Central extensions of rank 2 groups and applications

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Abstract

We show that the universal central extensions of the little projective group of any Moufang polygon is precisely the Steinberg group obtained from its defining commutator relations, provided the defining structure is not too small. As an application, we get that also the universal central extensions of the little projective group of any 2-spherical Moufang twin building is precisely the Steinberg group obtained from its defining commutator relations.

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1 Introduction

In his famous lecture notes [10], R. Steinberg determined the universal central extensions of all Chevalley groups over arbitrary commutative fields (provided the field has enough elements). Very quickly, this result was generalized in various ways. We mention the result by C.W. Curtis [2], which deals with groups admitting a Bruhat decomposition, but it is not shown that the obtained central extensions are universal and there is an additional assumption about automorphisms of the kernel. The results of J. Grover [5] do address this question, but only for a limited class of groups of Lie type, and in fact, the last two conditions in his definition of these groups seem to be designed specifically to avoid the main problems which occur in this question.

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The most important generalization of Steinberg's results is by V. Deodhar [4], who constructs the universal central extension for groups G_k^+ generated by the root groups of an absolutely (almost) simple algebraic group G of arbitrary positive rank over an arbitrary field k (with at least 7 elements). These groups G_k^+ can be realized as the automorphism group generated by all root elations of the corresponding Moufang building, but not all Moufang buildings arise from algebraic groups. However, it is known that all Moufang buildings of rank at least 2 arise from either an algebraic group, a classical group, or a so-called mixed group; in fact, all Moufang buildings of rank two (also known as Moufang polygons) are classified by J. Tits and R.M. Weiss [11].

Our goal is to construct the universal central extensions for all Moufang polygons, i.e. the Moufang buildings of rank 2 in the generic situation (i.e. unless the field is too small).

Main Theorem. *Let G be a group with a root datum $(G, (U_\alpha)_{\alpha\in\Psi})$ of type (W, S), where (W, S) is an irreducible spherical Coxeter system of rank 2 satisfying the restrictions given in Theorems 3.1, 4.2, 5.1 and 6.1 below. Let $\hat{G}$ be the Steinberg group of $(G, (U_\alpha)_{\alpha\in\Psi})$, i.e. the group (freely) generated by the U_α modulo the Steinberg relations. Then $\hat{G}$ is the universal central extension of G.*

It was shown in [1] that the Steinberg group of a 2-spherical twin root datum is the universal central extension of the little projective group of the associated building, provided this is true for all rank 2 subgroups. Thus we obtain the following corollary covering in particular the groups of Lie-type, see [1] for terminology:

Corollary. *Let G be a group of Kac-Moody type, with twin root datum $(G, (U_\alpha)_{\alpha\in\Psi})$ of type (W, S), where (W, S) is a 2-spherical Coxeter system without direct factors of type A_1. Assume that the rank 2 subgroups of G satisfy the restrictions given in Theorems 3.1, 4.2, 5.1 and 6.1 below. Let $\hat{G}$ be the Steinberg group of $(G, (U_\alpha)_{\alpha\in\Psi})$, i.e. the group (freely) generated by the U_α modulo the Steinberg relations for all prenilpotent pairs $\{\alpha, \beta\} \subseteq \Psi$ such that $s_\alpha s_\beta$ has finite order. Then $\hat{G}$ is the universal central extension of G.*

It follows in particular that over (sufficiently large) finite fields the universal central extension of such a group has finite kernel.

We do not know whether the given restrictions are sharp in general. They are sharp at least in the commutative case of Theorem 3.1, and in Theorem 6.1 (see [7, p. 48] and [12, 5.8.1]). For known results about classical groups in arbitrary dimension, we refer to [6].

We note that the Moufang rank one buildings (also known as Moufang sets) are not classified, and that many examples exist which do not arise from algebraic, classical or mixed groups (see, for example, [8]), and therefore it is natural to start at rank two, and to exclude direct factors of type A_1 in the Main Theorem.

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2 General setup

For a group G, we write commutators as $[g,h] = g^{-1}h^{-1}gh$ for all $g,h \in G$, and G^* for G without its neutral element.

Definition 2.1. A *central extension* of a group G is a couple (π, E) where E is a group, π is a homomorphism of E onto G, and $\ker(\pi) \subseteq Z(E)$, where $Z(E)$ denotes the center of E.

Definition 2.2. A central extension (π, E) of a group G is called *universal* if for any central extension $(\hat{\pi}, \hat{E})$ of G there is a unique homomorphism $\varphi : E \to \hat{E}$ such that $\hat{\pi}\varphi = \pi$.

If a group admits a universal central extension, then this extension is clearly unique. Moreover, it is a well known fact that a group G has a universal central extension if and only if G is perfect, i.e. $[G,G] = G$; see, for example, [10, §7].

The following theorem provides a powerful method to determine the universal central extension of a group, which is also the method that was used by Steinberg.

Theorem 2.3. *Let G and $\hat{G}$ be two perfect groups, and assume that $(\hat{\pi}, \hat{G})$ is a central extension of G. Suppose that every central extension (π, E) of $\hat{G}$ splits, i.e. there is a homomorphism $\theta : \hat{G} \to E$ such that $\pi\theta = \mathrm{id}_{\hat{G}}$. Then $(\hat{\pi}, \hat{G})$ is the universal central extension of G.*

Proof. See [10, §7]. □

We now turn our attention to the Moufang polygons; we refer to [11] for more details. Let Γ be a Moufang n-gon; so $n \in \{3,4,6,8\}$. Any Moufang n-gon can be constructed from a so-called *root group sequence* $(U_+; U_1, \dots, U_n)$, where U_+ is a group and $U_1, \dots, U_n$ are subgroups of U_+ satisfying certain conditions. In particular, the groups $U_1, \dots, U_n$ satisfy certain *commutator relations* of the form $[a_i, a_j] = f_{i+1}(a_i, a_j) \cdots f_{j-1}(a_i, a_j)$, where the f_k are maps from $U_i \times U_j$ to U_k; it will be convenient to write $f_k(a_i, a_j)$ as $[a_i, a_j]_k$. Note that $[U_i, U_{i+1}] = 1$ for all i.

In the Moufang polygon Γ which arises from this root group sequence, the groups $U_1, \ldots, U_n$ are realized as the *positive* root groups, and the group U_+ is the group generated by them, i.e. $U_+ = \langle U_1, \ldots, U_n \rangle$. Corresponding to the opposite roots, Γ also has *negative* root groups, denoted by $U_{n+1}, \ldots, U_{2n}$ (it is convenient to interpret the labels modulo $2n$), and we set $U_- := \langle U_{n+1}, \ldots, U_{2n} \rangle$. It then turns out that the *little projective group* of Γ, i.e., the group $G \leq Aut(\Gamma)$ generated by all the root groups, is already generated by these $2n$ root groups, i.e. $G = \langle U_+, U_- \rangle$. Apart from the commutator relations which already hold in U_+, G also satisfies commutator relations between the other root groups, all of the form

$$[a_i, a_j] = [a_i, a_j]_{i+1} \cdots [a_i, a_j]_{j-1} \,, \tag{2.1}$$

for all $i, j \in \mathbb{Z}$ with $0 < j - i < n$ modulo $2n$ (with some abuse of the $<$-sign). These relations can be derived from the defining relations by the "Shift Lemma" [11, (6.4)].

Now, except for the smallest cases of a Moufang quadrangle, hexagon or octagon defined over the field $\mathsf{GF}(2)$, the group G is a perfect group (in fact a simple group) [11, (37.3)], and hence admits a universal central extension. With the known results of Steinberg and Deodhar in mind, it is natural to guess that the group $\hat{G}$ (freely) generated by $U_1, \ldots, U_{2n}$ and subject to the commutator relations (2.1), is the universal central extension of G. We show that, if the defining field is not too small, this is indeed the case.

