hom(qA \boxtimes \mathcal{K}, B \boxtimes \mathcal{K}) \to Hom(qA \boxtimes \mathcal{K}, B)$$

is a weak homotopy equivalence.

PROOF. By the previous proposition it suffices to show that the corresponding map $Hom(qA, (B \boxtimes \mathcal{K}) \otimes \mathcal{K}) \to Hom(qA, B \otimes \mathcal{K})$ is a π_*-isomorphism.

Note that for any $Y \in \mathrm{Cpt}_*$ and a general $B' \in \mathbb{A}_\nu$ we have the following natural composition of bijections

$$\begin{aligned} Hom((qA \circledast Y) \boxtimes \mathcal{K}, B') &\overset{(6.1)}{\cong} Hom(qA \circledast Y, B' \otimes \mathcal{K}) \\ &\overset{(5.2)}{\cong} Hom(qA, \mathcal{C}_*(Y, B' \otimes \mathcal{K})) \overset{(3.2)}{\cong} \mathcal{C}_*(Y, Hom(qA, B' \otimes \mathcal{K})). \end{aligned}$$

Using naturality in Y it follows that for any $n \geq 0$ there is a canonical bijection

$$[(qA \circledast S^n) \boxtimes \mathcal{K}, B'] \cong [S^n, Hom(qA, B' \otimes \mathcal{K})]_R,$$

where the right hand side denotes right homotopy classes of maps, i.e. two pointed maps $f, g : S^n \to Hom(qA, B' \otimes \mathcal{K})$ are equivalent if there is a corresponding pointed map $H : S^n \to Hom(qA, \mathcal{C}(I, B' \otimes \mathcal{K}))$ such that the maps $S^n \to Hom(qA, B' \otimes \mathcal{K})$ obtained from H by composition with the evaluation maps $\mathcal{C}(I, B' \otimes \mathcal{K}) \to B' \otimes \mathcal{K}$ at 0 and 1 respectively are the maps f and g. However, using Lemma 3.3 and Proposition 3.4 one gets $\pi_n(Hom(qA, B' \otimes \mathcal{K})) \cong [S^n, Hom(qA, B' \otimes \mathcal{K})]_R$. Thus, taken together we have canonical isomorphisms

$$\pi_n(Hom(qA, B' \otimes \mathcal{K})) \cong [(qA \circledast S^n) \boxtimes \mathcal{K}, B'].$$

The statement of the proposition then follows from Proposition 6.8, applied to $qA \circledast S^n$ for the various $n \geq 0$. □

7. The Seminorm Extension Property

The point of this section is to introduce a class of maps which will play roughly the role of embeddings in topological categories.

DEFINITION 7.1. Let $f : A \to B$ be a continuous *-homomorphism between *l.m.c.*-C^*-algebras. We say f has the seminorm lifting property if for any C^*-seminorm $p \in S(A)$ there is a C^*-seminorm $q \in S(B)$ such that $p = f^*q$. In other words f has the semi-norm extension property if and only if $S(A) = f^*S(B)$.

First, we have the key property of maps with the seminorm extension property.

LEMMA 7.2. *All objects in $\mathbb{L}$ are small with respect to *-homomorphisms which have the seminorm extension property.*

A corresponding statement for colimits in the category $\mathbb{A}_\nu$ which will allow the so-called small object argument to proceed in Section 8 is the content of Lemma 7.5 below.

Before giving a proof of Lemma 7.2, we need a better understanding of the seminorm extension property.

LEMMA 7.3. *Let A, B be l.m.c.-C^*-algebras. If a *-homomorphism $f : A \to B$ has the seminorm extension property, then f is a topological inclusion.*

PROOF. If $x \neq y \in A$, then there exists $p \in S(A)$ with $p(x - y) \neq 0$ because A carries a Hausdorff *l.m.c.*-C^*-topology. Since f has the seminorm extension property, there exists $q \in S(B)$ with $f^*q = p$. Thus, $qf(x - y) = p(x - y) \neq 0$ so $f(x) \neq f(y)$ in B, or f is an injection.

To see that f is an inclusion, note that A equipped with the topology pulled back from B over f will be $A_{f^*S(B)}$. However, since f has the seminorm extension property we have $f^*S(B) = S(A)$ which implies $A_{f^*S(B)} = A_{S(A)} = A$, the last equality from the fact that A is an *l.m.c.*-C^*-algebra. □

LEMMA 7.4. *Let λ be an ordinal and suppose*

$$A_0 \to A_1 \to A_2 \to \dots \to A_\alpha \to \dots$$

is a λ-sequence[1] *over *-homomorphisms $A_\alpha \to A_{\alpha+1}$ in $\mathbb{L}$ which have the seminorm extension property. Then the underlying set of the colimit in $\mathbb{L}$ is the union of the underlying sets of the A_α, and each*

$$A_\alpha \to \operatorname*{colim}_{\alpha<\lambda} A_\alpha$$

has the seminorm extension property .

PROOF. By the previous lemma, all structure maps in the sequence are topological inclusions. Hence the colimit of the sequence in the category of associative complex *-algebras is given by the union $A = \bigcup_{\alpha<\lambda} A_\alpha$. On this union, there is a coarsest *l.m.c.*-C^*-topology and the colimit in $\mathbb{L}$ then is given by taking the Hausdorff quotient of A with respect to this topology. We now show that the topology on the union A is already Hausdorff.

We claim that any two distinct points in A are separated by a seminorm. To see this, let $x \in A_{\alpha'}$ and $y \in A_\alpha$ denote two distinct points of the union. Without loss of generality, assume $\alpha \geq \alpha'$ so $x \in A_\alpha$ as well. Since A_α is a Hausdorff *l.m.c.*-C^*-algebra, there is a C^*-seminorm $p \in S(A_\alpha)$ with $p(x-y) \neq 0$. We extend this to $q \in S(A)$ by transfinite induction. By assumption, we can extend any $p_\alpha \in S(A_\alpha)$ to $p_{\alpha+1} \in S(A_{\alpha+1})$ and this handles successor ordinals. Now, for a limit ordinal $\beta < \lambda$, we define a seminorm on the union $\bigcup_{\alpha<\beta} A_\alpha$ using the universal property of the union with respect to continuous maps and the continuous C^*-seminorms already defined. This yields a continuous map $\bigcup_{\alpha<\beta} A_\alpha \to \mathbb{R}$ which acts as a seminorm since each pair of points in the union lies in some A_α as above, thereby completing the transfinite induction. We now have $q \in S(A)$ with $q(x-y) = p(x-y) \neq 0$ for any pair of distinct points in A. Hence, $A_{S(A)}$ is a Hausdorff space, so taking the Hausdorff quotient is not necessary in constructing the colimit in $\mathbb{L}$. This implies the underlying set of the colimit in $\mathbb{L}$ is precisely A, the union of the sets A_α.

