UNBOUNDED FREDHOLM OPERATORS AND K-THEORY

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Abstract. Let $\mathcal{A}$ be a unital C^*-algebra. We show that the set of regular unbounded $\mathcal{A}$-Fredholm operators respectively the set of self-adjoint regular unbounded $\mathcal{A}$-Fredholm operators on the standard Hilbert-$\mathcal{A}$-module $l^2(A)$ equipped with the gap topology represent the functors which associate to a compact space X the groups $K^0(X;\mathcal{A})$ and $K^1(X;\mathcal{A})$.

1. Introduction

For a unital complex C^*-algebra $\mathcal{A}$ and a compact space X let $K^0(X;\mathcal{A})$ denote the Grothendieck group of isomorphism classes of locally trivial Banach space bundles over X, whose fibres are finitely generated projective $\mathcal{A}$-modules. Equivalently we can think of $K^0(X;\mathcal{A})$ as the K-homology of the C^*-algebra $X\mathcal{A}$ of continuous functions on X with values in $\mathcal{A}$ since there is a natural isomorphism $K^0(X;\mathcal{A}) \cong K_0(X\mathcal{A})$. The corresponding reduced groups are defined by

$$\widetilde{K}^0(X;\mathcal{A}) = Coker(K^0(\{*\};\mathcal{A}) \to K^0(X;\mathcal{A})),$$

where the unlabeled homomorphism is induced by the canoncial map from X onto the one-point space $\{*\}$. The functor $\widetilde{K}^0$ is used to define

$$K^{-n}(X;\mathcal{A}) = \widetilde{K}^0(S^n \wedge X_+;\mathcal{A}),\ n \in \mathbb{N},$$

where X_+ stands for the pointed space $X \amalg \{*\}$ with basepoint $*$. Moreover there is a canonical isomorphism $\mu_{Bott} : K^0(X;\mathcal{A}) \to K^{-2}(X;\mathcal{A})$, which in the end can be used to extend the functor $K^0(X;\mathcal{A})$ to a 2-periodic cohomology theory $X \mapsto K^*(X;\mathcal{A})$.

In [2] Atiyah showed that the space of bounded Fredholm operators on an infinite dimensional separable complex Hilbert space equipped with the operator norm topology represents the functor $X \mapsto K^0(X;\mathbb{C})$. Little later Atiyah and Singer [3] proved that the functor $X \mapsto K^1(X;\mathbb{C})$ can be represented by a specific path component of the space of self-adjoint bounded Fredholm operators, and about 15 years later when the theory of Hilbert modules was studied in more detail Mingo [9], Kasimov [7] and Troitskii [10] generalized these results to the general case where $\mathcal{A}$ is an arbitrary unital C^*-algebra.

However, from the applications point of view it seems to be more natural to consider unbounded operators. Many important invariants which take values in $K^0(X;\mathcal{A})$ or in $K^1(X;\mathcal{A})$ come from certain unbounded operators d; and one uses the bounded transform $F = d/\sqrt{1+d^*d}$ which then is a bounded operator to get the desired invariants. The unbounded operators that arise from applications are typically contained in a nice class of unbounded operators, the class of so-called regular[1] operators. All bounded operators are contained in that class, and there is a topology on the set of regular operators, the so-called *gap topology*, which restricts to the norm topology on the subset of bounded operators.

This shall motivate our main theorem, which is

Theorem 3.5. *Let $\mathcal{A}$ be a unital C^*-algebra, and let $H = l^2(\mathcal{A})$. Let $FredR(H)$ denote the space of regular $\mathcal{A}$-Fredholm operators, equipped with the gap topology, and let $FredR(H)_{sa}$ denote the subspace consisiting of the self-adjoint regular $\mathcal{A}$-Fredholm operators.*

(i) *The space $FredR(H)$ represents the functor which associates to a compact space X the group $K^0(X;\mathcal{A})$*
(ii) *The space $FredR(H)_{sa}$ represents the functor which associates to a compact space X the group $K^1(X;\mathcal{A})$.*

Note that in the case of (ii) it is no longer necessary to pay attention to path components as it was in the corresponding statement involving the bounded self-adjoint operators Fredholm operators ([3],[10]). The theorem also gives a positive answer to a question raised by Booss-Bavnbek, Lesch and Phillips (cf. Remark 3.6).

The paper is organized as follows. In section 1 we review material on operators on Hilbert modules. In particular we define the notion of a regular operator and introduce the gap topology on the set of regular operators acting on a specific Hilbert module H. We recall the concept of bounded transform which provides a map from the set of regular operators into the unit ball of the bounded adjointable operators. We then show that the gap topology can be obtained by restriction from suitable topologies on the unit ball. In section 2 we consider Fredholm operators, operators with compact resolvents as well as essentially unitary operators, and we compare the weak homotopy types of the according operator spaces. In section 3 we relate what we obtained so far with results in [5], which will give the theorem.

Convention. Throughout the paper we will deal with bounded and unbounded operators at the same time. To simplify notation we will, as a general rule, denote bounded operators by captial letters and unbounded operators by small letters.

[1] In case $\mathcal{A} = \mathbb{C}$ an unbounded operator is regular if it is closed and densely defined.

1. Regular operators

In this section we first recall some basic concepts concerning operators acting on a Hilbert module. In particular we recall the definition of a regular operator and introduce the gap topology on the set of regular operators acting on a particular Hilbert module, - the particular Hilbert module of interest being $H = l^2\mathcal{A}$ for a unital C^*-algebra $\mathcal{A}$. The space of regular operators on H we will denote $R(H)$. We then study the map which associates to a regular operator t its bounded transform $F(t) = t(1+t^*t)^{-1/2}$. The bounded transform of a regular operator by definition is contained in the unit ball $B(H)$ of the bounded adjointable operators. However the map $R(H) \to B(H)$ will not be continuous if we equip $B(H)$ with the norm topology. The main goal of this section is to introduce a suitable weaker topology on $B(H)$ such that the map $R(H) \to B(H)$ becomes a topological inclusion. There is an obvious candidate for such a topology, which we call the extended gap topology (Definition 1.12). However this topology will not be strong enough to let us perform the relevant homotopies in the following section. For that reason we will proceed and show that the gap topology also is the restriction of a stronger topology on $B(H)$, the strict extended gap topology (Definition 1.30).

Let $\mathcal{A}$ be a complex unital C^*-algebra, and let $H = l^2 \otimes \mathcal{A}$ be the canonical Hilbert-$\mathcal{A}$-module with the standard $\mathcal{A}$-sesquilinear scalar product, i.e. the scalar product of two elementary tensors $h \otimes a, h' \otimes a'$ in H is given by $\langle h \otimes a, h' \otimes a' \rangle = \langle h, h' \rangle \cdot a^* a'$, and H is the completion of the algebraic tensor product $l^2 \otimes_{alg} \mathcal{A}$ with respect to the norm given by $|x|^2 = \|\langle x, x\rangle\|_{\mathcal{A}}$ for $x \in H$. A map $F : H \to H$ is called adjointable if there exists a map $F^* : H \to H$ such that $\langle x, Fx' \rangle = \langle F^*x, x' \rangle$ holds for all $x, x' \in H$. If such a map F^* exists it is unique and it is called the adjoint of F. If F has an adjoint it follows that F^* is also adjointable with $(F^*)^* = F$, and F necessarily is a right $\mathcal{A}$-linear norm bounded operator on H. The composition of two adjointable operators G and F is adjointable with $(G \circ F)^* = F^* \circ G^*$. The set of adjointable operators equipped with the operator norm is a C^*-algebra, denoted $\mathbb{B}(H)$. Given elements $x, x' \in H$ we define $\Theta_{x',x} \in \mathbb{B}(H)$ by $\Theta_{x',x}(z) = x'\langle x, z\rangle$ for $z \in H$. The subalgebra of $\mathbb{B}(H)$ algebraically generated by the operators $\Theta_{x',x}$ is called the algebra of finite rank operators, and elements in the closure of the subalgebra of finite rank operators inside $\mathbb{B}(H)$ are called compact adjointable operators. The set of compact adjointable operators is a closed $*$-subalgebra in $\mathbb{B}(H)$; hence it forms a C^*-algebra which we denote $\mathbb{K}(H)$.

Now let us consider closed $\mathcal{A}$-linear operators $t : dom(t) \to H$, where $dom(t)$ is a dense $\mathcal{A}$-submodule of H. We define

$$dom(t^*) = \{y \in H \mid \exists y' \in H : \langle tx, y\rangle = \langle x, y'\rangle \text{ for all } x \in dom(t)\}.$$

This is the domain of a closed $\mathcal{A}$-linear operator $t^* : dom(t^*) \to H$ uniquely determined by

$$\langle tx, y\rangle = \langle x, t^*y\rangle \text{ for all } x \in dom(t), y \in dom(t^*).$$

Definition 1.1 (Regular operators)**.** An operator t as above is called *regular*, if t^* is densely defined and $1 + t^*t$ has dense range. We define $R(H)$ to be the set of all regular operators on H.