We first show that this group is indeed a central extension, and we start with a lemma. Note that every element $a_i \in U_i < G$ has a unique preimage in the corresponding subgroup $U_i < \hat{G}$, and we will continue to denote this element by a_i. Hence the maps $\mu : U_i^* \to U_{i+n}^* U_i^* U_{i+n}^*$ defined on G as in [11, (6.1)] induce similar maps on $\hat{G}$, in a unique way.

Lemma 2.4. *If the relation* $a_i^{\mu(b_j)} = c_{2j+n-i}$ *holds in* G*, for arbitrary* i, j*, then it also holds in* $\hat{G}$*. Moreover,* $\hat{G}$ *is perfect.*

Proof. Note that $U_i^{\mu(b_j)} = U_{2j+n-i}$ by [11, (6.1)]. If $j \neq i$ and $j \neq i+n$, then the first statement follows from the fact that the commutator relations (2.1) continue to hold in $\hat{G}$. If $j \in \{i, i+n\}$, then we can rewrite a_i as $[a_k, a_l]_i$ for some k, l different from i and $i + n$, and again this result follows.

The proof of [11, (37.3)] which shows that G is perfect, also shows without any change that $\hat{G}$ is perfect. □

Note that general relations between elements of U_i and U_{i+n} inside G do *not* necessarily continue to hold in $\hat{G}$.

Theorem 2.5. *The pair $(\pi, \hat{G})$, where π is the unique homomorphism from $\hat{G}$ to G whose restriction to each U_i is the identity map, is a central extension of G.*

Proof. Let $K := \ker(\pi)$; we have to show that $K \leq Z(\hat{G})$. Clearly, the group G acts faithfully on the Moufang polygon Γ since $G \leq \mathrm{Aut}(\Gamma)$. The group $\hat{G}$ acts on Γ as well, through the map π; this action is of course not faithful in general, and K is precisely the kernel of the action of $\hat{G}$ on Γ. Now let $H := \bigcap_{i=1}^{2n} N_{\hat{G}}(U_i)$, let B_+ be the stabilizer in $\hat{G}$ of the chamber $\{n, n+1\}$, and let B_- be the stabilizer in $\hat{G}$ of the opposite chamber $\{0, 1\}$, where $\{0, 1, \ldots, 2n-1\}$ is the standard apartment with respect to which the root groups U_i are defined. Then it follows from a deep result, to our knowledge only made explicit for the first time by B. Rémy [9, Th. 3.5.4] in the context of twin buildings and groups with a twin root datum, that $B_+ \cap B_- = H$. (Observe that the group $\hat{G}$ admits indeed a twin root datum; the only non-trivial fact, which is the existence of the μ-maps, follows from Lemma 2.4.) In particular, this implies that $K \leq H$, i.e. K normalizes all the root groups U_i. Of course, each root group U_i normalizes K, and since $U_i \cap K = 1$, this implies that K commutes with each root group U_i. Since $\hat{G}$ is generated by these root groups, this implies $K \leq Z(\hat{G})$. □

In each case, we will define certain *identity elements* $e_i \in U_i$ for each i, and we will define $h(a_i) := \mu(e_i)^{-1}\mu(a_i)$; these elements will normalize all root groups $U_1, \ldots, U_{2n}$. If x_k is a parameterization of the root group U_k, then we will also write $\mu_k(a)$ for $\mu(x_k(a))$ and $h_k(a)$ for $h(x_k(a))$. Let $H := \langle h(U_i) \mid i \in \mathbb{Z} \rangle$; this group H is known as the *Cartan subgroup*, the *diagonal subgroup*, or the *torus*, of G. Note that $H = \langle h(U_1), h(U_n) \rangle$; see [11, (33.7)], and see also the paragraph following its proof.

Let (π, E) be an arbitrary central extension of $\hat{G}$, and let $C := \ker(\pi)$. We have to show that this extension splits by constructing a *canonical* homomorphism θ from $\hat{G}$ to E. For this, we have to define a canonical lifting of the root groups $U_i < \hat{G}$ into E and show that all of the commutator relations (2.1) continue to hold in E. To define the lifting, we will make use of the maps h_i which we have just defined; the crucial observation (due to Steinberg) is that a commutator $[x, y]$ with $x, y \in E$ only depends on the classes modulo C to which x and y belong. The difference between our approach and the approach of Steinberg and others is that we will use commutator relations between elements of U_i and $h(U_j)$ for $i \neq j$. This will not only result in easier relations to work with, but it will also allow us to consider slightly smaller fields in most cases. (Of course, the reason that we can do this is precisely because we do not have to care about the rank one groups.)

3 Moufang triangles

Let Γ be an arbitrary Moufang triangle (i.e. a Moufang projective plane). Then the classification [11] tells us that Γ can be parameterized by an alternative division ring $(A, +, \cdot)$, in the following way. Let U_1, U_2 and U_3 be parameterized by $(A, +)$ (we will denote the explicit isomorphisms from A to U_i by x_i), subject to the relations $[U_1, U_2] = [U_2, U_3] = 1$ and

$$[x_1(a), x_3(b)] = x_2(a \cdot b)$$

for all $a, b \in A$. We let $e_i = x_i(1)$ for all i. By [11, (32.5)], we have $\mu_3(t) = x_0(t^{-1})x_3(t)x_0(t^{-1})$ and $\mu_1(t) = x_4(t^{-1})x_1(t)x_4(t^{-1})$, and we obtain

$$\begin{aligned} x_1(a)^{h_1(z)} &= x_1(zaz)\,; & x_1(a)^{h_3(z)} &= x_1(az^{-1})\,; \\ x_2(b)^{h_1(z)} &= x_2(zb)\,; & x_2(b)^{h_3(z)} &= x_2(bz)\,; \\ x_3(c)^{h_1(z)} &= x_3(z^{-1}c)\,; & x_3(c)^{h_3(z)} &= x_3(zcz)\,; \end{aligned}$$

for all $a, b, c \in A$ and all $t \in A^*$.

Now assume that A has at least five elements, and that the center $Z(A)$ of A has at least three elements. Take a fixed element $t \in Z(A) \setminus \{0, 1\}$. We will define a lifting λ of all the root groups U_i into E; we first define λ on the root group U_2, by the commutator relation

$$\tilde{x}_2\big((t-1)\cdot a\big) = [\tilde{x}_2(a), \tilde{h}_1(t)] \tag{3.1}$$

for all $a \in A$, where we write $\tilde{x}_k$ and $\tilde{h}_k$ for $\lambda \circ x_k$ and $\lambda \circ h_k$, respectively. Note that although this might look like a recursive definition, it is not, precisely because E is a *central* extension of $\hat{G}$, and hence the right hand side does not depend on the lifting λ. However, it is not clear yet whether the lifted object $\tilde{x}_2(A)$ is indeed a group, and whether this lifting depends on the choice of t.

Observe that $h_1(t) \in Z(H)$ since $t \in Z(A)$. We claim that

$$\tilde{x}_2(a)^{\lambda(h)} = \lambda\big(x_2(a)^h\big) \tag{3.2}$$

for all $a \in A$ and all $h \in H$. Indeed, we have $x_2(a)^h = x_2(b)$ for some element $b \in A$, and conjugation of equation (3.1) by $\lambda(h)$ yields $\tilde{x}_2\big((t-1)\cdot a\big)^{\lambda(h)} = [\tilde{x}_2(a)^{\lambda(h)}, \tilde{h}_1(t)^{\lambda(h)}]$; using the fact that commutators of elements in E only depend on their classes modulo C, this is in turn equal to $[\lambda\big(x_2(a)^h\big), \tilde{h}_1(t)] = [\tilde{x}_2(b), \tilde{h}_1(t)] = \tilde{x}_2\big((t-1)\cdot b\big)$. On the other hand, since h and $h_1(t)$ commute, we have

$$\begin{aligned} x_2((t-1)\cdot a)^h = \big(x_2(a)^{-1}x_2(a)^{h_1(t)}\big)^h &= \big(x_2(a)^h\big)^{-1}\big(x_2(a)^h\big)^{h_1(t)} \\ &= x_2(b)^{-1}\,x_2(b)^{h_1(t)} = x_2((t-1)\cdot b) \end{aligned}$$

as well. Replacing a by $(t-1)^{-1}a$ shows (3.2). We will often use this fact implicitly when we conjugate identities by elements of H.