Given an arbitrary C^*-seminorm $p \in S(A_\alpha)$ for some index α we can use the same procedure as above to obtain an extending C^*-seminorm $q \in S(\operatorname{colim}_\alpha A_\alpha)$, thus each of the maps $A_\alpha \to \operatorname{colim}_\alpha A_\alpha$ has the seminorm extension property. □

We can now give a proof of Lemma 7.2.

PROOF OF LEMMA 7.2. By the combination of Lemmas 7.3 and 7.4 each of the morphisms in a λ-sequence $A_\alpha \to \operatorname*{colim}_{\alpha<\lambda} A_\alpha$ is a topological inclusion. Hence it suffices to see that any object $B \in \mathbb{L}$, regarded as a set, is small with respect to set inclusions and that the underlying set of such a colimit is the colimit of the underlying sets. The former fact is well known (see [**Hov99**, Example 2.1.5]), while the latter is part of Lemma 7.4. □

We can now verify the version of Lemma 7.2 we will use to perform our small object arguments.

[1]For a definition of "λ-sequence into a cocomplete category" we refer the reader to [**Hov99**, 2.1.1.]

LEMMA 7.5. *All objects in $\mathbb{A}_\nu$ are small with respect to *-homomorphisms which have the seminorm extension property.*

PROOF. Let λ be an ordinal and suppose

$$A_0 \to A_1 \to A_2 \to \cdots \to A_\alpha \to \dots$$

is a λ-sequence over *-homomorphisms $A_\alpha \to A_{\alpha+1}$ in $\mathbb{A}_\nu$ which have the seminorm extension property. Let us write C for the colimit over this sequence taken in the category $\mathbb{L}$ which we considered in Lemma 7.4. The colimit in $\mathbb{A}_\nu$ then is the ν-completion $\overline{(C)}^\nu$ of C. In particular, the maps from A_α into the colimit $\overline{(C)}^\nu$ are all inclusions, and the elements of $\overline{(C)}^\nu$ by definition can be represented by Cauchy ν-sequences of elements of C.

For any *-homomorphism $f : B \to \overline{(C)}^\nu$ and $b \in B$ let $(f_{b,i})_{i\in\nu}$ be a Cauchy sequence in C which represents $f(b)$. By Lemma 7.4, for each pair (b, i) we can choose an ordinal $\alpha_{b,i} < \lambda$ such that $f_{b,i} \in A_{\alpha_{b,i}}$. Now assume that λ is $|B| \times |\nu|$-filtered (see [**Hov99**, Definition 2.1.2]). Let $\gamma = \sup\{\alpha_{b,i} \mid (b, i) \in B \times \nu\}$, so $\gamma < \lambda$ and $f(B) \subset A_\gamma$. Since f was arbitrary, we conclude $Hom(B, \overline{(C)}^\nu) \cong \operatorname{colim}_\alpha Hom(B, A_\alpha)$. □

REMARK 7.6. The previous lemma is the reason why we work with ν-complete *l.m.c.*-C^*-algebras instead of complete *l.m.c.*-C^*-algebras. Roughly speaking, it was crucial in the proof that we have an a priori set theoretic bound for the length of the Cauchy nets we are considering for the completion.

To apply Lemma 7.5, we will need to know certain formal constructions preserve the seminorm extension property. As a straight consequence of Lemma 7.4 transfinite compositions preserve the seminorm extension property and the next lemma gives a left cancellation result.

LEMMA 7.7. *Suppose $f : A \to B$ and $g : B \to C$ are continuous *-homomorphisms. Then $gf : A \to C$ having the seminorm extension property implies that f has the seminorm extension property.*

PROOF. Suppose $p \in S(A)$ and $q \in S(C)$ with $(gf)^*q = p$. Then $f^*g^*q = p$ so $g^*q \in S(B)$ is the required extension of p over f. □

LEMMA 7.8. *Let*

$$B \xleftarrow{f} A \xrightarrow{g} C$$

be a diagram in $\mathbb{A}_\nu$ with $f : A \to B$ having the seminorm extension property. Then the canonical map $h : C \to B *_A C$ has the seminorm extension property.

PROOF. Suppose $p \in S(C)$ and let $r = g^*p \in S(A)$ with $q \in S(B)$ an extension of r over f. Now let C_1 denote the completion of C with respect to p (or $\overline{C_p}$ from below Lemma 2.5), B_1 the completion of B with respect to q and A_1 the completion of A with respect to r. This implies A_1, B_1 and C_1 are C^*-algebras and the completed versions, f_1 and g_1 of f and g respectively, have become isometries. Now take the pushout in the category of C^*-algebras,

$$\begin{array}{ccc} A_1 & \xrightarrow{g_1} & C_1 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle h_1} \\ B_1 & \longrightarrow & P \end{array}$$

so that P represents the appropriate quotient of the amalgamated free product of C^*-algebras. We now have a commutative diagram of continuous *-homomorphisms

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[r] \arrow[d, "h"] & C_1 \arrow[dd, "h_1"] \\
B \arrow[r] \arrow[d] & B *_A C & \\
B_1 \arrow[rr] & & P
\end{tikzcd}$$

hence an induced continuous *-homomorphism from the universal property of $B*_A C$ which makes the diagram

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "h"'] & C_1 \arrow[d, "h_1"] \\
B *_A C \arrow[r] & P
\end{tikzcd}$$

commute. Since f_1 and g_1 are isometries, the induced *-homomorphism $h_1 : C_1 \to P$ is an isometry as well. In particular, the C^*-norm of P extends that of C_1, so commutativity of the last diagram implies pulling back the C^*-norm on P to a C^*-seminorm on $B *_A C$ extends the original C^*-seminorm $p \in S(C)$ as required. □

Corollary 7.9. *Let I be a set, and let $\{f_i\}_{i\in I}$ be a set of morphisms $f_i : A_i \to B_i$ in $\mathbb{A}_\nu$ which all have the seminorm extension property. Then the canonical map $*_{i\in I} f_i : *_{i\in I} A_i \to *_{i\in I} B_i$ of coproducts in $\mathbb{A}_\nu$ has the seminorm extension property.*

Proof. Suppose $f : A \to B$ and $g : C \to D$ are morphisms in $\mathbb{A}_\nu$ which have the seminorm extension property. Notice $B * C \approx B *_A (A * C)$, so the diagram

$$B \xleftarrow{f} A \longrightarrow A * C$$

and Lemma 7.8 imply $A * C \to B * C$ has the seminorm extension property. Since the symmetric argument implies $B * C \to B * D$ also has the seminorm extension property, and composition clearly preserves the seminorm extension property, we conclude that $A * C \to B * D$ has the seminorm extension property.