If t is regular so are t^* and t^*t, moreover t^*t is self-adjoint (cf. [8, 9.6,9.8]), and one can define

$$F(t) = t(1+t^*t)^{-1/2} \in \mathbb{B}(H), \tag{1.2}$$

$$\mathcal{R}(t) = (1+t^*t)^{-1} \in \mathbb{B}(H), \quad \text{and also} \quad t\mathcal{R}(t) \in \mathbb{B}(H). \tag{1.3}$$

The map $t \mapsto F(t)$ defines a bijection (see [8, 10.4])

$$R(H) \overset{\cong}{\to} \{F \in \mathbb{B}(H) \mid \|F\|_{\mathbb{B}(H)} \leq 1 \text{ and } (1 - F^*F) \text{ has dense image}\}. \tag{1.4}$$

There is a natural metric on the set of regular operators, the so-called *gap metric*. It is defined as follows. Given a regular operator t let $\Gamma(t) \subset H \times H$ denote its graph, and let $\wp_{\Gamma(t)} : H^2 \to H^2$ denote the orthogonal projection onto the graph $\Gamma(t)$.[2] The orthogonal projection can be described through the following matrix (compare [6, (22)])

$$\wp_{\Gamma(t)} = \begin{pmatrix} \mathcal{R}(t) & t^*\mathcal{R}(t^*) \\ t\mathcal{R}(t) & 1 - \mathcal{R}(t^*) \end{pmatrix}. \tag{1.5}$$

Hence by (1.3) we have $\wp_{\Gamma(t)} \in \mathbb{B}(H^2)$.

Definition 1.6 (Gap topology)**.** Let $R(H)$ denote the space that we obtain by equipping the set $R(H)$ with the topology induced by the metric

$$d_\wp(t,t') = \|\wp_{\Gamma(t)} - \wp_{\Gamma(t')}\|_{\mathbb{B}(H^2)}, \quad t, t' \in R(H).$$

This topology is called the *gap topology*. Since $(t\mathcal{R}(t))^* = t^*\mathcal{R}(t^*)$, using expression (1.5), the metric $d_\wp$ is equivalent to the metric d given by

$$d(t,t') = \sup \left\{ \begin{array}{c} \|\mathcal{R}(t) - \mathcal{R}(t')\|_{\mathbb{B}(H)}, \\ \|\mathcal{R}(t^*) - \mathcal{R}(t'^*)\|_{\mathbb{B}(H)}, \\ \|t\mathcal{R}(t) - t'\mathcal{R}(t')\|_{\mathbb{B}(H)} \end{array} \right\}. \tag{1.7}$$

If t is a regular operator we have $\|F(t)\|_{\mathbb{B}(H)} \leq 1$. If F is just any bounded adjointable operator of norm $\|F\|_{\mathbb{B}(H)} \leq 1$ the following operator is positive

$$\mathcal{F}(F) = 1 - F^*F. \tag{1.8}$$

[2] It is a non-trivial fact that the orthogonal projection onto the graph of a regular operator exists (cf. [8, 9.3]). Note that many statements familiar when working with Hilbert spaces do no longer hold when working with Hilbert modules.

Moreover we have

$$\mathcal{R}(t) = \mathcal{F}(F(t)), \tag{1.9}$$

$$\mathcal{R}(t^*) = \mathcal{F}(F(t)^*), \tag{1.10}$$

$$t\mathcal{R}(t) = F(t)\{\mathcal{F}(F(t))\}^{1/2} \tag{1.11}$$

for regular operators $t \in R(H)$.

Definition 1.12 (Extended gap topology)**.** Let $B(H)$ denote the set of bounded adjointable operators F satisfying $\|F\|_{\mathbb{B}(H)} \leq 1$. We define $B(H,d)$ to be the topological space that we obtain when we equip $B(H)$ with the topology induced by the pseudo metric

$$d(F,F') = \sup \left\{ \begin{array}{c} \|\mathcal{F}(F) - \mathcal{F}(F')\|_{\mathbb{B}(H)}, \\ \|\mathcal{F}(F^*) - \mathcal{F}(F'^*)\|_{\mathbb{B}(H)}, \\ \|F\mathcal{F}(F)^{1/2} - F'\mathcal{F}(F')^{1/2}\|_{\mathbb{B}(H)} \end{array} \right\}. \tag{1.13}$$

Proposition 1.14. *The map*

$$R(H) \longrightarrow B(H,d), \ t \longmapsto F(t)$$

is a topological inclusion.

Proof. By (1.4) the map is injective. The fact that $R(H)$ carries the induced topology follows from (1.9) - (1.11). □

As mentioned at the beginning of this section it will turn out that the extended gap topology will not be good enough for performing certain relevant homotopies that will come up in the next section. We therefore will show that the gap topology is also the restriction of a stronger topology on $B(H)$. This topology we call the *strict extended gap topology* (cf. Definition 1.30), and we simply write $B(H)$ for the corresponding space. By definition the strict extended gap topology is the weakest topology on the set $B(H)$ which contains the extended gap topology as well as the so-called strict topology (cf. page 9). To see that $R(H)$ also can be regarded as a subspace of $B(H)$ we need to show that the map $R(H) \to B(H)$ is strictly continuous. For doing so we first establish a different characterization of the extended gap topology (stated in Corollary 1.23), which also might be of independent interest.

We need to consider $\mathbb{Z}/2$-gradings on Hilbert modules and operators. This will allow us (by a standard argument) to concentrate on self-adjoint operators. A $\mathbb{Z}/2$-grading on a Hilbert-$\mathcal{A}$-module $\widehat{H}$ by definition is a symmetry $\varepsilon \in \mathbb{B}(\widehat{H})$, i.e. ε satisfies

$$\varepsilon^* = \varepsilon \text{ and } \varepsilon^2 = 1 \in \mathbb{B}(\widehat{H}).$$

The symmetry can be used to decompose H according to the ± 1-eigenspaces. The operator ε then allows to consider the two subsets

$$\begin{aligned} \mathbb{B}(\widehat{H})^0 &= \{F \in \mathbb{B}(\widehat{H}) \mid \varepsilon F = F\varepsilon\}, \\ \mathbb{B}(\widehat{H})^1 &= \{F \in \mathbb{B}(\widehat{H}) \mid \varepsilon F = -F\varepsilon\} \end{aligned}$$

of $\mathbb{B}(\widehat{H})$, called the even and the odd bounded adjointable operators on the $\mathbb{Z}/2$-graded Hilbert module $\widehat{H} = (\widehat{H}, \varepsilon)$. We will only consider the simple case where $\widehat{H} = H \oplus H$ and where ε acts by $1 \oplus (-1) \in \mathbb{B}(\widehat{H})$. Thus an odd operator $F \in \mathbb{B}(\widehat{H})$ is given by a matrix

$$F = \begin{pmatrix} 0 & F_1 \\ F_0 & 0 \end{pmatrix},$$

where $F_0, F_1 \in \mathbb{B}(H)$.

Lemma 1.15. *The map which associates to a bounded adjointable operator $F \in B(H,d)$ the self-adjoint operator*

$$\widehat{F} = \begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix}$$

defines a pseudo metric preserving homeomorphism from the space $B(H,d)$ onto the subspace of odd self-adjoint bounded operators $B(\widehat{H},d)^1_{sa} \subset B(\widehat{H},d)$.

Proof. Clearly the map $F \mapsto \widehat{F}$ is a bijection onto the subspace of odd self-adjoint bounded operators in $B(\widehat{H},d)$. Hence we need to check that the map preserves the pseudo metrics. We have

$$\begin{aligned} \mathcal{F}(\widehat{F}) &= \begin{pmatrix} 1 - F^*F & 0 \\ 0 & 1 - FF^* \end{pmatrix}, \\ \widehat{F}\mathcal{F}(\widehat{F})^{1/2} &= \begin{pmatrix} 0 & F^*(1 - FF^*)^{1/2} \\ F(1 - F^*F)^{1/2} & 0 \end{pmatrix}. \end{aligned}$$

Hence for $F, F' \in B(H,d)$ we have

$$\begin{aligned} \|\mathcal{F}(\widehat{F}) - \mathcal{F}(\widehat{F'})\|_{\mathbb{B}(\widehat{H})} &= \sup \left\{ \begin{array}{c} \|\mathcal{F}(F) - \mathcal{F}(F')\|_{\mathbb{B}(H)}, \\ \|\mathcal{F}(F^*) - \mathcal{F}(F'^*)\|_{\mathbb{B}(H)} \end{array} \right\}, \\ \|\widehat{F}\mathcal{F}(\widehat{F})^{1/2} - \widehat{F'}\mathcal{F}(\widehat{F'})^{1/2}\|_{\mathbb{B}(\widehat{H})} &= \|F\mathcal{F}(F)^{1/2} - F'\mathcal{F}(F')^{1/2}\|_{\mathbb{B}(H)}, \end{aligned}$$

and therefore $d(\widehat{F}, \widehat{F'}) = d(F, F')$ as claimed. □

Let $C_b(\mathbb{R})$ denote the C^*-algebra of continuous bounded functions on $\mathbb{R}$. If t is a self-adjoint regular operator we can use the bounded transform $F(t)$ to do functional calculus with t using functions $u \in C_b(\mathbb{R})$. More precisely, let $q : \mathbb{R} \to (-1,1)$ be given by $q(s) = s(1+s^2)^{-1/2}$, and let p denote its inverse. Given a function $u \in C_b(\mathbb{R})$ we use the function p to define $u(t) = (u \circ p)(F(t)) \in \mathbb{B}(H)$. We thus obtain a unital $*$-homomorphism

$$fc(t) : C_b(\mathbb{R}) \to \mathbb{B}(H), \quad u \mapsto u(t). \tag{1.16}$$

If $F \in B(H)$ is self-adjoint and $u \in C_b(\mathbb{R})$ has a continuous extension to $\overline{\mathbb{R}} = [-\infty, \infty]$ we also can define $u_q(F) = (u \circ p)(F)$. In particular, for any self-adjoint operator $F \in B(H)$ we obtain a unital $*$-homomorphism

$$fc_q(F) : C_0(\mathbb{R})^+ \to \mathbb{B}(H), \quad u \mapsto u_q(F), \tag{1.17}$$

from $C_0(\mathbb{R})^+$, the unitization of the C^*-algebra $C_0(\mathbb{R})$ of continuous functions vanishing at infinity, into $\mathbb{B}(H)$, the C^*-algebra of bounded adjointable operators. In the following we will give a description of the extended gap topology in terms of functional calculus. We need the following lemma.