We now show that

$$[\tilde{x}_2(a), \tilde{x}_2(b)] = 1 \tag{3.3}$$

for all $a, b \in A$. Since U_2 is abelian, we have $f(a,b) := [\tilde{x}_2(a), \tilde{x}_2(b)] \in C$ for all $a, b \in A$. Let $v \in A^*$; if we apply $\tilde{h}_1(v)$ on this equation, then we get $f(a,b) = [\tilde{x}_2(va), \tilde{x}_2(vb)] = f(va, vb)$. Also observe that the map $f : A \times A \to C$ is bi-additive. Since A has at least 5 elements, we can find some $v \in A$ such that the elements v, $1-v$ and $1-v+v^2$ are all non-zero. We obtain

$$\begin{aligned} f(a,b) &= f(a, vb) + f\big(a, (1-v)b\big) \\ &= f\big((1-v)a, (1-v)vb\big) + f\big(va, v(1-v)b\big) \\ &= f\big(a, (v-v^2)b\big) , \end{aligned}$$

and hence $f\big(a, (1-v+v^2)b\big) = 1$ for all $a, b \in A$. This implies that $f \equiv 1$, which shows (3.3).

The next step is to show that $\tilde{x}_2(-a) = \tilde{x}_2(a)^{-1}$ for all $a \in A$. Let $s := (t-1)^{-1} \in Z(A)$; then equation (3.1) can be rewritten as $\tilde{x}_2(sa)^{\tilde{h}_1(t)} = \tilde{x}_2(sa)\tilde{x}_2(a)$. We have $\tilde{x}_2(sa)\tilde{x}_2(-sa) \in C$, and conjugating this element by $\tilde{h}_1(t)$ yields $\tilde{x}_2(sa)\tilde{x}_2(-sa) = \tilde{x}_2(sa)\tilde{x}_2(a)\tilde{x}_2(-sa)\tilde{x}_2(-a)$; using (3.3), this implies $\tilde{x}_2(-a) = \tilde{x}_2(a)^{-1}$.

We will now verify that $\tilde{x}_2$ is indeed an isomorphism from A to a subgroup of E. So let $y := \tilde{x}_2(sa)\tilde{x}_2(sb)\tilde{x}_2(s(a+b))^{-1}$ for arbitrary $a, b \in A$; then $y \in C = \ker(\pi)$ since U_2 is a group. Since $C \leq Z(E)$, we have $y^{\tilde{h}_1(t)} = y$; on the other hand, it follows by (3.2) and (3.3) that

$$\begin{aligned} y^{\tilde{h}_1(t)} &= \tilde{x}_2(sa)\tilde{x}_2(a) \cdot \tilde{x}_2(sb)\tilde{x}_2(b) \cdot \big(\tilde{x}_2(s(a+b))\tilde{x}_2(a+b)\big)^{-1} \\ &= y\tilde{x}_2(a)\tilde{x}_2(b)\tilde{x}_2(a+b)^{-1} , \end{aligned}$$

and we conclude that $\tilde{x}_2(a+b) = \tilde{x}_2(a)\tilde{x}_2(b)$. In a completely similar fashion, we can lift the other root groups into E. We will write $\tilde{U}_i$ for the lifted root group $\lambda(U_i)$.

It remains to show that the commutator relations (2.1) lift to E. We first consider two subsequent root groups, and without loss of generality, we consider U_1 and U_2; we have to show that $[\tilde{U}_1, \tilde{U}_2] = 1$. Let $v \in A$ with $v^3 \notin \{0, 1\}$ (such an element exists since A has at least five elements), and let $g(a,b) := [\tilde{x}_1(a), \tilde{x}_2(b)] \in C$ for all $a, b \in A$. Observe that g is bi-additive. If we conjugate by $\tilde{h}_3(b^{-1})$, we obtain $g(a,b) = g(ab, 1)$ for all $a, b \in A$; we will write $g(c) := g(c, 1)$ for all $c \in A$. If, on the other hand, we conjugate $g(v, v^{-1}b)$ by $\tilde{h}_1(v)$, we obtain $g(v, v^{-1}b) = g(v^3, b)$, and hence $g(b) = g(v, v^{-1}b) = g(v^3, b) = g(v^3 b)$, and since g is additive, we get $g\big((v^3-1)b\big) = 1$ for all $b \in A$, and we conclude that $g \equiv 1$.

Finally, we consider two root groups "at distance two", and without loss of generality, we may work with U_1 and U_3. In a similar way as in the previous paragraph, we define $p(a,b) := [\tilde{x}_1(a), \tilde{x}_3(b)] \cdot \lambda([x_1(a), x_3(b)])^{-1} = [\tilde{x}_1(a), \tilde{x}_3(b)] \cdot \tilde{x}_2(ab)^{-1} \in C$ for all $a, b \in A$. Again, observe that p is bi-additive (using the fact that we already know that subsequent $\tilde{U}_i$'s commute!). Conjugating by $\tilde{h}_1(z)\tilde{h}_3(z)$ yields $p(a,b) = p(za, bz)$ for all $a, b, z \in A$ with $z \neq 0$. Using the same method as in our proof of equation (3.3), we can conclude that $p \equiv 1$, and hence all of the commutator relations which define $\hat{G}$ continue to hold in E, which shows that the extension (π, E) splits.

We are left to show that the section defined in this way is canonical, i.e. it does not depend on the choice of the chosen element $t \in Z(A)$ in the equations which define the liftings of the root groups. We will work out the details for the case of Moufang triangles; all the other cases are completely similar.

So consider two arbitrary elements $s, t \in Z(A) \setminus \{0, 1\}$, and let $a, b \in A$ be such that $(t-1) \cdot a = (s-1) \cdot b$. By (3.2) and using the fact that $\tilde{x}_2$ is an isomorphism, we obtain

$$\begin{aligned}[\tilde{x}_2(b), \tilde{h}_1(s)] &= \tilde{x}_2(b)^{-1}\ \tilde{x}_2(b)^{\tilde{h}_1(s)} = \tilde{x}_2(-b)\ \lambda\big(x_2(b)^{h_1(s)}\big) \\ &= \tilde{x}_2(-b)\ \tilde{x}_2(bs) = \tilde{x}_2\big((s-1) \cdot b\big)\,,\end{aligned}$$

and therefore $[\tilde{x}_2(b), \tilde{h}_1(s)] = [\tilde{x}_2(a), \tilde{h}_1(t)]$, which shows that the lifting (3.1) is independent of the choice of t.