The previous paragraph is the successor ordinal case and suggests the general case may be considered as a transfinite composition after any choice of ordering for I. Hence, the limit ordinal case follows from Lemma 7.4, and the claim follows by transfinite induction. □

The key to applying the results on the seminorm extension property to the expected model structures is the next proposition. We'll take zero as the basepoint of I in general.

Definition 7.10. Let A be an algebra in $\mathbb{A}_\nu$. We say A has the cone seminorm extension property if the canonical morphism $i_1 : A \to A \circledS I$ has the seminorm extension property. If $i_1 : A \circledS X \to (A \circledS X) \circledS I$ has the seminorm extension property for any $X \in \mathrm{Cpt}_*$ we will say that A has the stable cone seminorm extension property.

REMARK 7.11. The proof of the following proposition relies upon a result in KK-theory. Namely, for any separable $A \in C^*$-alg, any Hilbert space $\mathcal{H}$ and any $X \in \mathrm{Cpt}_*$ one has $KK(A, \mathcal{C}_*(X, \mathcal{B}(\mathcal{H}))) = 0$. This follows from an Eilenberg-swindle type of argument.

For the next statement recall from Definition 6.10 that qA for an $A \in \mathbb{A}_\nu$ denotes the kernel of the fold map $A * A \to A$.

PROPOSITION 7.12. *If A is a separable C^*-algebra, then $qA \boxtimes \mathcal{K}$ has the stable cone seminorm extension property.*

PROOF. For ease of reference, let $B = (qA \boxtimes \mathcal{K}) \oslash X$ and suppose $p \in S(B)$. We must show there is an extension to $q \in S(B \oslash I)$ over the map $i_1 : B \to B \oslash I$. Clearly, there is a composite morphism in $\mathbb{A}_\nu$

$$\psi : B \to \overline{B_p} \to \mathcal{B}(\mathcal{H})$$

with p the pullback under ψ of the norm on $\mathcal{B}(\mathcal{H})$ (since $\overline{B_p}$ is a C^*-alg). Using Proposition 5.4 and Proposition 6.3, this corresponds to a unique *-homomorphism between C^*-algebras

$$\psi' : qA \to \mathcal{C}_*(X, \mathcal{B}(\mathcal{H})) \otimes \mathcal{K}.$$

Since $KK(A, \mathcal{C}_*(X, \mathcal{B}(\mathcal{H})))$ is trivial, Cuntz's Theorem 1.1 implies that ψ' is null-homotopic. Hence, there is a corresponding null homotopy

$$H : qA \to \mathcal{C}_*(I, \mathcal{C}_*(X, \mathcal{B}(\mathcal{H})) \otimes \mathcal{K})$$

of *-homomorphisms. Recall that there is an isomorphism of C^*-algebras

$$\mathcal{C}_*(I, \mathcal{C}_*(X, \mathcal{B}(\mathcal{H})) \otimes \mathcal{K}) \approx \mathcal{C}_*(X, \mathcal{C}_*(I, \mathcal{B}(\mathcal{H}) \otimes \mathcal{K}))$$

since both $X, I \in \mathrm{Cpt}_*$. Using Proposition 5.4 and Proposition 6.3 again, H yields a corresponding *-homomorphism

$$H' : B \oslash I \to \mathcal{B}(\mathcal{H})$$

which factors ψ as follows

$$\psi : B \xrightarrow{i_1} B \oslash I \xrightarrow{H'} \mathcal{B}(\mathcal{H}).$$

Hence, the pullback of the norm on $\mathcal{B}(\mathcal{H})$ along H' yields the desired $q \in S(B \oslash I)$ extending p over i_1. □

8. Building Model Structures on $\mathbb{A}_\nu$

A standard technique in model categories involves building a new model structure from a known model structure together with an adjoint pair. This technique is referred to as lifting, and results saying when such operations are successful are generally referred to as lifting lemmas.

For the basic object(s) $A \in \mathbb{A}_\nu$, we will make the following two assumptions throughout this section.

(1) $\mathbb{R}_A = \mathrm{Hom}(A, ?)$ has $\mathbb{L}_A = A \oslash ?$ as a partial left adjoint
(2) there is a natural homeomorphism

$$\mathbb{R}_A(\mathcal{C}_*(X, B)) \to \mathcal{C}_*(X, \mathbb{R}_A(B))$$

We would like to give a lifting lemma under these assumptions, so we should begin by pointing out that we have a variety of objects which satisfy both assumptions. We know that for any separable $A' \in C^*\text{-}alg$, the object $A = qA' \boxtimes \mathcal{K}$ satisfies assumption (1) by Proposition 6.6 and assumption (2) by Proposition 6.4 equation (6.4). Near the end of this section, we will also need to assume A has the stable cone seminorm extension property in order to produce our model structures, which holds for $A = qA' \boxtimes \mathcal{K}$ by Proposition 7.12.

Although this lifting technique is standard, the proofs are provided in detail because we do not quite consider an adjoint pair. (Compare with [**IJ02**, Section 8].) The following is not particularly surprising but is a technical necessity for the argument that follows.

Lemma 8.1.

(1) *Suppose A satisfies our second assumption. Then for any $B \in \mathbb{A}_\nu$ and $X \in Cpt_*$, there is a natural pointed continuous map*

$$\mathbb{R}_A(B) \wedge X \to \mathbb{R}_A(B \owedge X).$$

In fact, this is given by $f \wedge x \longmapsto i_x \circ f$.