Given two C^*-algebras $\mathcal{C}$ and $\mathcal{B}$ let us denote by $Hom(\mathcal{C}, \mathcal{B})$ the set of $*$-homomorphisms equipped with the weak topology. If $\mathcal{C}$ and $\mathcal{B}$ are unital let $Hom_{(u)}(\mathcal{C}, \mathcal{B}) \subset Hom(\mathcal{C}, \mathcal{B})$ denote the subspace of unital $*$-homomorphims.

Lemma 1.18.
(A) *Let $q_0, q_1 \in C_0(\mathbb{R})$ be the functions given by $q_0(s) = (1+s^2)^{-1}$ and $q_1(s) = sq_0(s)$ respectively. The functions q_0 and q_1 generate $C_0(\mathbb{R})$ as a C^*-algebra. Moreover, for any C^*-algebra $\mathcal{B}$ the topology on $Hom(C_0(\mathbb{R}), \mathcal{B})$ is the topology induced from the evaluation map*

$$ev : Hom(C_0(\mathbb{R}), \mathcal{B}) \xrightarrow{(ev_{q_0}, ev_{q_1})} \mathcal{B} \times \mathcal{B}, \tag{1.19}$$

which maps $f \in Hom(C_0(\mathbb{R}), \mathcal{B})$ to the pair $(f(q_0), f(q_1))$.
(B) *Let $q : \mathbb{R} \to \mathbb{R}$ be given by $q(s) = s(1+s^2)^{-1/2}$. Then $C_0(\mathbb{R})^+$ as a C^*-algebra is generated by the functions $1, q^2, q - q^3$. Moreover, for any unital C^*-algebra $\mathcal{B}$ the topology on $Hom_{(u)}(C_0(\mathbb{R})^+, \mathcal{B})$ is the topology induced from the evaluation map*

$$ev_{(u)} : Hom_{(u)}(C_0(\mathbb{R})^+, \mathcal{B}) \xrightarrow{(ev_{q^2}, ev_{q-q^3})} \mathcal{B} \times \mathcal{B}, \tag{1.20}$$

which maps $f \in Hom_{(u)}(C_0(\mathbb{R})^+, \mathcal{B})$ to the pair $(f(q^2), f(q - q^3))$.

Proof. Let $C = C([-\infty, \infty])$ denote the C^*-algebra of continuous functions on $\mathbb{R}$ which have a continuous extension to $[-\infty, \infty]$. By the Weierstraß approximation theorem we know that 1 and q generate C as a C^*-algebra, i.e. the polynomials in q form a dense linear subspace of C. Now consider the map $ev_{\pm\infty} : C \to \mathbb{C} \oplus \mathbb{C}$ given by evaluating a function in C at $\pm\infty$. Clearly the sub-C^*-algebra $C^*(q_0, q_1) \subset C$ generated by q_0 and q_1 is contained in $Ker(ev_{\pm\infty}) = C_0(\mathbb{R})$. On the other hand we have $q^2 = 1 - q_0$ and $qq_0 = q_1 {q_0}^{1/2}$, and hence for $n \in \mathbb{N}$ we have

$$\begin{aligned} q^{2n} &= (1-q_0)^n = 1 - nq_0 + \binom{n}{2} {q_0}^2 + \cdots + (-1)^n {q_0}^n, \\ q^{2n+1} &= qq^{2n} = q - nq_1 {q_0}^{1-1/2} + \binom{n}{2} q_1 {q_0}^{2-1/2} + \cdots + (-1)^n q_1 {q_0}^{n-1/2}. \end{aligned}$$

We therefore have $\mathbb{C}1 + \mathbb{C}q + C^*(q_0, q_1) = C$, and as $ev_{\pm\infty}$ is surjective, we can conclude that $Ker(ev_{\pm\infty}) = C_0(\mathbb{R}) = C^*(q_0, q_1)$, as claimed.

To show the second part of the first statement let $f \in Hom(C_0(\mathbb{R}), \mathcal{B})$. We need to show that for a given function $u \in C_0(\mathbb{R})$ and any real number $\varepsilon > 0$ there is a set $U \subset Hom(C_0(\mathbb{R}), \mathcal{B})$ which is open in the topology induced from ev such that $f \in U$ and such that $g \in U$ implies $\|f(u) - g(u)\|_{\mathcal{B}} < \varepsilon$. Since $C_0(\mathbb{R}) = C^*(q_0, q_1)$ the polynomials in q_0 and q_1 are dense in $C_0(\mathbb{R})$, and hence we can find $k \in \mathbb{N}$, and $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ and $m_1, \ldots m_k, n_1, \ldots n_k \in \mathbb{N}_0$ such that $m_i + n_i \geq 1$ for all $1 \leq i \leq k$ and

$$\|u - \sum\nolimits_{i=1}^k \lambda_i {q_0}^{m_i} {q_1}^{n_i}\|_{C_0(\mathbb{R})} < \varepsilon/4.$$

Let $L \geq \sup_i\{|\lambda_i|\}$ and $M = \sup_i\{m_i + n_i\}$. Without loss of generality assume that $L \geq 1$ and define $\varepsilon' = (2kLM)^{-1}\varepsilon$. We then put

$$U \quad = \quad \{g \in Hom(C_0(\mathbb{R}), \mathcal{B}) \mid \|f(q_0) - g(q_0)\|_{\mathcal{B}},\ \|f(q_1) - g(q_1)\|_{\mathcal{B}} < \varepsilon'\}$$

which is open in the topology of $Hom(C_0(\mathbb{R}), \mathcal{B})$ induced from ev. Now, if $l \in \mathbb{N}$ and $a_1, \ldots, a_l$ and $b_1, \ldots, b_l$ are elements of a C^*-algebra $\mathcal{C}$ such that all norms $\|a_j\|_{\mathcal{C}}, \|b_j\|_{\mathcal{C}}$ for $j = 1, \ldots, l$ are bounded by 1 one can easily prove by induction over l that

$$\left\| \prod_{j=1}^l a_j - \prod_{j=1}^l b_j \right\|_{\mathcal{C}} \leq \sum_{j=1}^l \|a_j - b_j\|_{\mathcal{C}}$$

For $g \in U$ this then implies $\|f({q_0}^{m_i}{q_1}^{n_i}) - g({q_0}^{m_i}{q_1}^{n_i})\|_{\mathcal{B}} \leq (m_i + n_i)\varepsilon' \leq M\varepsilon'$ since f and g are norm decreasing. Hence we obtain

$$\begin{aligned} \|f(u) - g(u)\|_{\mathcal{B}} &\leq \varepsilon/2 + \left\| \sum\nolimits_{i=1}^k \lambda_i (f({q_0}^{m_i}{q_1}^{n_i}) - g({q_0}^{m_i}{q_1}^{n_i})) \right\|_{\mathcal{B}} \\ &\leq \varepsilon/2 + \sum\nolimits_{i=1}^k |\lambda_i| \cdot \|(f({q_0}^{m_i}{q_1}^{n_i}) - g({q_0}^{m_i}{q_1}^{n_i}))\|_{\mathcal{B}} \\ &\leq \varepsilon/2 + kLM\varepsilon' \; = \varepsilon. \end{aligned}$$

This completes the proof of statement (A); statement (B) follows essentially by the same line of argument. □

Remark 1.21. Assertion (B) of the previous lemma will not be needed in this section and is stated here for future reference. Note that in the case where $\mathcal{B}$ is unital we have a canonical homeomorphism $Hom(C_0(\mathbb{R}), \mathcal{B}) \cong Hom_{(u)}(C_0(\mathbb{R})^+, \mathcal{B})$. Hence the lemma provides two characterizations of the topology on $Hom(C_0(\mathbb{R}), \mathcal{B})$. The second characterization is in some sense easier to deal with and will be used in the following section.

Let $B(H,d)_{sa} \subset B(H,d)$ denote the subspace of self-adjoint operators in $B(H,d)$.

Proposition 1.22. *The map*

$$fc_q : B(H,d)_{sa} \to Hom_{(u)}(C_0(\mathbb{R})^+, \mathbb{B}(H))$$

which maps an operator $F \in B(H,d)_{sa}$ to $fc_q(F) \in Hom_{(u)}(C_0(\mathbb{R})^+, \mathbb{B}(H))$ as defined in (1.17) is a topological inclusion.