We conclude:

Theorem 3.1. *Let Γ be a Moufang triangle $\mathcal{T}(A)$ defined over an alternative division ring A, let G be the little projective group of Γ, and let $\hat{G}$ be defined by the commutator relations* (2.1) *as explained above. Assume that $|A| \geq 5$ and $|Z(A)| \geq 3$. Then $\hat{G}$ is the universal central extension of G.*

4 Moufang quadrangles

Let Γ be an arbitrary Moufang quadrangle. The Moufang quadrangles have been classified in [11], but this classification distinguishes between six different families. For our purposes, it is more convenient to use the *quadrangular systems*, an algebraic structure which uniformly parametrizes all Moufang quadrangles [3], although we cannot avoid to invoke their classification at certain points. So let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system, with corresponding bilinear maps F and H; let U_2 and U_4 be parameterized by $(V, +)$, and let U_1 and U_3 be parameterized by $(W, \boxplus)$. The defining relations are given by $[U_1, U_2] = [U_2, U_3] = [U_3, U_4] = 1$, and

$$\begin{aligned}
[x_1(w), x_3(z)^{-1}] &= x_2(H(w,z)) \\
[x_2(v), x_4(c)^{-1}] &= x_3(F(v,c)) \\
[x_1(w), x_4(v)^{-1}] &= x_2(\tau_V(v,w))x_3(\tau_W(w,v)) = x_2(vw)x_3(wv)
\end{aligned}$$

for all $v, c \in V$ and all $w, z \in W$. We let $e_i = x_i(\epsilon)$ for $i = 2, 4$, and $e_i = x_i(\delta)$ for $i = 1, 3$. By [3, Chapter 4], we have $\mu_4(v) = x_0(v^{-1})x_4(v)x_0(v^{-1})$ and $\mu_1(w) = x_5(\kappa(w))x_1(w)x_5(\lambda(w))$, and we obtain

$$\begin{aligned}
x_1(w)^{h_1(z)} &= x_1\big(\Pi_z\big(w \cdot (-\epsilon)(\boxminus z)\big)\big)\ ; & x_1(w)^{h_4(c)} &= x_1(wc^{-1})\,; \\
x_2(v)^{h_1(z)} &= x_2(vz)\,; & x_2(v)^{h_4(c)} &= x_2(-\pi_c(\overline{v}))\,; \\
x_3(w)^{h_1(z)} &= x_3(\Pi_z(w))\,; & x_3(w)^{h_4(c)} &= x_3(wc)\,; \\
x_4(v)^{h_1(z)} &= x_4(v\kappa(z))\,; & x_4(v)^{h_4(c)} &= x_4\left(-\pi_c(\overline{v \cdot \delta c})\right)\,;
\end{aligned} \tag{4.1}$$

for all $v \in V$, $c \in V^*$, $w \in W$, $z \in W^*$, where Π_z and π_c are generalized reflections in Ω. The following technical lemma will be used in the sequel.

Lemma 4.1. *Suppose that Ω satisfies the following restrictions.*

- *If $\Omega \cong \mathcal{Q}_Q(k, V, q)$, then we require $|k| \geq 5$;*
- *if $\Omega \cong \mathcal{Q}_I(K, K_0, \sigma)$ or $\Omega \cong \mathcal{Q}_P(K, K_0, \sigma, X, \pi)$, then we require $|Z(K) \cap K_0| \geq 5$;*
- *if $\Omega \cong \mathcal{Q}_D(k, k_0, l_0)$, then we require $|k| \geq 5$;*
- *if $\Omega \cong \mathcal{Q}_E(k, V, q)$ or $\Omega \cong \mathcal{Q}_F(k, V, q)$, then we do not require anything.*

Then there exists an element $y \in R^*$ (where $R := \mathrm{Rad}(H)$, the radical of H), such that the maps

$$\begin{aligned}
\alpha &: V \to V : v \mapsto vy - v\,, \\
\beta &: V \to V : v \mapsto vy - vyy - v\,,
\end{aligned}$$

are surjective, and such that $h_1(y) \in Z(H)$. Moreover, there exists an element $j = \epsilon r \in \epsilon R^* \setminus \{\epsilon\}$ such that the maps

$$\begin{aligned}
\zeta &: W \to W : w \mapsto \boxminus w \boxplus wj\,, \\
\zeta_{|R} &: R \to R : w \mapsto \boxminus w \boxplus wj\,,
\end{aligned}$$

are surjective, such that $-\pi_j(\overline{u}) = u$ for all $u \in V$, and such that $h_4(j) \in Z(H)$. Also, there exists an element $i \in V \setminus \{0, \epsilon\}$ such that the maps

$$\begin{aligned}
\eta &: R \to R : b \mapsto bi(\epsilon - i) \boxminus b \boxplus F(ib, \epsilon - i)\,, \\
\theta &: W/R \to W/R : b \boxplus R \mapsto bi(\epsilon - i) \boxminus b \boxplus R\,,
\end{aligned}$$

are surjective, and such that $h_4(i) \in Z(H)$.

Proof. We refer to [3, Chapter 7] for a description of the different quadrangular systems we will encounter.

(i) First, let $\Omega \cong \mathcal{Q}_Q(k, V, q)$ for some anisotropic quadratic space (k, V, q) with $|k| \geq 5$. Then there exists an element $t \in k \setminus \{0, 1, -1\}$ such that $t - t^2 - 1 \neq 0$, and an element $u \in k \setminus \{0, 1\}$ such that $u^4 - 2u^3 - u^2 + 2u - 1 \neq 0$. Let $y = [t]$, let $j = [t\epsilon]$, and let $i = [u\epsilon]$. Then $\alpha[v] = [(t-1)v]$, $\beta[v] = [(t - t^2 - 1)v]$, $\zeta[s] = [(t^2 - 1)s]$, $\eta[s] = [(u^4 - 2u^3 - u^2 + 2u - 1)s]$, and all of these maps are surjective. Moreover, it follows from the formulas (4.1) that $h_1(W) \subseteq Z(H)$ and $h_4(\epsilon R) \subseteq Z(H)$, and $\pi_j[v] = \pi_{[t\epsilon]}[v] = -\overline{v}$ for all $v \in V$.

(ii) Let $\Omega \cong \mathcal{Q}_P(K, K_0, \sigma, X, \pi)$ for some anisotropic pseudoquadratic space (K, K_0, σ, X, π). Note that this includes the $\mathcal{Q}_I$ case (which occurs when $X = 0$). Let $L := Z(K) \cap K_0$, and assume $|L| \geq 5$. As in the previous case, there exists an element $t \in L \setminus \{0, 1, -1\}$ such that $t - t^2 - 1 \neq 0$, and an element $u \in L \setminus \{0, 1\}$ such that $u^4 - 2u^3 - u^2 + 2u - 1 \neq 0$. Let $y = [0, t]$, let $j = [t]_V = \epsilon[0, t]$, and let $i = [u]_V$. Then $\alpha[v] = [(t-1)v]$ and $\beta[v] = [(t-t^2-1)v]$ for all $v \in k$. Moreover, $\zeta[a, \pi(a) + s] = [a(t-1), \pi(a)(t^2 - t) + \pi(a)^\sigma (t-1)] \boxplus [0, s(t^2 - 1)]$ for all $a \in X$ and all $s \in k_0$, so in particular $\zeta[0, s] = [0, s(t^2 - 1)]$. Also $\eta[0, s] = [0, (u^4 - 2u^3 - u^2 + 2u - 1)s]$ and $\theta[a] = [a(u - u^2 - 1)]$. All of these maps are surjective. Moreover, it follows from the formulas (4.1) that $h_1(y), h_4(i), h_4(j) \in Z(H)$, and $\pi_j[v] = \pi_{[t]_V}[v] = -\overline{v}$ for all $v \in V$.

(iii) Let $\Omega \cong \mathcal{Q}_D(k, k_0, l_0)$ for some indifferent set (k, k_0, l_0) with $|k| \geq 5$. Note that k_0 and l_0 contain all squares of k and that $\mathrm{char}(k) = 2$, hence $|k_0| \geq 5$ and $|l_0| \geq 5$ as well. There exists an element $t \in l_0 \setminus \{0, 1\}$ such that $t^2 + t + 1 \neq 0$, Let $y = [t]_W$, and let $j = i = [t]_V$. Then $\alpha[a] = [(t^2 + 1)a]$, $\beta[a] = [(t^4 + t^2 + 1)a]$, $\zeta[s] = [(t + 1)s]$, $\eta[a] = [(t^4 + t^2 + 1)a]$, and all of these maps are surjective. Moreover, H is abelian, and all generalized reflections are trivial.