(2) *For any $B \in \mathbb{A}_\nu$ and $X,Y \in Cpt_*$, there is a natural *-homomorphism*

$$(B \owedge X) \owedge Y \to B \owedge (X \wedge Y).$$

Proof. Recall from the construction of $\owedge$ that there is a natural pointed continuous map $j : B \wedge X \to B \owedge X$. Since X is compact, it follows that

$$\iota_B : \mathcal{C}_* (B \wedge X, B \owedge X) \to \mathcal{C}_* (B, \mathcal{C}_* (X, B \owedge X))$$

is a homeomorphism, as in the proof of Lemma 3.3. Thus, the adjoint of j,

$$\iota_B(j) : B \to \mathcal{C}_* (X, B \otimes X)$$

is pointed, continuous and compatible with the algebraic structure, i.e. it is an element in $\mathrm{Hom}(B, \mathcal{C}_* (X, B \otimes X))$. This induces a pointed continuous map (by postcomposition)

$$\mathbb{R}_A(B) \to \mathbb{R}_A(\mathcal{C}_* (X, B \otimes X)).$$

By our second assumption on A, we have a natural homeomorphism

$$\mathbb{R}_A(\mathcal{C}_* (X, B \owedge X)) \to \mathcal{C}_* (X, \mathbb{R}_A(B \owedge X))$$

hence, an induced pointed continuous map

$$\mathbb{R}_A(B) \to \mathcal{C}_* (X, \mathbb{R}_A(B \owedge X)).$$

Finally,

$$\iota_{\mathbb{R}_A(B)} : \mathcal{C}_* (\mathbb{R}_A(B) \wedge X, \mathbb{R}_A(B \owedge X)) \to \mathcal{C}_* (\mathbb{R}_A(B), \mathcal{C}_* (X, \mathbb{R}_A(B \owedge X)))$$

is once again a homeomorphism, since X is compact. Taking $(\iota_{\mathbb{R}_A(B)})^{-1}$ of the composition described above yields the expected pointed map

$$\mathbb{R}_A(B) \wedge X \to \mathbb{R}_A(B \owedge X)$$

and the formula given in the statement may be traced from the construction.

For the second claim, the natural isomorphism

$$\mathcal{C}_* (X \wedge Y, D) \approx \mathcal{C}_* (X, \mathcal{C}_* (Y, D))$$

together with several variations of the natural bijection of Proposition 5.4 yields a string of natural bijections

$$\begin{tikzcd} \mathrm{Hom}(B \owedge (X \wedge Y), D) \arrow[r] & \mathrm{Hom}(B, \mathcal{C}_*(X \wedge Y, D)) \arrow[d] \\ \mathrm{Hom}(B \owedge X, \mathcal{C}_*(Y, D)) \arrow[r] & \mathrm{Hom}(B, \mathcal{C}_*(X, \mathcal{C}_*(Y, D))) \\ \mathrm{Hom}((B \owedge X) \owedge Y, D) \arrow[u] & \end{tikzcd}$$

Choosing $D = B \owedge (X \wedge Y)$ and taking the unique *-homomorphism corresponding to the identity map of D yields the expected natural *-homomorphism. ☐

LEMMA 8.2. *Suppose $A, B \in \mathbb{A}_\nu$ with A satisfying our second assumption. Then the identity $\mathbb{R}_A(B \owedge I) \to \mathbb{R}_A(B \owedge I)$ is null homotopic in Top_*.*

PROOF. A pointed null homotopy for I,

$$H : I \wedge I_+ \to I$$

induces a morphism

$$B \owedge H : B \owedge (I \wedge I_+) \to B \owedge I$$

in $\mathbb{A}_\nu$. Now applying the functor $\mathbb{R}_A$ and working with the two natural maps of Lemma 8.1 yields

$$\begin{tikzcd} \mathbb{R}_A(B \owedge I) \wedge I_+ \arrow[r] & \mathbb{R}_A((B \owedge I) \owedge I_+) \arrow[r] & \mathbb{R}_A(B \owedge (I \wedge I_+)) \arrow[d, "\mathbb{R}_A(B \owedge H)"] \\ & & \mathbb{R}_A(B \owedge I). \end{tikzcd}$$

This composite is a pointed null homotopy of the identity on $\mathbb{R}_A(B \owedge I)$. ☐

DEFINITION 8.3.

(1) A *-homomorphism $i : B \to D$ is the inclusion of a retract if there exists a *-homomorphism $r : D \to B$ such that $r \circ i = id_B$. In the language of lifting diagrams, this is equivalent to the existence of a lift (the dotted arrow) in the following commutative square:

$$\begin{tikzcd} B \arrow[r, "="] \arrow[d, "i"'] & B \arrow[d] \\ D \arrow[r] \arrow[ur, dotted] & 0. \end{tikzcd}$$

(2) An inclusion of a retract is the inclusion of a deformation retract if there exists a lift h in the following diagram:

$$\begin{tikzcd} B \arrow[r, "i"] \arrow[d, "i"'] & D \arrow[r, "j"] & \mathcal{C}_*(I_+, D) \arrow[d] \\ D \arrow[r, "(id, i\circ r)"'] \arrow[urr, dotted, "h"] & D \times D \arrow[r] & \mathcal{C}_*\left(S^0_+, D\right) \end{tikzcd}$$

where precomposition by the map $I_+ \to S^0$ which collapses the interval to the non-basepoint gives the *-homomorphism

$$j : D \approx \mathcal{C}_*\left(S^0, D\right) \to \mathcal{C}_*\left(I_+, D\right).$$

REMARK 8.4. This definition in terms of lifting diagrams, convenient for our application below, does agree with the usual notion of deformation retract for the category Top_*.

A morphism $f : A \to B$ will be said to have the left lifting property with respect to a class of morphisms if in each commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & D \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & E \end{tikzcd}$$

with p in the specified class of morphisms there exists a dotted arrow (or lift) which makes both triangles commute. For example, for any topological space the map

$$X \to X \times I$$

has the left lifting property with respect to the class of Hurewicz fibrations. Recall that the Serre cofibrations are defined as the maps in Top_* with the left lifting property with respect to the class of Serre fibrations that are also weak homotopy equivalences. We will mean Serre fibration by the word fibration in what follows and acyclic fibrations will refer to Serre fibrations which are also weak homotopy equivalences.

LEMMA 8.5. *If $j : X \to Y$ is a Serre cofibration between compact spaces in Top_* and $Z \in Top_*$, then*

$$j^* : \mathcal{C}_*\left(Y, Z\right) \to \mathcal{C}_*\left(X, Z\right)$$

is a Serre fibration in Top_*.

PROOF. For a convenient category, like Steenrod's category of compactly generated spaces, this would be the topological analog of Quillen's simplicial model category axiom SM7. In that case, the result holds as in [**Hov99**].

For Steenrod's category, we would instead have

$$k\mathcal{C}_*\left(Y, k(Z)\right) \to k\mathcal{C}_*\left(X, k(Z)\right)$$

is a fibration. However, it follows from [**Ste67**, Lemma 5.3] that $k(Z)$ may be replaced by Z up to homeomorphism, since Y and X are compact. Now notice the condition of being a fibration is checked by mapping out of compact spaces, so the k on the outside is also not necessary. □

The following two lemmas are the keys to the existence of the model category structure. The first is essentially due to Quillen.