Proof. The topology on $B(H,d)_{sa}$ is induced by the pseudo metric d, defined in (1.13). Hence it can equivalently be described as being induced by the map $B(H,d)_{sa} \to \mathbb{B}(H) \times \mathbb{B}(H)$ given by $F \mapsto (\mathcal{F}(F), F\mathcal{F}(F)^{1/2})$. The map can be factored through $Hom_{(u)}(C_0(\mathbb{R})^+, \mathbb{B}(H))$. The factorization is given by $ev \circ fc_q$, where $F \mapsto fc_q(F)$ is defined by (1.17) and ev is as in (1.19). As we have already seen in the previous lemma, the topology on $Hom_{(u)}(C_0(\mathbb{R})^+, \mathbb{B}(H))$ is induced from ev and hence it follows that the topology on $B(H,d)_{sa}$ is induced from fc_q. $\square$

Corollary 1.23. *The map $B(H,d) \to Hom_{(u)}(C_0(\mathbb{R})^+, \mathbb{B}(\widehat{H}))$ which maps F to $fc_q(\widehat{F})$ is a topological inclusion. In particular it follows that the gap topolgy on $R(H)$ is the one induced from the map*

$$R(H) \to Hom(C_0(\mathbb{R}), \mathbb{B}(\widehat{H})),\ t \mapsto fc(\hat{t}) = fc_q(\widehat{F(t)}). \tag{1.24}$$

Finally we want to relate the gap topology on $R(H)$ to the strict topology on $\mathbb{B}(H)$, which is defined as follows. The C^*-algebra of bounded adjointable operators $\mathbb{B}(H)$ can be identified with the multiplier algebra $\mathcal{M}(\mathbb{K}(H))$ of the compact adjointable operators $\mathbb{K}(H)$. In general, given a C^*-algebra $\mathcal{B}$ the *strict topology* on the multiplier algebra $\mathcal{M}(\mathcal{B})$ is the weakest topology for which the multiplication maps

$$l_b : \mathcal{M}(\mathcal{B}) \to \mathcal{B}, \quad m \mapsto mb$$

$$r_b : \mathcal{M}(\mathcal{B}) \to \mathcal{B}, \quad m \mapsto bm$$

are continuous for all $b \in \mathcal{B}$. Given two C^*-algebras $\mathcal{C}$ and $\mathcal{B}$ then a $*$-homomorphisms $f : \mathcal{C} \to \mathcal{M}(\mathcal{B})$ is called *non-degenerate* if $f(\mathcal{C})\mathcal{B}$ is dense in $\mathcal{B}$. We define $Mor(\mathcal{C}, \mathcal{B})$ to be the set of non-degenerate $*$-homomorphism from $\mathcal{C}$ to $\mathcal{M}(\mathcal{B})$. It is well-known (e.g. see [8, 2.1]) that a non-degenerate $*$-homomorphism $f : \mathcal{C} \to \mathcal{M}(\mathcal{B})$ uniquely lifts to $*$-homomorphism between the multiplier algebras

$$\mathcal{M}(f) : \mathcal{M}(\mathcal{C}) \to \mathcal{M}(\mathcal{B}).$$

Lemma 1.25. *The set map*

$$Mor(\mathcal{C}, \mathcal{B}) \longrightarrow Hom(\mathcal{M}(\mathcal{C}), \mathcal{M}(\mathcal{B})),\ f \longmapsto \mathcal{M}(f)$$

becomes a continuous map if we equip the homomorphism sets with the point-open topology[3] *with $\mathcal{M}(\mathcal{B})$ carrying the strict topology.*

Proof. We need to show that for any tuple $(f, m, b, \varepsilon) \in Mor(\mathcal{C}, \mathcal{B}) \times \mathcal{M}(\mathcal{C}) \times \mathcal{B} \times \mathbb{R}_+$ we can find an open set $U \subset Mor(\mathcal{C}, \mathcal{B})$ such that $f \in U$ and such that $g \in U$ implies

$$\|((\mathcal{M}(f) - \mathcal{M}(g))(m))b\|_{\mathcal{B}} < \varepsilon \text{ and } \|b((\mathcal{M}(f) - \mathcal{M}(g))(m))\|_{\mathcal{B}} < \varepsilon.$$

Without loss of generality we may assume that $m \neq 0$, and we put $M = \|m\|_{\mathcal{M}(\mathcal{C})}$. As f is non-degenerate we can find $k \in \mathbb{N}$ and elements $c_i \in \mathcal{C}, b_i \in \mathcal{B}$ for $i = 1, \ldots, k$ such that

$$\left\| b - \sum_{i=1}^{k} f(c_i) b_i \right\|_{\mathcal{B}} < \varepsilon/(4M).$$

Without loss of generality we may assume that $b_i \neq 0$ for all i. We then define the following subspaces of $Mor(\mathcal{C}, \mathcal{B})$

$$U_L = \left\{ g \;\middle|\; \begin{array}{ccc} \|f(c_i) - g(c_i)\|_{\mathcal{B}} & < & \varepsilon/(4kM\|b_i\|_{\mathcal{B}}) \\ \|f(mc_i) - g(mc_i)\|_{\mathcal{B}} & < & \varepsilon/(4k\|b_i\|_{\mathcal{B}}) \end{array} \text{ for } i = 1, \ldots, k \right\}$$

$$U_R = \left\{ g \;\middle|\; \begin{array}{ccc} \|f(c_i) - g(c_i)\|_{\mathcal{B}} & < & \varepsilon/(4kM\|b_i\|_{\mathcal{B}}) \\ \|f(c_i m) - g(c_i m)\|_{\mathcal{B}} & < & \varepsilon/(4k\|b_i\|_{\mathcal{B}}) \end{array} \text{ for } i = 1, \ldots, k \right\}$$

Now, if $g \in U_L$ we have

$$\begin{aligned}
&\|((\mathcal{M}(f) - \mathcal{M}(g))(m))b\|_{\mathcal{B}} \\
&\leq \quad 2M\varepsilon/(4M) + \|((\mathcal{M}(f) - \mathcal{M}(g))(m)) \sum_{i=1}^{k} f(c_i) b_i\|_{\mathcal{B}} \\
&\leq \quad \varepsilon/2 + \|\sum_{i=1}^{k} (\mathcal{M}(f)(m)(fc_i) - \mathcal{M}(g)(m)(gc_i)) b_i\|_{\mathcal{B}} \\
&\qquad\quad + M\|\sum_{i=1}^{k} (fc_i - gc_i) b_i\|_{\mathcal{B}} \\
&\leq \quad \varepsilon/2 + \sum_{i=1}^{k} \|f(mc_i) - g(mc_i)\|_{\mathcal{B}} \|b_i\|_{\mathcal{B}} + M \sum_{i=1}^{k} \|fc_i - gc_i\|_{\mathcal{B}} \|b_i\|_{\mathcal{B}} \\
&< \quad \varepsilon/2 + k\varepsilon/(4k) + Mk\varepsilon/(4Mk) = \varepsilon.
\end{aligned}$$

Similarly if $g \in U_R$ we obtain $\|b((\mathcal{M}(f) - \mathcal{M}(g))(m))\|_{\mathcal{B}} < \varepsilon$. Hence we can put $U = U_L \cap U_R$. $\square$

---

[3]Given spaces X and Y the *point-open topology* on the set $map(X, Y)$ of continuous maps from X to Y is generated by the sets $U(x, V) = \{f \in map(X, Y) \mid f(x) \in V\}$ where x varies over the points of X and V varies over the open sets of Y.

As already stated, in the case where $\mathcal{B} = \mathbb{K}(H)$ we have an identification of sets $\mathcal{M}(\mathbb{K}(H)) \cong \mathbb{B}(H)$. On norm bounded sets, and hence in particular on the set $B(H)$, the strict topology on $\mathbb{B}(H)$ coincides with the strong $*$-topology (cf. [8, 8.1]). It is easy to see that the norm topology on $\mathbb{B}(H)$ is stronger than the strict topology.

Definition 1.26. Let $\mathbb{B}(H)_\beta$ denote the C^*-algebra of bounded adjointable operators on H equipped with the strict topology.

Lemma 1.27. *The map $\mathbb{B}(H)_\beta \to \mathbb{B}(\widehat{H})_\beta$ which sends F to the self-adjoint odd operator*

$$\widehat{F} = \begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix}$$

defines a homeomorphism onto the subspace consisting of the odd self-adjoint operators $\mathbb{B}(\widehat{H})^1_{\beta,sa} \subset \mathbb{B}(\widehat{H})_\beta$.

Proof. Observe that we can identify $\mathbb{K}(\widehat{H}) = \mathbb{K}(H \oplus H)$ with the C^*-algebra of 2×2-matrices over $\mathbb{K}(H)$; for $a, b, c, d \in \mathbb{K}(H)$ we then have

$$\widehat{F} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} F^*c & F^*d \\ Fa & Fb \end{pmatrix} \in \mathbb{K}(\widehat{H}).$$

Using this equation it is easy to verify the claim. □

Proposition 1.28. *The map $R(H) \to \mathbb{B}(H)_\beta, t \mapsto F(t)$ is continuous.*

Proof. Consider the map $R(H) \to Hom(C_0(\mathbb{R}), \mathbb{B}(\widehat{H}))$ provided by (1.24). Since $q_0(t) = \mathcal{R}(t)$ has dense image (cf. Definition 1.1) this map factors set-theoretically through $Mor(C_0(\mathbb{R}), \mathbb{K}(\widehat{H}))$. Since the strict topology is weaker than the norm topology we obtain that the corresponding map $R(H) \to Mor(C_0(\mathbb{R}), \mathbb{K}(\widehat{H}))$ is continuous. We now compose this map with the map provided by Lemma 1.25. In case $\mathcal{C} = C_0(\mathbb{R})$ the multiplier algebra can be identified with the C^*-algebra of bounded continuous functions $C_b(\mathbb{R})$. We obtain the map

$$\begin{array}{cccccc} & R(H) & \longrightarrow & R(\widehat{H}) & \longrightarrow & Hom(C_b(\mathbb{R}), \mathbb{B}(\widehat{H})_\beta) \\ (1.29) & t & \longmapsto & \widehat{t} = \begin{pmatrix} 0 & t^* \\ t & 0 \end{pmatrix} & \longmapsto & fc(\widehat{t}), \end{array}$$

where $fc(\widehat{t})$ is given by (1.16). By Lemma 1.25 evaluating $fc(\widehat{t})$ at a specific function in $C_b(\mathbb{R})$ yields a continuous map $R(H) \to \mathbb{B}(\widehat{H})_\beta$. We now choose the function $q \in C_b(\mathbb{R})$ given by $s \mapsto s(1 + s^2)^{-1/2}$, so that the composition of the map (1.29) with the corresponding evaluation map yields the map which associates to $t \in R(H)$ the bounded adjointable operator $\widehat{F(t)}$, i.e. $t \mapsto \widehat{F(t)}$ is strictly continuous. The statement of the proposition now follows from the previous lemma. □

Definition 1.30 (strict extended gap topology)**.** Let the space $B(H)$ be the set $B(H)$ equipped with the weakest topology which contains the extended gap topology (Definition 1.12) as well as the strict topology (Definition 1.26), i.e. the set is equipped with the weakest topology such that the maps $id : B(H) \to B(H,d)$ and $id : B(H) \to \mathbb{B}(H)_\beta$ are both continuous.