(iv) Let $\Omega \cong \mathcal{Q}_E(k, V, q)$ for some quadratic space (k, V, q) of type E_6, E_7 or E_8. In particular, k is infinite. Choose an element $t \in k \setminus \{0, 1, -1\}$ such that $t - t^2 - 1 \neq 0$, and an element $u \in k \setminus \{0, 1\}$ such that $u^4 - 2u^3 - u^2 + 2u - 1 \neq 0$. Let $y = [t]$, let $j = [t\epsilon]$, and let $i = [u\epsilon]$. Then $\alpha[v] = [(t-1)v]$, $\beta[v] = [(t - t^2 - 1)v]$, $\zeta[a, s] = [a(s-1), t(s^2 - 1) + Q(a)(s-1)]$, $\eta[0, s] = [0, (u^4 - 2u^3 - u^2 + 2u - 1)s]$, $\theta[a] = [a(u - u^2 - 1)]$, and all of these maps are surjective. Moreover, it follows from the formulas (4.1) that $h_1(R) \subseteq Z(H)$ and $h_4(\epsilon R) \subseteq Z(H)$, and $\pi_j[v] = \pi_{[t\epsilon]}[v] = -\overline{v}$ for all $v \in V$.

(v) Let $\Omega \cong \mathcal{Q}_F(k, V, q)$ for some quadratic space (k, V, q) of type F_4. In particular, k is infinite. Choose an element $t \in k \setminus \{0, 1\}$ such that

$t^2+t+1 \neq 0$, Let $y = [0,t]_W$, let $i = j = [0,t^2]_V = \epsilon[0,t]_W$. Then $\alpha[b,u] = [(t{+}1)b, (t{+}1)^2u]$, $\beta[b,u] = [(t{+}t^2{+}1)b, (t{+}t^2{+}1)^2u]$, $\zeta[a,s] = [(t^2+1)a, (t^2+1)s]$, $\eta[0,s] = [0, (t^4+t^2+1)s]$, $\theta[a] = [(t^4+t^2+1)a]$, and all of these maps are surjective. Moreover, $h_1(y), h_4(i), h_4(j) \in Z(H)$. □

We are now ready to define a lifting λ of all the root groups U_i into E. We define λ on U_2 and U_3 by the commutator relations

$$\tilde{x}_2(vy - v) = [\tilde{x}_2(v), \tilde{h}_1(y)]\,; \tag{4.2}$$

$$\tilde{x}_3(\boxminus w \boxplus wj) = [\tilde{x}_3(w), \tilde{h}_4(j)]\,; \tag{4.3}$$

for all $v \in V$, $w \in W$. Observe that this defines a lifting of all of U_2 and U_3, precisely because α and ζ are surjective. Moreover, we have

$$\tilde{x}_2(v)^{\lambda(h)} = \lambda\big(x_2(v)^h\big)\,, \qquad \tilde{x}_3(w)^{\lambda(h)} = \lambda\big(x_3(w)^h\big)\,,$$

for all $v \in V, w \in W, h \in H$; the reason is exactly the same as for equation (3.2) above, and again, we will use these facts implicitly when we conjugate certain identities by elements of H.

Now let $f(u,v) = [\tilde{x}_2(u), \tilde{x}_2(v)] \in C$ for all $u,v \in V$; then f is bi-additive. Conjugating by $\tilde{h}_1(z)$ yields $f(u,v) = f(uz,vz)$ for all $u,v \in V$ and all $z \in W^*$, and hence

$$\begin{aligned} f(u,v) &= f\big(u, v - vz\big) + f(u, vz) \\ &= f\big(uz, (v-vz)z\big) + f\big(u(\delta \boxminus z), vz(\delta \boxminus z)\big) \\ &= f\big(uz, vz - vzz\big) + f\big(u - uz, vz - vzz\big) \\ &= f\big(u, vz - vzz\big)\,, \end{aligned}$$

and therefore $f(u, vz - vzz - v) = 1$ for all $u,v \in V$ and all $z \in W^*$. Since β is surjective, choosing $z = y$ yields $f \equiv 1$.

Next let $g(w,b) = [\tilde{x}_3(w), \tilde{x}_3(b)] \cdot \lambda\left([x_3(w), x_3(b)]\right)^{-1} \in C$. Conjugating by $\tilde{h}_4(c^{-1})$ yields $g(w,b) = g(wc,bc)$ for all $w,b \in W$ and all $c \in V^*$. Now, because U_3 is not abelian in general, g is not bi-additive in general either; instead, we get the identities

$$\begin{aligned} g(w \boxplus z, b) &= g(w,b)\, g(z,b)\, g(F(H(b,w),\epsilon), z)\,, \\ g(w, b \boxplus d) &= g(w,b)\, g(w,d)\, g(F(H(b,w),\epsilon), d)\,, \end{aligned}$$

for all $w,z,b,d \in W$. Showing that $g(w,b) = 1$ for all $w,b \in W$ will be somewhat more difficult in this case. Therefore, we will first consider the case where w and b belong to the radical $R := \mathrm{Rad}(H)$ (which we know to be non-trivial), then the case where one of the two belongs to R, and finally the general case.

So assume first that $w, b \in R$, and note that $(R, \boxplus)$ is abelian, that $RV = R$, and that g restricted to $R \times R$ is bi-additive (we will denote this restriction by g_R). At this point, we will have to use the classification of the (reduced) quadrangular systems again, and distinguish between the three different cases $\mathcal{Q}_D$ (indifferent type), $\mathcal{Q}_Q$ (quadratic form type), and $\mathcal{Q}_I$ (involutory type). So let Δ be the reduced quadrangular system obtained from Ω by restriction to R, as in [3, 8.4].

The case $\mathcal{Q}_D$ follows from what we have shown for $\tilde{U}_2$ above, since these quadrangular systems have a dual (obtained by interchanging the roles of V and W), which is again a quadrangular system of type $\mathcal{Q}_D$.

Now assume $\Delta \cong \mathcal{Q}_Q(k, V, q)$ for some anisotropic quadratic space (k, V, q) with base point $\epsilon \in V^*$, i.e. $q(\epsilon) = 1$. We have $R \cong (k, +)$, $V \cong (V, +)$, and τ_R is given by $[t][v] = [tq(v)]$ for all $t \in k$ and all $v \in V$ [3, 7.1]. So we have $g_R([s], [t]) = g_R([sq(v)], [tq(v)])$ for all $s, t \in k$ and all $v \in V^*$. Suppose first that $\dim_k V = 1$. Then this gives us $g_R([s], [t]) = g_R([sr^2], [tr^2])$ for all $s, t \in k$, $r^* \in k$. As in [10, Theorem 10 (5c)], we can conclude that $g_R \equiv 1$ if $|k| \geq 5$ and $|k| \neq 9$. If $\dim_k V \geq 2$, then we can do slightly better. We may of course assume that k is finite; but then $\dim_k V = 2$, and $q : V \to k$ is surjective (see, for example, [11, (34.1) and (34.3)]). We thus get $g_R([s], [t]) = g_R([sr], [tr])$ for all $s, t \in k$, $r^* \in k$. As in [10, Theorem 10 (5b)], we can conclude that $g_R \equiv 1$ if $|k| \geq 5$.