LEMMA 8.6. *Suppose $j : B \to D$ is a morphism in $\mathbb{A}_\nu$ which has the left lifting property with respect to each morphism p with $\mathbb{R}_A(p)$ a fibration in Top_*. If A satisfies our second assumption, then $\mathbb{R}_A(j)$ is a weak homotopy equivalence.*

Proof. We will show that $\mathbb{R}_A(j)$ is the inclusion of a deformation retract, hence a weak homotopy equivalence in Top_*. To this end, it suffices to show that j is the inclusion of a deformation retract, since by our second assumption there is a natural homeomorphism $\mathbb{R}_A(\mathcal{C}_*(I_+, D)) \cong \mathcal{C}_*(I_+, \mathbb{R}_A(D))$ and similarly for $\mathcal{C}_*\left(S^0_+, D\right)$.

The inclusion $S^0_+ \to I_+$ is a cofibration between compact spaces, hence

$$\mathcal{C}_*(I_+, \mathbb{R}_A(D)) \to \mathcal{C}_*\left(S^0_+, \mathbb{R}_A(D)\right)$$

is a fibration in Top_* by Lemma 8.5. Thus, the homeomorphic (again by our second assumption) map

$$\mathbb{R}_A(\mathcal{C}_*(I_+, D)) \to \mathbb{R}_A\left(\mathcal{C}_*\left(S^0_+, D\right)\right)$$

is a fibration as well.

Now j is assumed to have the left lifting property with respect to *-homomorphisms p for which $\mathbb{R}_A(p)$ is a fibration. We have just verified that $\mathcal{C}_*(I_+, D) \to \mathcal{C}_*\left(S^0_+, D\right)$ is a *-homomorphism of this type. Since $\mathbb{R}_A$ sends $0 \in \mathbb{A}_\nu$ to the basepoint and every map $X \to *$ in Top_* is a fibration, $B \to 0$ is also a *-homomorphism of this type. Hence, there exist lifts in the two diagrams which define j as the inclusion of a deformation retract in $\mathbb{A}_\nu$. □

Lemma 8.7. *Suppose $A, B \in \mathbb{A}_\nu$ and A satisfies our second assumption, while the canonical morphism $i_1 : B \to B \oslash I$ has the seminorm extension property. Furthermore, assume $j : B \to D$ is a morphism in $\mathbb{A}_\nu$ which has the left lifting property with respect to any morphism q with $\mathbb{R}_A(q)$ an acyclic fibration in Top_*. Then j has the seminorm extension property.*

Proof. By Lemma 8.2, one has that $\mathbb{R}_A$ applied to the *-homomorphism $B \oslash I \to 0$ is an acyclic fibration in Top_*. Now, by assumption there exists a lift in the diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d, "j"'] & B \oslash I \arrow[d] \\ D \arrow[r] \arrow[ur, dotted] & 0 \end{tikzcd}$$

which implies j is the first factor in a factorization of $i_1 : B \to B \oslash I$, hence it has the seminorm extension property by Lemma 7.7. □

We say the functor $\mathbb{R}_A$ reflects weak equivalences if p is a weak equivalence in $\mathbb{A}_\nu$ precisely when $\mathbb{R}_A(p)$ is a weak equivalence in Top_* and similarly for fibrations. We also remind the reader that all three of the assumptions here are satisfied by the object $A = qA' \boxtimes \mathcal{K}$ for any separable $A' \in C^*\text{-}alg$, as discussed at the beginning of this section.

Proposition 8.8. *Suppose $A \in \mathbb{A}_\nu$ satisfies:*

(1) $\mathbb{R}_A = \mathrm{Hom}(A, ?)$ *has* $\mathbb{L}_A = A \oslash ?$ *as a partial left adjoint*

(2) *there is a natural homeomorphism*

$$\mathbb{R}_A(\mathcal{C}_*(X, B)) \to \mathcal{C}_*(X, \mathbb{R}_A(B))$$

(3) *A has the stable cone seminorm extension property.*

Then there is a cofibrantly generated model structure on $\mathbb{A}_\nu$ with fibrations and weak equivalences reflected by $\mathbb{R}_A$ and with A itself cofibrant. Furthermore, this structure *is right proper (all objects are fibrant), and the generating cells are of the form $\mathbb{L}_A$ applied to the inclusions $i' : S^{n-1}_+ \to D^n_+$.*

This proof has become standard, with the exception of the fact that $(\mathbb{L}_A, \mathbb{R}_A)$ is not quite an adjoint pair. Thus, we present the proof in full detail. Compare with the proof of [**IJ02**, 8.7] and notation from [**Hov99**].

Proof. The axioms (from [**Qui69**, Definition 3.3]) will be verified directly.

(1) The existence of limits and colimits is handled by Lemmas 2.4 and 2.5.
(2) The 2 of 3 property for weak equivalences follows from that of weak equivalences in Top_*.
(3) The class of cofibrations is defined as the retracts of relative I-cell complexes, hence is closed under retracts by definition. Here I is the set of generating cells, $\mathbb{L}_A$ applied to the inclusions $i' : S^{n-1}_+ \to D^n_+$. The fact that $\mathbb{R}_A$ preserves retracts (as a functor) and the fact that the classes of fibrations and weak equivalences in Top_* are closed under retracts implies the same for $\mathbb{A}_\nu$.
(4) Let

$$\begin{tikzcd} B \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ D \arrow[r] & Y \end{tikzcd}$$

be a potential lifting square.

If i is a retract of a relative I-cell complex, and p is an acyclic fibration, one must verify the existence of a lift. Consider first the case that i is a generating cofibration. Then $i = \mathbb{L}_A(i')$ for a generating cofibration i' in Top_* (whose source and target lie in Cpt_*). Thus, i' has the left lifting property with respect to $\mathbb{R}_A(p)$, which is an acyclic fibration in Top_* by assumption. Considering the unique square

$$\begin{tikzcd} S^{n-1}_+ \arrow[r] \arrow[d, "i'"'] & \mathbb{R}_A(X) \arrow[d, "\mathbb{R}_A(p)"] \\ D^n_+ \arrow[r] \arrow[ur, dotted, "f"] & \mathbb{R}_A(Y) \end{tikzcd}$$

weakly adjoint to the original square, there must exist a lift f, which implies there was a lift in the original square via the weak adjoint of f. In building a retract of a relative I-cell complex, one uses cobase change (pushout), coproducts, transfinite composition (sequential colimits) and retracts, all operations which preserve the left lifting property with respect to a class of morphisms. Hence, the arbitrary cofibration case follows from that of the generating cofibration.