Proposition 1.31. *The map*

$$R(H) \longrightarrow B(H),\ t \longmapsto F(t)$$

is a topological inclusion.

Proof. This follows from Proposition 1.14 and Proposition 1.28. □

We will use this proposition to occasionally think of $R(H)$ as being a subspace of $B(H)$.

2. Fredholm operators and operators with compact resolvents

In this section we consider regular $\mathcal{A}$-Fredholm operators and operators with compact resolvents acting on H. They define two subspaces $FredR(H), CRR(H) \subset R(H)$. Both subspaces have corresponding analogues $FredB(H)$ and $CRB(H)$ contained in $B(H)$, where $B(H)$ denotes the unit ball of the adjointable bounded operators on H, equipped with the strict extended gap topology (cf. Definition 1.30). We will see that there is a commutative diagram of topological inclusions

$$\begin{tikzcd}
CRR(H) \arrow[r] \arrow[d] & CRB(H) \arrow[d] \\
FredR(H) \arrow[r] & FredB(H),
\end{tikzcd}$$

where the maps from the left to the right are given by taking the bounded transform ((1.3)). The main goal of this section is to show that all these topological inclusions are weak homotopy equivalences, and that the analog statement is also true for the diagram of subspaces that we obtain when we restrict to self-adjoint operators.

In the following we need to compose regular operators. The composition of two regular operators $t, t' \in R(H)$ is the unbounded operator $t' \circ t$ with $dom(t' \circ t) = \{x \in dom(t) \mid t(x) \in dom(t')\}$ given by $(t' \circ t)(x) = t'(tx))$ for all $x \in dom(t' \circ t)$. Note that in general the composition of regular operators will not be a regular operator.

Definition 2.1. Let $t \in R(H)$ be a regular operator. An adjointable bounded operator $G \in \mathbb{B}(H)$ is called a *pseudo left inverse* of t if Gt is closable and its closure $\overline{Gt}$ satisfies $\overline{Gt} \in \mathbb{B}(H)$ and $\overline{Gt} = 1 \ mod\ \mathbb{K}(H)$. Analogously G is called a *pseudo right inverse* if tG is closable and its closure

$\overline{tG}$ satisfies $\overline{tG} \in \mathbb{B}(H)$ and $\overline{tG} = 1 \; mod \; \mathbb{K}(H)$. The regular operator t is called *$\mathcal{A}$-Fredholm*, if it has a pseudo left as well as a pseudo right inverse. We write $FredR(H)$ for the subspace of $R(H)$ consisting of the $\mathcal{A}$-Fredholm operators, and $FredR(H)_{sa}$ for the subspace consisting of the self-adjoint $\mathcal{A}$-Fredholm operators. Similarly we write $FredB(H)$ for the subspace of $\mathcal{A}$-Fredholm operators in $B(H)$, and $FredB(H)_{sa}$ for the corresponding subspace of self-adjoint operators $FredB(H) \cap B(H)_{sa}$.

Lemma 2.2. *Let $t \in R(H)$ be a regular operator. Then t is $\mathcal{A}$-Fredholm if and only if $F(t)$ is.*

Proof. Assume that $F(t)$ is $\mathcal{A}$-Fredholm. By definition $F(t)$ has a pseudo left inverse, i.e. there is $G \in \mathbb{B}(H)$ and $K \in \mathbb{K}(H)$ such that $GF(t) = 1 + K$. Since t is regular so is t^* and therefore $(1 + tt^*)^{-1/2} \in \mathbb{B}(H)$. Hence $G' = G(1 + tt^*)^{-1/2} \in \mathbb{B}(H)$ and $\overline{G't} = GF(t) = 1 + K$, i.e. t has a pseudo left inverse. Since $F(t)$ is $\mathcal{A}$-Fredholm it also has a pseudo right inverse, i.e. there is $G \in \mathbb{B}(H)$ and a $K \in \mathbb{K}(H)$ such that $F(t)G = 1 + K \in \mathbb{B}(H)$. Since t is regular we have that $(1 + t^*t)^{-1} \in \mathbb{B}(H)$. Hence $G' = (1 + t^*t)^{-1/2}G \in \mathbb{B}(H)$, and $\overline{tG'} = F(t)G = 1 + K$, i.e. t has a pseudo right inverse. Thus t is an $\mathcal{A}$-Fredholm operator.

Conversely let us assume that t is $\mathcal{A}$-Fredholm, and let $G \in \mathbb{B}(H)$ and $K \in \mathbb{K}(H)$ be such that $\overline{Gt} = 1 + K$. Formally we then obtain that $G' = G \circ (1 + tt^*)^{1/2}$ is a pseudo left inverse of $F(t)$. Thus we need to check that the operator $G \circ (1 + tt^*)^{1/2}$ has a bounded extension. To see this we first notice that the operator is densely defined, which follows from the fact that $1 + tt^*$ is densely defined; actually we have $dom(G'(1 + tt^*)^{1/2}) = dom((1 + tt^*)^{1/2}) = dom(t^*)$ (cf. [8, (10.4)]). We have to check that G' is bounded on its domain. This follows from the following calculation, valid for $x \in dom(1 + tt^*)^{1/2} = dom(t^*)$

$$
\begin{aligned}
&\langle G(1+tt^*)^{1/2}x, G(1+tt^*)^{1/2}x\rangle \\
&\quad = \langle G(1+tt^*)(1+tt^*)^{-1/2}x, G(1+tt^*)(1+tt^*)^{-1/2}x\rangle \\
&\quad = \langle (1+tt^*)(1+tt^*)^{-1/2}x, G^*G(1+tt^*)(1+tt^*)^{-1/2}x\rangle \\
&\quad = \langle (1+tt^*)^{-1/2}x, G^*G(1+tt^*)(1+tt^*)^{-1/2}x\rangle \\
&\qquad + \langle tt^*(1+tt^*)^{-1/2}x, G^*G(1+tt^*)(1+tt^*)^{-1/2}x\rangle \\
&\quad = \langle G^*G(1+tt^*)^{-1/2}x, (1+tt^*)(1+tt^*)^{-1/2}x\rangle \\
&\qquad + \langle G^*Gtt^*(1+tt^*)^{-1/2}x, (1+tt^*)(1+tt^*)^{-1/2}x\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \langle G^*G(1+tt^*)^{-1/2}x, (1+tt^*)^{-1/2}x\rangle \\
&\quad + \langle G^*Gtt^*(1+tt^*)^{-1/2}x, (1+tt^*)^{-1/2}x\rangle \\
&\quad + \langle G^*G(1+tt^*)^{-1/2}x, tt^*(1+tt^*)^{-1/2}x\rangle \\
&\quad + \langle G^*Gtt^*(1+tt^*)^{-1/2}x, tt^*(1+tt^*)^{-1/2}x\rangle \\
&= \langle G(1+tt^*)^{-1/2}x, G(1+tt^*)^{-1/2}x\rangle \\
&\quad + \langle Gtt^*(1+tt^*)^{-1/2}x, G(1+tt^*)^{-1/2}x\rangle \\
&\quad + \langle G(1+tt^*)^{-1/2}x, Gtt^*(1+tt^*)^{-1/2}x\rangle \\
&\quad + \langle Gtt^*(1+tt^*)^{-1/2}x, Gtt^*(1+tt^*)^{-1/2}x\rangle
\end{aligned}
$$

since (using the Cauchy-Schwarz inequality) this calculation implies

$$
\begin{aligned}
&|G(1+tt^*)^{1/2}x|^2 \leq \\
&\qquad \{\|G\|_{\mathbb{B}(H)} \cdot \|(1+tt^*)^{-1/2}\|_{\mathbb{B}(H)} + \|\overline{Gt}\|_{\mathbb{B}(H)} \cdot \|F(t^*)\|_{\mathbb{B}(H)}\}^2 \; |x|^2.
\end{aligned}
$$

Thus $F(t)$ has a pseudo left inverse.