Finally, assume that $\Delta \cong \mathcal{Q}_I(K, K_0, \sigma)$ for some involutory set (K, K_0, σ) (see [3, 7.2]), let $k := Z(K)$, the center of the (skew) field K, let $F := \mathrm{Fix}_k\, \sigma$ (so F might or might not be equal to k), and let $M := \{a^\sigma a \mid a \in k\} \subseteq F$. We have $R \cong (K_0, +)$, $V \cong (K, +)$, and τ_R is given by $[s][a] = [a^\sigma sa]$ for all $s \in K_0$ and all $a \in K$. So we have $g_R([s], [t]) = g_R([a^\sigma sa], [a^\sigma ta])$ for all $s, t \in K_0$ and all $a \in K^*$. In particular, $g_R([s], [t]) = g_R([sm], [tm])$ for all $s, t \in K_0$ and all $m \in M^*$.

We now require that $|k| \geq 5$ and $|k| \neq 9$, and if $F \neq k$, i.e. if σ is an involution of the second kind, then we require in addition that $|k| \neq 16$. We first point out that if $\mathrm{char}(k) \notin \{2, 3, 5\}$, then $4 \in M^*$, and we get $g_R([s], [t]) = g_R([4s], [4t]) = g_R([s], [t])^{16}$, hence $g_R([15s], [t]) = 1$, and since we can divide by 15, we get $g_R \equiv 1$ in this case.

Now assume $\mathrm{char}(k) \in \{2, 3, 5\}$. We claim that we can find an element $v \in k$ such that $v \in M^*$, $1 - v \in M^*$ and $1 - v + v^2 \neq 0$. If $F = k$, then $M = \{a^2 \mid a \in k\}$, and then the claim follows as in [10, Theorem 10 (5c)]. Assume $F \neq k$; if k is infinite, then so is F, and again the claim follows in the same way (now take elements $a \in F$). So we may assume that k is finite, but then, since k/F is a (separable) quadratic extension, $|k| = |F|^2$ is a square. Also, $a^\sigma a = N_{k/F}(a)$ for all $a \in k$, and since the norm is surjective [11, (34.1)], we have $M = F$. By our assumptions, $|M| = |F| \geq 5$, and so we can indeed find an element $v \in M$ satisfying our claim.

We can now apply the argument of [10, Theorem 10 (5b)] again, with an element $v \in M$, and we conclude that $g_R \equiv 1$ in this case as well.

We now consider the case $w \in W$ and $b \in R$, and we will again show that $g(w,b) = 1$. Let $v \in V \setminus \{0, \epsilon\}$; then $b = b\big((\epsilon - v) + v\big) = b(\epsilon - v) \boxplus bv \boxplus F(vb, \epsilon - v)$. Note that all three terms in this expression belong to $R = \mathrm{Rad}(H)$. Therefore, the "bi-additivity formula" for g yields

$$\begin{aligned} g(w,b) &= g\big(w, b(\epsilon - v)\big) \cdot g(w, bv) \cdot g\big(w, F(vb, \epsilon - v)\big) \\ &= g\big(wv, b(\epsilon - v)v\big) \cdot g\big(w(\epsilon - v), bv(\epsilon - v)\big) \cdot g\big(w, F(vb, \epsilon - v)\big)\,, \end{aligned}$$

where we have used the fact that $g(w,b) = g(wc, bc)$ for all $c \in V^*$. Now, since $b \in R$, we have $bv(\epsilon - v) = b(\epsilon - v)v$; denote this element by $d \in R$. Then, using the facts that $\mathrm{Im}(F) \le R$ and $g(R,R) = 1$, we get

$$\begin{aligned} g(w,b) &= g(wv, d) \cdot g\big(w(\epsilon - v), d\big) \cdot g\big(w, F(vb, \epsilon - v)\big) \\ &= g\big(wv \boxplus w(\epsilon - v), d\big) \cdot g\big(w, F(vb, \epsilon - v)\big) \\ &= g\big(w \boxminus F(w(\epsilon - v)d, wv), d\big) \cdot g\big(w, F(vb, \epsilon - v)\big) \\ &= g(w, d) \cdot g\big(w, F(vb, \epsilon - v)\big)\,; \end{aligned}$$

since $b \in R$, $d \in R$ and $\mathrm{Im}(F) \le R$, it follows from the bi-additivity formula for g that

$$g\big(w, bv(\epsilon - v) \boxminus b \boxplus F(vb, \epsilon - v)\big) = 1$$

for all $w \in W$, $b \in R$, $v \in V \setminus \{0, \epsilon\}$. Since the map η is surjective, we can choose $v = j$ to conclude that $g(W, R) = 1$. Similarly, we also have $g(R, W) = 1$. In particular, the map g is bi-additive since the last factors of the bi-additivity formulas vanish.

Finally, we consider the general case $g(w,b)$ with $w, b \in W$. In this case, $b(\epsilon - v)v \neq bv(\epsilon - v)$ in general, but we have

$$\begin{aligned} b(\epsilon - v)v &= bv \boxminus bvv \boxplus F(vb, v - \epsilon)v\,; \\ bv(\epsilon - v) &= bv \boxminus bvv \boxplus F(v \cdot bv, v - \epsilon)\,; \end{aligned}$$

since $g(W, R) = 1$, we can still conclude that $g\big(w, b(\epsilon - v)v\big) = g\big(w, bv(\epsilon - v)\big)$ for all $w \in W$. Again using the fact that $g(W, R) = g(R, W) = 1$, we get

$$\begin{aligned} g(w,b) &= g\big(w, b(\epsilon - v)\big) \cdot g(w, bv) \\ &= g\big(wv, b(\epsilon - v)v\big) \cdot g\big(w(\epsilon - v), bv(\epsilon - v)\big) \\ &= g\big(wv \boxplus w(\epsilon - v), bv(\epsilon - v)\big) \\ &= g\big(w, bv(\epsilon - v)\big)\,; \end{aligned}$$

hence $g\big(w, bv(\epsilon - v) \boxminus b\big)$. Since θ is surjective, we can conclude that $g \equiv 1$. We have shown that

$$[\tilde{x}_3(w), \tilde{x}_3(b)] = \lambda\left([x_3(w), x_3(b)]\right) = \tilde{x}_3(\boxminus w \boxminus b \boxplus w \boxplus b) \qquad (4.4)$$

for all $w, b \in W$.

The next step is to show that $\tilde{x}_2$ and $\tilde{x}_3$ are group isomorphisms. The proof of this statement for $\tilde{x}_2$ is completely similar to the proof of this fact in the Moufang triangle case, since $\tilde{U}_2$ is abelian. To show the statement about $\tilde{U}_3$, we have to show that $\tilde{x}_3(w)\tilde{x}_3(b) = \tilde{x}_3(w \boxplus b)$ for all $w, b \in W$. Since ζ is surjective, we can find elements $\hat{w}, \hat{b} \in W$ such that $\zeta(\hat{w}) = w$ and $\zeta(\hat{b}) = b$, and if $b \in R$, we can choose $\hat{b} \in R$ since $\zeta_{|R}$ is surjective as well.