Now suppose that i is both a retract of a relative I-cell complex and a weak equivalence, while p is a fibration. It is shown below in (5) that i can be factored as a relative J-cell complex $j : B \to F$ followed by a fibration $q : F \to D$, where J is the set of maps $\mathbb{L}_A(j')$ for $j' : D^n_+ \to (D^n \times I)_+$ the inclusion of the base of the cylinder with a disjoint basepoint added. The argument in the previous paragraph then implies that j has the left lifting property with respect to all fibrations by construction, hence is a weak equivalence by Lemma 8.6. This left lifting property with respect to all fibrations also implies the existence of a morphism $h : F \to X$ which makes the right portion of

$$\begin{tikzcd}
B \arrow[rr] \arrow[dr, "j"] \arrow[dd, "i"'] & & X \arrow[dd, "p"] \\
& F \arrow[ur, dotted, "h"] \arrow[d, "q"] & \\
D \arrow[ur, dotted, "f"] \arrow[r, "="] & D \arrow[r] & Y
\end{tikzcd}$$

commute. Also, the 2 of 3 property established in (2) implies that q is actually an acyclic fibration. However, the argument in the previous paragraph then implies there is a lift of i against q, $f : D \to F$ making the left portion of the previous diagram commute. Now the composition hf acts as a lift in the original square.

(5) Suppose $k : X \to Y$ is a morphism in $\mathbb{A}_\nu$. Construct a relative I-cell complex $X = X_0 \to X_\gamma$ as follows. (Here γ is a cardinal such that each of the $\mathbb{L}_A(S^{n-1}_+)$ is γ-small with respect to morphisms in $\mathbb{A}_\nu$ satisfying the seminorm extension property, which is possible by Lemma 7.5.) For limit ordinals $\delta < \gamma$, define $X_\delta = \underset{\sigma<\delta}{\operatorname{colim}} X_\sigma$. For successor ordinals, define $X_{\delta+1}$ to be the pushout in the diagram

$$\begin{tikzcd}
\coprod \mathbb{L}_A(S^{n-1}_+) \arrow[r] \arrow[d] & \coprod \mathbb{L}_A(D^n_+) \arrow[d] \\
X_\delta \arrow[r] & X_{\delta+1}
\end{tikzcd}$$

where the coproducts are indexed over the set of commutative squares of the form

$$\begin{tikzcd}
\mathbb{L}_A(S^{n-1}_+) \arrow[r] \arrow[d] & \mathbb{L}_A(D^n_+) \arrow[d] \\
X_\delta \arrow[r] & Y.
\end{tikzcd}$$

The universal properties of colimits and pushouts then yields a factorization of k as $X \to X_\gamma \to Y$ with $X \to X_\gamma$ a relative I-cell complex. It remains to verify that the morphism $p : X_\gamma \to Y$ is an acyclic fibration. Since Top_* is a cofibrantly generated model category, it suffices to verify that $\mathbb{R}_A(p)$ has the RLP with respect to each $S^{n-1}_+ \to D^n_+$. Equivalently, one can consider the weak adjoint of each such diagram and show that each

$$\mathbb{L}_A(i') : \mathbb{L}_A(S^{n-1}_+ \to \mathbb{L}_A(D^n_+)$$

has the left lifting property with respect to p. Thus, we consider a lifting diagram

$$\begin{tikzcd} \mathbb{L}_A(S^{n-1}_+) \arrow[r, "f"] \arrow[d, "i"'] & X_\gamma \arrow[d, "p"] \\ \mathbb{L}_A(D^n_+) \arrow[r] & Y. \end{tikzcd}$$

The object $\mathbb{L}_A(S^{n-1}_+)$ is γ-small with respect to morphisms in $\mathbb{A}_\nu$ satisfying the seminorm extension property by assumption, while every relative I-cell complex satisfies the seminorm extension property by the combination of Lemmas 7.4, 7.8 and 8.7 with Corollary 7.9. Thus, one may factor f as

$$\mathbb{L}_A(S^{n-1}_+) \xrightarrow{g} X_\delta \xrightarrow{h} X_\gamma$$

for some X_δ with $\delta < \gamma$. However, one then has a commutative diagram of the form

$$\begin{tikzcd} \mathbb{L}_A(S^{n-1}_+) \arrow[r, "g"] \arrow[d, "i"'] & X_\delta \arrow[d, "ph"] \\ \mathbb{L}_A(D^n_+) \arrow[r] & Y \end{tikzcd}$$

which must have been used to build $X_{\delta+1}$. Hence, the composition

$$\mathbb{L}_A(D^n_+) \to X_{\delta+1} \to X_\gamma$$

acts as a lift in the original square

$$\begin{tikzcd} \mathbb{L}_A(S^{n-1}_+) \arrow[r, "g"] \arrow[d, "i"'] & X_\delta \arrow[r, "h"] \arrow[d] & X_\gamma \arrow[d, "p"] \\ \mathbb{L}_A(S^{n-1}_+) \arrow[r] & X_{\delta+1} \arrow[r] \arrow[ur] & Y. \end{tikzcd}$$

The other factorization of k is produced similarly, using J in place of I, with one additional difficulty. Namely, one must verify that the relative J-cell complex j produced is actually a retract of a relative I-cell complex as well as a weak equivalence. The lifting argument from the proof of the first half of (4) implies that j has the left lifting property with respect to each fibration, hence is a weak equivalence by Lemma 8.6. Now factor j as a relative I-cell i followed by an acyclic fibration, as in the previous paragraph. Then the left lifting property just mentioned for j implies the existence of a lift in the square

$$\begin{tikzcd} B \arrow[r, "i"] \arrow[d, "j"'] & F \arrow[d, "q"] \\ D \arrow[r, "="'] \arrow[ur, dotted] & D \end{tikzcd}$$

which exhibits j as a retract of i, hence as a retract of a relative I-cell complex.

$\square$

There is a relatively straightforward notion of intersection of cofibrantly generated model category structures introduced in [**IJ02**]. This makes the following lifting lemma a corollary of the previous proposition. We say the collection of functors $\{\mathbb{R}_A\}$ reflects the fibrations (resp. weak equivalences) when a *-homomorphism p is a fibration if and only if each $\mathbb{R}_A(p)$ is a fibration (resp. weak equivalence) in Top_*.

Corollary 8.9. *Suppose $\{A_\alpha\}$ is a set of objects in $\mathbb{A}_\nu$ all satisfying the three conditions in Proposition 8.8. Then there is a cofibrantly generated model structure on $\mathbb{A}_\nu$ with fibrations and weak equivalences reflected by the set of functors $\{\mathbb{R}_A\}$ and with the property that each A_α is cofibrant.*

Proof. This follows from the intersection construction of [**IJ02**, Proposition 8.7]. The relative smallness follows from the fact that cell complexes built from the collection of generating cells in each structure have the seminorm extension property and from Lemma 7.5. The other condition is handled by Lemma 8.6 and the uniformity of the cylinder chosen for each structure. □

The following technical lemmas will prove valuable in identifying weak equivalences in specific examples considered in the next section.