To see that $F(t)$ has a pseudo right inverse let G be a pseudo right inverse of t, i.e. we assume there is $K \in \mathbb{K}(H)$ such that $\overline{tG} = 1 + K$. Similarly as above we obtain a formal pseudo right inverse for $F(t)$ by $G' = (1+t^*t)^{1/2}G$; again we have to show that it has a bounded closure. This follows from the following calculation, valid for $x \in dom((1+t^*t)^{1/2}G) = dom(tG)$

$$
\begin{aligned}
|(1+t^*t)^{1/2}Gx|^2 &= \|\langle (1+t^*t)^{1/2}Gx, (1+t^*t)^{1/2}Gx\rangle\|_{\mathcal{A}} \\
&= \|\langle Gx, (1+t^*t)^{1/2}(1+t^*t)^{1/2}Gx\rangle\|_{\mathcal{A}} \\
&= \|\langle Gx, (1+t^*t)Gx\rangle\|_{\mathcal{A}} \\
&= \|\langle Gx, Gx\rangle + \langle Gx, t^*tGx\rangle\|_{\mathcal{A}} \\
&= \|\langle Gx, Gx\rangle + \langle tGx, tGx\rangle\|_{\mathcal{A}} \\
&\leq \{\|G\|^2_{\mathbb{B}(H)} + \|\overline{tG}\|^2_{\mathbb{B}(H)}\} \; |x|^2
\end{aligned}
$$

Thus $F(t)$ has a pseudo right inverse. □

Definition 2.3. We say that a regular operator $t \in R(H)$ has *compact resolvents* if $\mathcal{R}(t) = (1+t^*t)^{-1} \in \mathbb{K}(H)$ and $\mathcal{R}(t^*) = (1+tt^*)^{-1} \in \mathbb{K}(H)$. We define $CRR(H)$ to be the subspace of $R(H)$ consisting of the regular operators with compact resolvents, and we define $CRR(H)_{sa}$ to be the subspace consisting of the self-adjoint regular operators with compact resolvents.

An operator $T \in B(H)$ is called *essentially unitary* if $\mathcal{F}(T) = 1 - T^*T \in \mathbb{K}(H)$ and $\mathcal{F}(T^*) = 1 - TT^* \in \mathbb{K}(H)$. We define $CRB(H)$ to be the subspace of essentially unitary operators in $B(H)$, and we define $CRB(H)_{sa}$ to be the subspace of self-adjoint essentially unitary operators in $B(H)$.

Lemma 2.4. *Let X be a space and let $B(H,d)$ denote the unit ball of the adjointable bounded operators, equipped with the extended gap topology*

(Definition 1.12). A set map $F : X \to B(H,d)$ is continuous if and only if the three maps $X \to \mathbb{B}(H)$ given by

$$\begin{array}{rcl} x & \mapsto & F^*(x)F(x) \\ x & \mapsto & F(x)F^*(x) \\ x & \mapsto & F(x)(1 - F^*(x)F(x)) \end{array}$$

are norm continuous.

Proof. Given a continuous map $X \to B(H,d)$ then the first two functions $X \to \mathbb{B}(H)$ above are continuous by the very definition of the topology of $B(H,d)$, and hence so is $x \mapsto \sqrt{1 - F^*(x)F(x)}$. Furthermore by definition of the topology of $B(H,d)$ the map $x \mapsto F(x)\sqrt{1 - F(x)^*F(x)}$ is continuous. Therefore so is

$$x \mapsto F(x)(1 - F(x)^*F(x)) = F(x)\sqrt{1 - F(x)^*F(x)}\sqrt{1 - F(x)^*F(x)}.$$

Conversely, let us assume that $F : X \to B(H,d)$ is a set map and assume that the three functions above are continuous. By Corollary 1.23 the function F is continuous if the map $x \mapsto fc_q(\widehat{F(x)})$ is continuous. By Lemma 1.18 (B) this is equivalent to showing that the maps

$$x \mapsto \widehat{F(x)}^2 \quad \text{and} \quad x \mapsto \widehat{F(x)}(1 - \widehat{F(x)}^2)$$

are norm continuous. Continuity of F then follows as we have

$$\begin{array}{rcl} \widehat{F(x)}^2 & = & \begin{pmatrix} F(x)^*F(x) & 0 \\ 0 & F(x)F(x)^* \end{pmatrix} \\ \widehat{F(x)}(1 - \widehat{F(x)}^2) & = & \begin{pmatrix} 0 & F(x)^*(1 - F(x)F(x)^*) \\ F(x)(1 - F(x)^*F(x)) & 0 \end{pmatrix} \end{array}$$

□

Lemma 2.5. *Let X be a compact space and let $B(H)$ denote the unit ball of the adjointable bounded operators, equipped with the strict extended gap topology (Definition 1.30). Let $F : X \to B(H)$ and $K : X \to \mathbb{K}(H)$ be continuous maps.*

(i) *The function $X \to \mathbb{K}(H)$, $x \mapsto F(x)K(x)$ is continuous;*
(ii) *If $F(x) + K(x) \in B(H)$ for all $x \in X$, then the map*

$$X \longrightarrow B(H),\ x \longmapsto F(x) + K(x)$$

is continuous.

Proof. In order to prove the first statement it suffices to know that F is strictly continuous and norm bounded. By a theorem of Akemann, Pedersen and Tomiyama ([1, 3.4], see also [11, 2.H & 2.I]) the function F then corresponds to an element in the multiplier algebra $\mathcal{M}(X\mathbb{K}(H))$ of the C^*-algebra $X\mathbb{K}(H)$ of continuous functions on X with values in $\mathbb{K}(H)$. By the definition of the multiplier algebra it follows that FK gives an element in $X\mathbb{K}(H)$, which proves the first statement.

Since the topology on $B(H)$ is stronger than the topology of $B(H,d)$ it follows from the previous lemma that the functions F^*F, FF^* and $F(1-F^*F)$ are norm continuous. From the first statement we obtain that the maps FK, FK^*, F^*K and F^*K^* are all norm continuous. It then follows that the maps $(F+K)^*(F+K), (F+K)(F+K)^*$ and $(F+K)(1-(F+K)^*(F+K))$ are also norm continuous. Using the previous lemma again we conclude that $F+K : X \to B(H,d)$ is continuous. But with F also $F+K$ is strictly continuous, hence the second statement. □

Proposition 2.6. *The two inclusions*

$$CRB(H) \subset FredB(H) \quad \textit{and} \quad CRB(H)_{sa} \subset FredB(H)_{sa}$$

are weak homotopy equivalences.

Proof. In order to proof that an inclusion of topological spaces $A \subset Y$ is a weak homotopy equivalence one may show the following two statements

(ABS) If $F : X \to Y$ is any map with X being compact then the map is homotopic to a map whose image is contained in A;

(REL) If $h : X \times I \to Y$ is any map with X being compact such that the image of h_0 and h_1 is contained in A then h is homotopic relative boundary to a map whose image is contained in A.

We will prove the absolute case (ABS) for the inclusions $CRB(H) \subset FredB(H)$ and $CRB(H)_{sa} \subset FredB(H)_{sa}$; the corresponding relative case (REL) can be obtained in a similar fashion.

Let $X\mathcal{A}$ denote the C^*-algebra of continuous function on X with values in $\mathcal{A}$, and let XH denote the Hilbert $X\mathcal{A}$-module consisting of continuous functions on X with values in H: the scalar product on XH is given by $\langle u, u'\rangle : x \mapsto \langle u(x), u'(x)\rangle$ for functions $u, u' \in X\mathcal{A}$. Given an arbitrary continuous map $F : X \to FredB(H)$ it determines a corresponding adjointable operator $F \in B(XH)$, given by $(F(g))(x) = F(x)g(x)$. Since $F(x)$ is $\mathcal{A}$-Fredholm for every $x \in X$ it follows that $F \in \mathbb{B}(XH)$ maps to an invertible element in the Calkin algebra $\mathbb{B}(XH)/\mathbb{K}(XH)$. By [11, 17.1.4] we can find a compact perturbation G of F which is given as a product $G = WR$ with R being positive and invertible and W being a partial isometry satisfying $1 - W^*W, 1 - WW^* \in \mathbb{K}(XH)$. Moreover we may assume that $\|R\|_{\mathbb{B}(XH)} \leq 1$. If this is not already so we can change the self-adjoint operator H in the proof of [11, 17.1.4] by $-|H|$ to achieve this property. As already stated, the difference $G - F$ is compact. Using the canonical isomorphism $\mathbb{K}(XH) \cong X\mathbb{K}(H)$ the difference $G-F$ corresponds to a (norm continuous) map $K : X \to \mathbb{K}(H)$. By the previous lemma the one parameter family

$$F_s(x) = F(x) + sK(x)$$

defines a homotopy between functions F and G. Since $F_s(x)$ only differs from $F(x)$ by a compact operator it is clear that with $F(x)$ also $F_s(x)$ is an $\mathcal{A}$-Fredholm operator. Moreover we have $\|F_s\|_{\mathbb{B}(XH)} \leq 1$ since

$$\|F_s\|_{\mathbb{B}(XH)} = \|(1-s)F + sG\|_{\mathbb{B}(XH)} \leq (1-s)\|F\|_{\mathbb{B}(XH)} + s\|G\|_{\mathbb{B}(XH)} \leq 1.$$

By construction the kernels of the operators G and G^* are finitely generated projective $X\mathcal{A}$-modules (cf. [11, 17.1.4 & 15.4.2]). Let $P \in \mathbb{K}(XH)$ and $Q \in \mathbb{K}(XH)$ be the compact projections onto the kernels of G and G^* respectively. We then have

$$QG = 0 = GP \quad \text{and} \quad PG^* = 0 = G^*Q.$$

Via the isomorphism $\mathbb{K}(XH) \cong X\mathbb{K}(H)$ these projections can be identified with (norm continuous) maps $P, Q : X \to \mathbb{K}(H)$. We use them to define maps $U, V : X \to FredB(H)$ by

$$U = G^*G + P \quad \text{and} \quad V = GG^* + Q.$$

These maps are norm continuous and they satisfy

$$VG = GU = GG^*G, \quad UG^* = G^*V = G^*GG^*. \tag{2.7}$$

By construction the maps U, V take values in the subset of positive invertible operators (cf. [11, 15.3.8]). In particular we can define U^α and V^α for any real number α. It follows from (2.7) that we have

$$V^\alpha G = GU^\alpha \quad \text{and} \quad U^\alpha G^* = G^*V^\alpha \quad \text{for all } \alpha. \tag{2.8}$$

We use the function V to define

$$T = V^{-1/2}G : X \longrightarrow CRB(H) \subset FredB(H).$$

The function T^*T maps $x \in X$ to the projection onto the complement of the kernel of $G(x)$ which can be checked easily using equation (2.9) below. Similary one sees that TT^* maps $x \in X$ to the projection onto the kernel of $G^*(x)$. In particular we have $\|T(x)\|_{\mathbb{B}(H)} \leq 1$ for all $x \in X$. We claim that the function T is continuous. Clearly T is strictly continuous. Using Lemma 2.4 it remains to check that the functions T^*T, $T(1-T^*T)$, TT^* are norm continuous. Using (2.8) we have

$$T^*T = U^{-1/2}G^*V^{-1/2}G = U^{-1}G^*G, \tag{2.9}$$

$$TT^* = V^{-1/2}GU^{-1/2}G^* = V^{-1}GG^*. \tag{2.10}$$

The right hand sides are products of norm continuous functions, which shows that the left hand sides are norm continuous. To see that $T(1-T^*T)$ is norm continuous recall that T is strictly continuous and that $1 - T^*T$ is norm continuous and compact, so $T(1 - T^*T)$ is norm continuous by Lemma 2.5(i).