We will first consider the case where $b \in R$ (and $\hat{b} \in R$). Note that, in this case, $\zeta(\hat{w} \boxplus \hat{b}) = \zeta(\hat{w}) \boxplus \zeta(\hat{b}) = w \boxplus b$, since $\hat{b}$ and $\hat{b}j$ belong to $R \leq Z(W)$. Also observe that $[\tilde{x}_3(w), \tilde{x}_3(\hat{b})] = 1$ by (4.4). Using the identity $[ab, c] = [a, c]^b[b, c]$, we now obtain

$$\begin{aligned}\tilde{x}_3(w)\tilde{x}_3(b) &= \tilde{x}_3(w)^{\tilde{x}_3(\hat{b})} \cdot \tilde{x}_3(b) \\ &= [\tilde{x}_3(\hat{w}), \tilde{h}_4(j)]^{\tilde{x}_3(\hat{b})} \cdot [\tilde{x}_3(\hat{b}), \tilde{h}_4(j)] \\ &= [\tilde{x}_3(\hat{w})\tilde{x}_3(\hat{b}), \tilde{h}_4(j)]\,,\end{aligned}$$

and since E is a central extension, this is in turn equal to

$$\tilde{x}_3(w)\tilde{x}_3(b) = [\tilde{x}_3(\hat{w} \boxplus \hat{b}), \tilde{h}_4(j)] = \tilde{x}_3\big(\zeta(\hat{w} \boxplus \hat{b})\big) = \tilde{x}_3(w \boxplus b)$$

for all $w \in W$ and all $b \in R$, which is what we wanted to show.

We now consider the general case $w, b \in W$. Note that $[W, W] \leq R$, and therefore $[\tilde{U}_3, \tilde{U}_3] \in Z(\tilde{U}_3)$ by (4.4), but also

$$\tilde{x}_3(w) \cdot [\tilde{x}_3(w), \tilde{x}_3(\hat{b})] = \tilde{x}_3(\boxminus \hat{b} \boxplus w \boxplus \hat{b})$$

by (4.4) and the previous paragraph. Using the identity $a = a^b[b, a]$, we now obtain

$$\tilde{x}_3(w)\tilde{x}_3(b) = \tilde{x}_3(w)^{\tilde{x}_3(\hat{b})} \cdot [\tilde{x}_3(\hat{b}), \tilde{x}_3(w)] \cdot \tilde{x}_3(b)\,,$$

or equivalently, since $[a, b]^{-1} = [b, a]$,

$$\tilde{x}_3(w) \cdot [\tilde{x}_3(w), \tilde{x}_3(\hat{b})] \cdot \tilde{x}_3(b) = \tilde{x}_3(w)^{\tilde{x}_3(\hat{b})} \cdot \tilde{x}_3(b)\,,$$

which we can now, just as in the previous paragraph, rewrite as

$$\begin{aligned}\tilde{x}_3(\boxminus \hat{b} \boxplus w \boxplus \hat{b}) \cdot \tilde{x}_3(b) &= \tilde{x}_3\big(\zeta(\hat{w} \boxplus \hat{b})\big) \\ &= \tilde{x}_3(\boxminus \hat{b} \boxminus \hat{w} \boxplus \hat{w}j \boxplus \hat{b}j) \\ &= \tilde{x}_3(\boxminus \hat{b} \boxplus w \boxplus \hat{b}j) \\ &= \tilde{x}_3(\boxminus \hat{b} \boxplus w \boxplus \hat{b} \boxplus b)\,.\end{aligned}$$

Substituting $\hat{b} \boxplus w \boxminus \hat{b}$ for w (which is still an arbitrary element of W since it is a conjugate of w) now yields

$$\tilde{x}_3(w) \cdot \tilde{x}_3(b) = \tilde{x}_3(w \boxplus b)$$

for all $w, b \in W$, which shows that $\tilde{x}_3$ is an isomorphism.

Next, we have to consider commutator relations between $\tilde{U}_2$ and $\tilde{U}_3$; so let $\chi(v, w) := [\tilde{x}_2(v), \tilde{x}_3(w)] \in C$. Clearly, χ is bi-additive. Conjugating by $\tilde{h}_1(y)$ yields $\chi(v, w) = \chi(vy, w)$ since $\Pi_y \equiv 1$, and hence $\chi(vy - v, w) = 1$. Since α is surjective, we conclude that $\chi \equiv 1$.

We now consider commutator relations between $\tilde{U}_2$ and $\tilde{U}_4$; so let $\psi(u, v) := [\tilde{x}_2(u), \tilde{x}_4(v)] \cdot \lambda\big([x_2(u), x_4(v)]\big)^{-1} \in C$. Using the fact that $\chi \equiv 1$, we see again that ψ is bi-additive. Conjugating by $\tilde{h}_4(c)$ yields $\psi(u, v) = \psi\big(-\pi_c(\overline{u}), -\pi_c(\overline{v \cdot \delta c})\big)$. By Lemma 4.1, if we take $c = e$, we get $\psi(u, v) = \psi(u, v \cdot \delta c)$, and hence $\psi\big(u, v(\delta c \boxminus \delta)\big) = 1$. Since $\delta c \boxminus \delta \neq 0$, it follows that $\psi \equiv 1$.

We continue with the commutator relations between $\tilde{U}_1$ and $\tilde{U}_3$; so let $\phi(w, b) := [\tilde{x}_1(w), \tilde{x}_3(b)] \cdot \lambda\big([x_1(w), x_3(b)]\big)^{-1} \in C$, and observe that ϕ is bi-additive. Let $r \in R$ be as in Lemma 4.1. Then conjugating by $h_1(r)$ yields $\phi(w, b) = \phi(w \cdot \epsilon r, b)$, and hence $\phi(\boxminus w \boxplus w \cdot \epsilon r, b) = 1$ for all $w, b \in W$. Since ζ' is surjective, we can conclude that $\phi \equiv 1$.

Finally, we consider the longest commutator relations, between $\tilde{U}_1$ and $\tilde{U}_4$. Let $\rho(w, v) := [\tilde{x}_1(w), \tilde{x}_4(v)] \cdot \lambda\big([x_1(w), x_4(v)]\big)^{-1} \in C$, and using the fact that all the shorter commutator relations already lift to E, we see again that ρ is bi-additive. If we conjugate by $\tilde{h}_4(j)\tilde{h}_1(\delta j)$, we obtain $\rho(w, v) = \rho\big(wj^{-1} \cdot \epsilon(\delta j), v \cdot \delta j \cdot (\delta j)^{-1}\big)$, and hence $\rho(\boxminus w \boxplus wj^{-1} \cdot \epsilon(\delta j), v) = 1$ for all $v \in V$ and all $w \in W$. But $wj^{-1} \cdot \epsilon(\delta j) = wj$ since $j \in \epsilon R$ (this can be derived using identities of quadrangular systems, but it might be easier to check this directly for each of the six classes). Since ζ is surjective, we conclude once again that $\rho \equiv 1$.

We have shown that all the commutator relations lift to E. To summarize:

Theorem 4.2. *Let Γ be a Moufang quadrangle $\mathcal{Q}(\Omega)$ defined over a quadrangular system Ω, let G be the little projective group of Γ, and let $\hat{G}$ be defined by the commutator relations* (2.1) *as explained above. Assume:*

- *if $\Omega \cong \mathcal{Q}_Q(k, V, q)$, then $|k| \geq 5$, and if $\dim_k V = 1$, then in addition $|k| \neq 9$;*
- *if $\Omega \cong \mathcal{Q}_I(K, K_0, \sigma)$ or $\Omega \cong \mathcal{Q}_P(K, K_0, \sigma, X, \pi)$, then $|Z(K) \cap K_0| \geq 5$ and $|Z(K)| \neq 9$, and if σ is an involution of the second kind, then in addition $|Z(K)| \neq 16$;*
- *if $\Omega \cong \mathcal{Q}_D(k, k_0, l_0)$, then $|k| \geq 5$.*

Then $\hat{G}$ is the universal central extension of G.

5 Moufang hexagons

The Moufang hexagons and octagons are considerably easier than the previous case of the Moufang quadrangles, and we will not go into the same level of detail.