Lemma 8.10. *Given any $B \in \mathbb{A}_\nu$, one has $\mathcal{C}_*(I_+, B)$ acting as a path object in the model category sense for any model structure constructed with Proposition 8.8 or Corollary 8.9.*

Proof. Consider the composition $i : S^0_+ \to I_+$ followed by $f : I_+ \to S^0$ in Top_*, which gives a factorization of the fold map as a cofibration between compact spaces followed by a homotopy equivalence. Then precomposition with i,

$$i^* : \mathcal{C}_*(I_+, \mathbb{R}_A(B)) \to \mathcal{C}_*\left(S^0_+, \mathbb{R}_A(B)\right)$$

yields a fibration by Lemma 8.5 and precomposition with f,

$$f^* : \mathcal{C}_*\left(S^0, \mathbb{R}_A(B)\right) \to \mathcal{C}_*(I_+, \mathbb{R}_A(B))$$

yields a homotopy equivalence, hence a weak equivalence. However, our second assumption on A then implies $\mathcal{C}_*(I_+, B) \to \mathcal{C}_*\left(S^0_+, B\right)$ is a fibration in $\mathbb{A}_\nu$ and $\mathcal{C}_*\left(S^0, B\right) \to \mathcal{C}_*(I_+, B)$ is a weak equivalence in $\mathbb{A}_\nu$. Since $\mathcal{C}_*\left(S^0, B\right) \approx B$, and $\mathcal{C}_*\left(S^0_+, B\right) \approx B \times B$, this yields a factorization of the diagonal map

$$B \to \mathcal{C}_*(I_+, B) \to B \times B$$

with the first morphism a weak equivalence and the second map a fibration in $\mathbb{A}_\nu$. This is the definition of a path object in a model category. □

Lemma 8.11. *Suppose $A_\alpha \in \mathbb{A}_\nu$ is one of the generating objects in a structure given by Proposition 8.8 or Corollary 8.9. Then $\pi_0 \operatorname{Hom}(A_\alpha, B)$, $[A_\alpha, B]$ and $\operatorname{Ho}(A_\alpha, B)$ are all naturally isomorphic as sets.*

Proof. To see that $\pi_0 \operatorname{Hom}(A_\alpha, B) \approx [A_\alpha, B]$ it suffices to note that there is, by our second assumption on A_α a natural homeomorphism $\mathcal{C}_*(I_+, \operatorname{Hom}(A_\alpha, B)) \approx \operatorname{Hom}(A_\alpha, \mathcal{C}_*(I_+, B))$ and similarly for S^0 in place of I_+. Thus, one has the same set modulo the same equivalence relation on both sides, since $\mathcal{C}_*(I_+, B) = \mathcal{C}(I, B)$.

In order to see that $[A_\alpha, B] \approx \operatorname{Ho}(A_\alpha, B)$ under these conditions, it suffices to note that $\mathcal{C}(I, B) = \mathcal{C}_*(I_+, B)$ acts as a path object in the model category sense by Lemma 8.10. □

9. The Main Theorems

We now have the technical tools to build our required model category structure.

Definition 9.1 (K-model category structure). We define the K-theory model category structure on $\mathbb{A}_\nu$ to be the model category structure obtained from Proposition 8.8 using the object $q\mathbb{C} \boxtimes \mathcal{K}$. A weak equivalence for the K-model category structure we will refer to as a weak K-equivalence.

Theorem 9.2. *The* K*-model structure is cofibrantly generated with all objects fibrant. In addition, a morphism $f : B \to B'$ of C^*-algebras is a weak* K*-equivalence if and only if it is a K_*-equivalence, i.e. if and only if the induced map $K_*(B) \to K_*(B')$ is an isomorphism. Moreover, for C^*-algebras B there is a natural isomorphism*

$$Ho(\mathbb{C}, B) \cong K_0(B).$$

Proof. By Proposition 8.8, f is a weak K-equivalence if and only if the induced map

$$Hom(q\mathbb{C} \boxtimes \mathcal{K}, B) \to Hom(q\mathbb{C} \boxtimes \mathcal{K}, B')$$

is a weak homotopy equivalence. However, since for σ-unital C^*-algebras B there is a natural isomorphism $K_*(B) \cong KK_*(\mathbb{C}, B)$ Proposition 6.11 implies that a map of σ-unital C^*-algebras is a weak K-equivalence if and only if it is a K_*-equivalence. Thus, we have verified the first statement in case B are B' are σ-unital C^*-algebras.

For an arbitrary C^*-algebra B', consider the directed system of σ-unital sub-C^*-algebras B of B'. Since $q\mathbb{C}$ is separable the image of a *-homomorphism into $B' \otimes \mathcal{K}$ is contained in $B \otimes \mathcal{K}$ for a separable (and hence σ-unital) sub-C^*-algebra B of B', so

$$\operatorname{colim}_B Hom(q\mathbb{C}, B \otimes \mathcal{K}) \cong Hom(q\mathbb{C}, B' \otimes \mathcal{K}),$$

where the colimit is taken over the system of σ-unital sub-C^*-algebras B in B'. Continuity of the functor K_* then yields

$$\pi_* Hom(q\mathbb{C}, B' \otimes \mathcal{K}) \cong \operatorname{colim}_B \pi_* Hom(q\mathbb{C}, B \otimes \mathcal{K}) \cong \operatorname{colim}_B K_*(B) \cong K_*(B')$$

and the general case follows. Here we interchanged taking homotopy groups with taking a colimit. In general the two operations do not commute, but in our special case they do. The reason being that for a separable compact space X the canonical map

$$\operatorname{colim}_B \mathcal{C}(X, Hom(q\mathbb{C}, B \otimes \mathcal{K})) \to \mathcal{C}(X, Hom(q\mathbb{C}, B' \otimes \mathcal{K}))$$

is a continuous bijection. Injectivity of this map is clear. To see surjectivity let $\underline{X} = \{x_1, x_2, ...\}$ be a countable dense set of X, $\underline{q\mathbb{C}} = \{q_1, q_2, ...\}$ be a countable dense set of $q\mathbb{C}$, and let the functions $x_{ij}^{B'} : B' \otimes \mathcal{K} \to B'$ and the compact operators $e_{ij} \in \mathcal{K}$ be defined as in the proof of Lemma 3.2. Given a map $f : X \to Hom(q\mathbb{C}, B' \otimes \mathcal{K})$ define the subset $\underline{B}_f \subset B'$ by

$$\underline{B}_f = \{x_{ij}^{B'}(f(x_k)(q_l)) \mid i, j, k, l \in \mathbb{N}\}$$

Then $\underline{B}_f$ is countable and therefore generates a separable sub-C^*-algebra $B_f \subset B'$, and the given map f is in the image of $\mathcal{C}(X, Hom(q\mathbb{C}, B_f \otimes \mathcal{K}))$.