We then obtain a path joining G and T by defining

$$G_s = (s + (1-s)V^{-1})^{1/2}G, \quad s \in [0,1].$$

Note that the function $(s+(1-s)V^{-1})$ takes values in the positive invertible operators for all $s \in [0,1]$. Thus $(s+(1-s)V^{-1})^{\alpha}$ is defined for all real numbers α, and similarly for V replace by U. Moreover we have

$$(s+(1-s)V^{-1})^{\alpha}G = G(s+(1-s)U^{-1})^{\alpha}; \tag{2.11}$$

$$(s+(1-s)U^{-1})^{\alpha}G^* = G^*(s+(1-s)V^{-1})^{\alpha} \tag{2.12}$$

Below we will verify that $G_s(x) \in FredB(H)$ for all $(x,s) \in X \times I$. To see that the corresponding map $X \times I \to FredB(H)$, $(x,s) \mapsto G_s(x)$ is continuous first note that $G_s = (s+(1-s)V^{-1})^{1/2}G$ is strictly continuous. Thus it essentially remains to check (cf. Lemma 2.4) that the maps $G_s^*G_s$, $G_sG_s^*$ and $G_s(1-G_s^*G_s)$ are norm continuous. Using (2.11) and (2.12) we have

$$\begin{aligned}
G_s^*G_s &= (s+(1-s)U^{-1})G^*G; \\
G_sG_s^* &= (s+(1-s)V^{-1})GG^*; \\
G_s(1-G_s^*G_s) &= G_s(1-(s+(1-s)U^{-1})G^*G) \\
&= G_s(s(1-G^*G)+(1-s)(1-T^*T)) \\
&= (s+(1-s)V^{-1})^{1/2}Gs(1-G^*G)+G_s(1-T^*T).
\end{aligned}$$

We see that $G_s^*G_s$ and $G_sG_s^*$ are norm continuous since they are the product of norm continuous functions. To see norm continuity of $G_s(1-G_s^*G_s)$ note that $G(1-G^*G)$ is norm continuous as it is a compact perturbation of F; norm continuity of $G_s(1-T^*T)$ follows from Lemma 2.5(i) since G_s is strictly continuous and $1-T^*T$ is norm continuous and compact.

If G' is a pseudo left inverse of $G(x)$ then $G'(x)(s+(1-s)V(x)^{-1})^{-1/2}$ is a pseudo left inverse of $G_s(x)$, and if G' is a pseudo right inverse of $G(x)$ then $G'(x)(s+(1-s)V(x)^{-1})^{-1/2}$ is a pseudo right inverse of $G_s(x)$. Thus all $G_s(x)$ are $\mathcal{A}$-Fredholm operators. Moreover we have $\|G_s(x)\|_{\mathbb{B}(H)} \leq 1$ since

$$\begin{aligned}
\|G_s^*G_s\|_{\mathbb{B}(XH)} &\leq \|sG^*G\|_{\mathbb{B}(XH)} + \|(1-s)U^{-1}G^*G\|_{\mathbb{B}(XH)} \\
&= s\|G^*G\|_{\mathbb{B}(XH)} + (1-s) \leq 1.
\end{aligned}$$

We therefore finally obtain a homotopy $h: X \times I \to FredB(H)$, proving (ABS) for the inclusion $CRB(H) \subset FredB(H)$, by defining

$$h(x,s) = \begin{cases} F_{2s}(x) & 0 \leq s \leq 1/2 \\ G_{2-2s}(x) & 1 \geq s \geq 1/2. \end{cases}$$

Given a map $F: X \to FredB(H)_{sa}$ we define $h: X \times I \to FredB(H)$ as above. We claim that $F(x) = F(x)^*$ for $x \in X$ for all $x \in X$ implies that $h(x,s) \equiv h(x,s)^* \mod \mathbb{K}(H)$ for all $x \in X$ and $s \in I$. This is equivalent to saying that the map $\pi: \mathbb{B}(H) \to \mathbb{B}(H)/\mathbb{K}(H)$ maps $h(x,s)$ and $h(x,s)^*$ to the same element. This is clear for all $s \leq 1/2$. Since G is a compact perturbation of F the images of $h(x,s)$ and $h(x,s)^*$ for $s = 1 - s'/2 \geq 1/2$ are the products

$$\begin{aligned}\pi(h(x,s)) &= (s' + (1-s')\pi(F(x)^*F(x))^{-1})^{1/2}\pi(F(x));\\ \pi(h(x,s)^*) &= \pi(F(x)^*)(s' + (1-s')\pi(F(x)^*F(x))^{-1})^{1/2}\end{aligned}$$

In case $F = F^*$ the two products coincide which verifies the claim. We therefore get a continuous map

$$h - h^* : X \times I \longrightarrow \mathbb{K}(H),\ (x,s) \longmapsto h(x,s) - h(x,s)^*.$$

Hence, by Lemma 2.5(ii)

$$\frac{h+h^*}{2} : X \times I \longrightarrow FredB(H)_{sa},\ (x,s) \longmapsto \frac{h(x,s)+h(x,s)^*}{2}$$

is continuous, which proves (ABS) for $CRB(H)_{sa} \subset FredB(H)_{sa}$. □

Proposition 2.13. *The following topological inclusions are weak homotopy equivalences*

(A) $CRR(H) \subset CRB(H)$ *and* $CRR(H)_{sa} \subset CRB(H)_{sa}$.
(B) $FredR(H) \subset FredB(H)$ *and* $FredR(H)_{sa} \subset FredB(H)_{sa}$.

Proof. As H is countably generated we can choose a strictly positive compact operator $K \in \mathbb{B}(H)$ of norm $\|K\|_{\mathbb{B}(H)} \leq 1/4$. For any $F \in B(H)$ and $s \in [0,1]$ we define

$$F_s = (1-sK)F(1-sK).$$

Using arguments similar to the ones used in the proof of Lemma 2.5 we obtain that $h : B(H) \times I \to B(H)$, $(F,s) \mapsto F_s$ is a continuous map. Moreover $1 - F_s^*F_s$ has dense image for all $s \in (0,1]$, i.e. $h(B(H)) \times (0,1])$ is contained in $R(H) \subset B(H)$. To see the latter it suffices (cf. [8, 10.1]) to show that $1 - F_s^*F_s$ is strictly positive mod $\mathbb{K}(H)$ for $1 \geq s > 0$. Let ϱ be a state of $\mathbb{B}(H)$ which is strictly continuous on the unit disk. Let

$$P = (1-sK)^{1/2} \quad \text{and} \quad Q = P^{-1}F_s^*F_s.$$

The Cauchy-Schwartz inequality yields

$$\begin{aligned}\varrho(F_s^*F_s)^2 = \varrho(PQ)^2 &\leq \varrho(PP)\varrho(Q^*Q)\\ &\leq \varrho(1-sK)\|Q^*Q\|_{\mathbb{B}(H)} \leq 1 - s\varrho(K).\end{aligned}$$

Hence we obtain

$$\varrho(1 - F_s^*F_s) = 1 - \varrho(F_s^*F_s) \geq 1 - \sqrt{1 - s\varrho(K)} > 0 \quad \text{for } s > 0.$$

Thus $1 - F_s^*F_s$ is strictly positive mod $\mathbb{K}(H)$ for $1 \geq s > 0$.

We can use the homotopy h to see that the restriction $h_1 : B(H) \to R(H)$ provides a homotopy inverse to the inclusion $R(H) \subset B(H)$. If we restrict h_1 to the subspaces $CRB(H)$ and $FredB(H)$ we obtain corresponding maps

$$h_1 : CRB(H) \to CRR(H), \quad h_1 : FredB(H) \to FredR(H)$$

which are homotopy inverses to the inclusions $CRR(H) \subset CRB(H)$ and $FredR(H) \subset FredB(H)$ respectively. Since h is compatible with the $*$-operation the statement for the corresponding subspaces consisting of self-adjoint operators also follows. □

3. The natural topology on the set of Kasparov cycles

In this section we prove our main theorem. To do so we show that the strict extended gap topology (Definition 1.30) on the set $CRB(H)$ consisting of the essentially unitary operators in $B(H)$ agrees with what we like to call the *Kasparov topology.* This topology was used in Bunke-Joachim-Stolz [5] to obtain classifying spaces for $\mathbb{Z}/2$-graded σ-unital C^*-algebras. The fact that the two topologies agree allows us to combine results of [5] with the results of the previous section. The main theorem will be a direct consequence of this.