To prove the second statement of the theorem, note that

$$Hom(q\mathbb{C} \boxtimes \mathcal{K}, q\mathbb{C} \boxtimes \mathcal{K}) \to Hom(q\mathbb{C} \boxtimes \mathcal{K}, q\mathbb{C})$$

given by composing with the map $q_{q\mathbb{C}} : q\mathbb{C} \boxtimes \mathcal{K} \to q\mathbb{C}$ (defined in 6.7) is a weak equivalence by Proposition 6.12. Since $q\mathbb{C}\boxtimes\mathcal{K}$ is cofibrant by Proposition 8.8, it then forms a cofibrant approximation of $q\mathbb{C}$. Moreover, the canonical map $q\mathbb{C} \to \mathbb{C}$ is a K_*-equivalence. So, using the first statement above, it follows that the composition $q\mathbb{C} \boxtimes \mathcal{K} \to q\mathbb{C} \to \mathbb{C}$ is a cofibrant approximation for $\mathbb{C}$. Consequently, Lemma 8.11 followed by Proposition 6.5 and Cuntz's Theorem 1.1 imply

$$Ho(\mathbb{C}, B) \cong \pi_0(Hom(q\mathbb{C} \boxtimes \mathcal{K}, B)) \cong \pi_0(Hom(q\mathbb{C}, B \otimes \mathcal{K})) \cong K_0(B).$$

□

Remark 9.3. We expect that the last two statements of Theorem 9.2 also hold for more general B. We have restricted our attention to C^*-algebras for simplicity.

Remark 9.4. A natural question to ask (which also was asked by the referee) is wether there is a natural isomorphism $Ho(\mathcal{C}_*(\mathbb{R}^n), B) \cong K_n(B)$ for C^*-algebras B, or more generally any $B \in \mathbb{A}_\nu$. We currently do not know the answer to this question, as the cofibrant approximation of $\mathcal{C}_*(\mathbb{R}^n)$ is not easy to deal with in the K-model structure. We could modify the definition of the K-model structure so that it will have this property at least for C^*-algebras B. Instead of using the single object $q\mathbb{C}\boxtimes\mathcal{K}$ we then would use the family of objects $\{q\mathcal{C}_*(\mathbb{R}^n)\boxtimes\mathcal{K}, n \geq 0\}$. In the associated model structure the composition $q\mathcal{C}_*(\mathbb{R}^n) \boxtimes \mathcal{K} \to q\mathcal{C}_*(\mathbb{R}^n) \to \mathcal{C}_*(\mathbb{R}^n)$ would be a cofibrant approximation, so that we would obtain $Ho(\mathcal{C}_*(\mathbb{R}^n), B) = [q\mathcal{C}_*(\mathbb{R}^n) \boxtimes \mathcal{K}, B] \cong K_n(B)$. The isomorphism would be obtained essentially as in the proof of Theorem 9.2, but we also would need to implement the Bott periodicity isomorphism $KK(\mathcal{C}_*(\mathbb{R}^n), B) \cong KK(\mathbb{C}, \mathcal{C}_*(\mathbb{R}^n) \otimes B)$ for σ-unital C^*-algebras B.

We now focus on a further model structure on $\mathbb{A}_\nu$, which we will refer to as the KK-structure.

Definition 9.5 (K K-model category structure). We define the K K-model category structure on $\mathbb{A}_\nu$ to be the model category structure which comes from Corollary 8.9 using the *l.m.c.*-C^*-algebras $qA\boxtimes\mathcal{K}$ as A varies over a set of representatives, one for each isomorphism class of separable C^*-algebras. Clearly, there is only a set of isomorphism classes of separable C^*-algebras since the separable condition is equivalent to having a faithful representation in a countable-dimensional Hilbert space. A weak equivalence for the K K-model category structure we call a weak K K-equivalence.

Theorem 9.6. *The* K K-*model structure is cofibrantly generated with all objects fibrant. In addition, a *-homomorphism $f : B \to B'$ of σ-unital C^*-algebras is a weak* K K-*equivalence if and only if it is a "(left) KK_*-equivalence", i.e. if and only if the induced map $KK_*(A, B) \to KK_*(A, B')$ is an isomorphism for all separable C^*-algebras A. Moreover, for a separable C^*-algebra A and a σ-unital C^*-algebra B there is a natural isomorphism*

$$Ho(A, B) \cong KK_0(A, B),$$

while the composition

$$qA \boxtimes \mathcal{K} \to qA \to A$$

yields a cofibrant approximation.

Proof. By definition a *-homomorphism $f : B \to B'$ is a weak K K-equivalence if and only if the induced maps

$$Hom(qA \boxtimes \mathcal{K}, B) \to Hom(qA \boxtimes \mathcal{K}, B')$$

are weak equivalences for all separable C^*-algebras A. However, by Proposition 6.11 this is equivalent to saying f induces an isomorphism

$$KK_*(A, B) \to KK_*(A, B')$$

for all separable C^*-algebras A, or equivalently that f is a (left) KK_*-equivalence.

To see the second statement of the theorem, recall from Proposition 6.12 that composition with $q_{qB} : qB \boxtimes \mathcal{K} \to qB$ (defined in 6.7) induces weak equivalences

$$Hom(qA \boxtimes \mathcal{K}, qB \boxtimes \mathcal{K}) \to Hom(qA \boxtimes \mathcal{K}, qB)$$

for all separable C^*-algebras A. Moreover, it is well-known that $qA \to A$ is a (left) KK_*-equivalence for all separable C^*-algebras A (cf. [**Bla98**, 19.1.2.(i)]). Thus, the composition $qA \boxtimes \mathcal{K} \to qA \to A$ is a cofibrant approximation for A if A is a separable C^*-algebra. Consequently, as in the proof of Theorem 9.2 we get

$$Ho(A, B) \cong \pi_0 Hom(qA \boxtimes \mathcal{K}, B) \cong \pi_0 Hom(qA, B \otimes \mathcal{K}) \cong KK_0(A, B).$$

□

Remark 9.7. When defining $KK(A, B)$ one typically assumes A to be a separable C^*-algebra and B to be a σ-unital C^*-algebra. KK-groups also have been defined in more general settings by various authors. A new approach for defining a set $KK(A, B)$ for algebras $A, B \in \mathbb{A}_\nu$ is given by simply defining it by $KK(A, B) = Ho(A, B)$.

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