Definition 3.1. Recall that the underlying set of the space $CRB(H)$ consists of the essentially unitary operators $T \in B(H)$, i.e. T satisfies $1 - T^*T,\ 1 - TT^* \in \mathbb{K}(H)$. Such operators we also call *Kasparov cycles*, as they define elements in the Kasparov group $KK(\mathbb{C}, \mathcal{A})$. Following [5] we define the *Kasparov topology* on the set $KC(H)$ of Kasparov cycles to be the weakest topology for which the maps

$$\begin{aligned} T &\mapsto Tx \in H,\ x \in H, \\ T &\mapsto T^*x \in H,\ x \in H, \\ T &\mapsto 1 - T^*T \in \mathbb{K}(H), \\ T &\mapsto 1 - TT^* \in \mathbb{K}(H) \end{aligned}$$

are continuous. Thus it is the weakest topology on $KC(H)$ which contains the strict topology and for which the maps $T \mapsto 1 - T^*T,\ T \mapsto 1 - TT^*$ are norm continuous.

The following statement is the analogue of Lemma 1.15.

Lemma 3.2. *The map which associates to an adjointable bounded operator $T \in KC(H)$ the self-adjoint operator*

$$\widehat{T} = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$$

defines a homeomorphism from the space $KC(H)$ onto the subspace of odd self-adjoint adjointable bounded operators $KC(\widehat{H})^1_{sa} \subset KC(\widehat{H})$.

Proof. First of all it is clear that the map of sets $KC(H) \to KC(\widehat{H})^1_{sa}$ is bijective. To see that the map is a homeomorphism observe that for

$(x_0, x_1) \in \widehat{H} = H \oplus H$

$$\begin{aligned} |\widehat{T}(x_0 + x_1)|^2 &= |Tx_0|^2 + |T^* x_1|^2 \quad \text{and} \\ 1 - \widehat{T}^* \widehat{T} &= \begin{pmatrix} 1 - T^*T & 0 \\ 0 & 1 - TT^* \end{pmatrix}. \end{aligned}$$

□

Proposition 3.3. *The map $id : CRB(H) \to KC(H)$ is a homeomorphism.*

Proof. It follows immediately from the definition of the topology of $CRB(H)$ that $id : CRB(H) \to KC(H)$ is continuous. To see that the map is a homeomorphim we just need to show that the map $KC(H) \to \mathbb{K}(H), T \mapsto T(1 - T^*T)$ is norm continuous (cf. Lemma 2.4). This follows from the fact that T is strictly continuous and that $1 - T^*T$ is norm continuous and compact. □

Theorem 3.4 (cf. [5])**.** *Let $\mathcal{A}$ be a unital C^*-algebra, and $H = l^2(\mathcal{A})$.*

(i) *The space $KC(H)$ represents the functor which associates to a compact space X the Grothendieck group of finitely generated projective Hilbert-$\mathcal{A}$-bundles $K^0(X; \mathcal{A})$.*
(ii) *The space $KC(H)_{sa}$ represents the functor which associates to a compact space X the abelian group $K^1(X; \mathcal{A})$.*

Proof. Consider the map $KC(H) \to KC(\widehat{H})$ which maps $T \in KC(H)$ to

$$\widehat{T} = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}.$$

By Lemma 3.2 the map is a homeomorphism onto its image, which is

$$\begin{aligned} KC(\widehat{H})^1_{sa} &= KC(\widehat{H}) \cap R(\widehat{H})^1_{sa} \\ &= \{T \in KC(\widehat{H}) \mid T = T^*, T \text{ is an odd operator}\}. \end{aligned}$$

Now let X be a compact space, let $X\mathcal{A}$ denote the C^*-algebra of continuous functions on X with values in $\mathcal{A}$, and let XH denote the Hilbert-$X\mathcal{A}$-module of continuous functions on X with values in H. Proposition 2.3 of [5] applied to the C^*-algebra $\mathcal{A}$ says that there is a natural homeomorphism

$$KC(\widehat{XH})^1_{sa} \cong map(X, KC(\widehat{H})^1_{sa}),$$

where $map(X, Y)$ for an arbitrary topological space Y (here and below) shall denote the mapping space of continuous functions from X to Y with the compact-open topology. It follows that we have a commutative diagram of homeomorphisms

$$\begin{tikzcd} KC(XH) \arrow[r, "\cong"] \arrow[d, "\cong"] & map(X, KC(H)) \arrow[d, "\cong"] \\ KC(\widehat{XH})^1_{sa} \arrow[r, "\cong"] & map(X, KC(\widehat{H})^1_{sa}) \end{tikzcd}$$

Theorem 2.4 of [5] applied to $X\mathcal{A}$ says that $\pi_0(KC(\widehat{XH})^1_{sa}) \cong K_0(X\mathcal{A})$. Using these results we obtain

$$\begin{aligned} [X, KC(H)] &\cong \pi_0(map(X, KC(H))) \\ &\cong \pi_0(map(X, KC(\widehat{H})^1_{sa})) \\ &\cong \pi_0(KC(\widehat{XH})^1_{sa}) \\ &\cong K_0(X\mathcal{A}) \cong K^0(X; \mathcal{A}). \end{aligned}$$

Thus $KC(H)$ represents $K^0(X; \mathcal{A})$.

The second statement of the theorem can be deduced in a similar fashion. Let $Cl(1)$ denote the $\mathbb{Z}/2$-graded C^*-algebra generated by one odd symmetry e, i.e. $Cl(1)$ is the Clifford algebra of a one dimensional vector space with a positive definite quadratic form. We identify $\widehat{H} = H \oplus H$ with $H \otimes Cl(1)$ by the map $(h, h') \mapsto h \otimes 1 + h' \otimes e$. Consider the map

$$KC(H)_{sa} \to KC(\widehat{H}), \quad T \mapsto \widetilde{T},$$

where $\widetilde{T}$ is given by

$$\begin{aligned} \widetilde{T}(h \otimes 1) &= Th \otimes e, \\ \widetilde{T}(h \otimes e) &= Th \otimes 1. \end{aligned}$$

As above the map is a homeomorphism onto its image

$$KC(H \otimes Cl(1))^1_{sa} = \{\widetilde{T} \in KC(\widehat{H})^1_{sa} \mid \widetilde{T}(xe) = (\widetilde{T}(x))e \text{ for all } x \in \widehat{H}\}.$$

Again one can use [5, Proposition 2.3] and [5, Theorem 2.4], this time for the $\mathbb{Z}/2$-graded C^*-algebra $\mathcal{A} \otimes Cl(1)$ instead of $\mathcal{A}$, to deduce that

$$\begin{aligned} [X, KC(H)_{sa}] &\cong \pi_0(map(X, KC(H \otimes Cl(1))^1_{sa})) \\ &\cong \pi_0(KC(XH \otimes Cl(1))^1_{sa}) \\ &\cong K_0(X(\mathcal{A} \otimes Cl(1))) \cong K_{-1}(X\mathcal{A}) \cong K^1(X; \mathcal{A}). \end{aligned}$$

Thus $KC(H)_{sa}$ represents $K^1(X; \mathcal{A})$. □

Finally we can state and prove our main theorem.

Theorem 3.5. *Let $\mathcal{A}$ be a unital C^*-algebra, and $H = l^2(\mathcal{A})$.*

(i) *The space $FredR(H)$ represents the functor which associates to a compact space X the abelian group $K^0(X; \mathcal{A})$.*

(ii) *The space* $FredR(H)_{sa}$ *represents the functor which associates to a compact space* X *the abelian group* $K^1(X;\mathcal{A})$.

Proof. The statements follow immediately from Proposition 2.6, Proposition 2.13, Proposition 3.3 and the previous theorem. □

Remark 3.6. In case $\mathcal{A} = \mathbb{C}$ Booss-Bavnbek, Lesch and Philipps have shown that the space $FredR(H)_{sa}$ has just one path component ([4, Theorem 1.10 (a)]), and they conjectured (ibid. Remark 1.11) that $FredR(H)_{sa}$ would be a classifying space for the functor $X \mapsto K^1(X) = K^1(X;\mathbb{C})$. Thus we answer their conjecture in the affirmative by Theorem 3.5 (2) (applied to $\mathcal{A} = \mathbb{C}$).

We would also like to emphasize that the induced topology on the set of adjointable bounded Fredholm operators $\mathbb{F}(H) \subset FredR(H)$ is the norm topology. Kasimov [7] and Mingo [9] both have shown that $\mathbb{F}(H)$ represents the functor $X \mapsto K^0(X;\mathcal{A})$; and it is then easy to see that the inclusion $\mathbb{F}(H) \subset FredR(H)$ is a weak homotopy equivalence.

On the other hand it is well-known that the space $\mathbb{F}(H)_{sa} \subset \mathbb{F}(H)$ consisting of the self-adjoint bounded operators in general does not qualify as a space representing the functor $X \mapsto K^1(X,\mathcal{A})$. E.g. in case $\mathcal{A} = \mathbb{C}$ the space $\mathbb{F}(H)_{sa}$ has three path components, and one of these path components represents the functor $X \mapsto K^1(X) = K^1(X;\mathbb{C})$. In the general case where $\mathcal{A}$ is an arbitrary unital C^*-algebra the corresponding statement is even a little more difficult to state. For a discussion of this matter we refer the reader to [10].

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