So let Γ be an arbitrary Moufang hexagon. Then Γ can be parametrized by a so-called *hexagonal system*, which is, in fact, equivalent to a *unital quadratic Jordan algebra of degree three.* So let $(J, k, \sharp)$ be a hexagonal system (which we will simply denote by J), with corresponding (bilinear) trace T, norm N, Freudenthal cross product $\times$, unit element $1 \in J^*$, and U-operators U_a, $a \in J^*$. Let U_1, U_3 and U_5 be parameterized by the additive group of the vector space J, and let U_2, U_4 and U_6 be parameterized by the additive group of the commutative field k. The defining relations are given by

$$\begin{aligned}
[x_1(a), x_3(b)] &= x_2\big(T(a,b)\big) \\
[x_3(a), x_5(b)] &= x_4\big(T(a,b)\big) \\
[x_1(a), x_5(b)] &= x_2\big(-T(a^\sharp, b)\big)\, x_3(a \times b)\, x_4\big(T(a, b^\sharp)\big) \\
[x_2(s), x_6(t)] &= x_4(st) \\
[x_1(a), x_6(t)] &= x_2\big(-TN(a)\big)\, x_3(ta^\sharp)\, x_4\big(t^2 N(a)\big)\, x_5(-ta)\,,
\end{aligned}$$

for all $v, c \in V$ and all $w, z \in W$; the commutator relations which do not occur here are trivial (i.e. the corresponding root groups commute). We let $e_i = x_i(1)$ (where 1 is the unit element in J) for $i = 1, 3, 5$, and $e_i = x_i(1)$ (where 1 is the unit element in k) for $i = 2, 4, 6$. By [11, (32.12)], we have $\mu_6(t) = x_0(t^{-1})x_6(t)x_0(t^{-1})$ and $\mu_1(a) = x_7(a^{-1})x_1(a)x_7(a^{-1})$, and we obtain

$$\begin{aligned}
x_1(a)^{h_1(c)} &= x_1\big(U_c(a)\big)\,; & x_1(a)^{h_6(u)} &= x_1(u^{-1}a)\,; \\
x_2(t)^{h_1(c)} &= x_2\big(N(c)t\big)\,; & x_2(t)^{h_6(u)} &= x_2(u^{-1}t)\,; \\
x_3(a)^{h_1(c)} &= x_3\left(N(c)^{-1}U_{c^\sharp}(a)\right)\,; & x_3(a)^{h_6(u)} &= x_3(a)\,; \\
x_4(t)^{h_1(c)} &= x_4(t)\,; & x_4(t)^{h_6(u)} &= x_4(ut)\,; \\
x_5(a)^{h_1(c)} &= x_5\left(N(c)^{-1}U_c(a)\right)\,; & x_5(a)^{h_6(u)} &= x_5(ua)\,; \\
x_6(t)^{h_1(c)} &= x_6\left(N(c)^{-1}t\right)\,; & x_6(t)^{h_6(u)} &= x_6(u^2 t)\,;
\end{aligned}$$

for all $a \in J$, $c \in J^*$, $t \in k$, $u \in k^*$. Note that $h_6(u) \in Z(H)$ for all $u \in k^*$. We will assume from now on that k has at least five elements (and this is the only assumption we will make).

We will now define a lifting λ of all the root groups U_i into E. Choose a fixed element $u \in k \setminus \{0, 1\}$, and define λ on U_4 and U_5 by the commutator relations

$$\tilde{x}_4(ut - t) = [\tilde{x}_4(t), \tilde{h}_6(u)]\,; \tag{5.1}$$

$$\tilde{x}_5(ua - a) = [\tilde{x}_5(a), \tilde{h}_6(u)]\,; \tag{5.2}$$

for all $v \in V$, $w \in W$. Since $u - 1 \neq 0$, this defines a lifting of all of U_4 and U_5, Just as in the previous cases, we have

$$\tilde{x}_4(t)^{\lambda(h)} = \lambda\big(x_4(t)^h\big)\,, \qquad \tilde{x}_5(a)^{\lambda(h)} = \lambda\big(x_5(a)^h\big)\,,$$

for all $t \in k, a \in J, h \in H$.

We will now show that all the commutator relations lift to E, and we will gradually increase the "distance" between the root groups. In each case, the maps that we will define will be bi-additive (by induction on this distance), so we will not repeat this over and over.

Now let $\psi_{4,4}(s,t) := [\tilde{x}_4(s), \tilde{x}_4(t)] \in C$. Conjugating by $\tilde{h}_6(u)$ yields $\psi_{4,4}(s,t) = \psi_{4,4}(us, ut)$, and since $|k| \geq 5$, the standard "Steinberg argument" (as for equation (3.3)) then shows that $\psi_{4,4} \equiv 1$. Similarly, let $\psi_{5,5}(a,b) := [\tilde{x}_5(a), \tilde{x}_5(b)] \in C$. Conjugating by $\tilde{h}_6(u)$ yields $\psi_{5,5}(a,b) = \psi_{5,5}(ua, ub)$, and hence $\psi_{5,5} \equiv 1$.

Let $\psi_{4,5}(t,a) := [\tilde{x}_4(t), \tilde{x}_5(a)] \in C$. Conjugating by $\tilde{h}_6(u)$ yields $\psi_{4,5}(t,a) = \psi_{4,5}(ut, ua)$, and again $\psi_{4,5} \equiv 1$.

Now let $\psi_{3,5}(a,b) := [\tilde{x}_3(a), \tilde{x}_5(b)] \cdot \lambda\big([x_3(a), x_5(b)]\big)^{-1} \in C$. Conjugating by $\tilde{h}_6(u)$ yields $\psi_{3,5}(a,b) = \psi_{3,5}(a, ub)$, hence $\psi_{3,5}(a, (u-1)b) = 1$ and therefore $\psi_{3,5} \equiv 1$. Let $\psi_{4,6}(s,t) := [\tilde{x}_4(s), \tilde{x}_6(t)] \in C$. This time, we conjugate by $\tilde{h}_1(u \cdot 1)$ to get $\psi_{4,6}(s,t) = \psi_{4,6}(s, u^3 t)$, and since $|k| \geq 5$, there exists a $u \in k^*$ with $u^3 \neq 1$. Therefore $\psi_{4,6} \equiv 1$.

Since no new arguments arise in the remaining commutator relations, we will only mention by which elements we have to conjugate. For $\psi_{3,6}$, conjugate by $\tilde{h}_6(u)$; for $\psi_{2,6}$, conjugate by $\tilde{h}_6(u^3) \cdot \tilde{h}_1(u \cdot 1)$; for $\psi_{1,5}$, conjugate by $\tilde{h}_6(u) \cdot \tilde{h}_1(u \cdot 1)$; for $\psi_{1,6}$, conjugate by $\tilde{h}_6(u^2) \cdot \tilde{h}_1(u \cdot 1)$. We conclude that all the commutator relations lift to E, so we have shown:

Theorem 5.1. *Let Γ be a Moufang hexagon $\mathcal{H}(J)$ defined over an hexagonal system J over a commutative field k, let G be the little projective group of Γ, and let $\hat{G}$ be defined by the commutator relations* (2.1) *as explained above. Assume that $|k| \geq 5$. Then $\hat{G}$ is the universal central extension of G.*

6 Moufang octagons

Finally, let Γ be an arbitrary Moufang octagon. Then Γ can be parametrized by a so-called *octagonal set*, which is determined by a commutative field k

with $\mathrm{char}(k) = 2$, and a Tits endomorphism σ, i.e. an endomorphism σ such that $x^{\sigma^2} = x^2$ for all $x \in k$. So let (k, σ) be an octagonal set, let U_1, U_3, U_5 and U_7 be parameterized by $(k, +)$, and let U_2, U_4, U_6 and U_8 be parameterized by the group $k_\sigma^{(2)}$ as defined in [11, (10.15)], i.e. the group with underlying set $k \times k$ and with group operation given by

$$(s, w) \boxplus (u, v) = (s + u + w^\sigma v, w + v)$$

for all $s, w, u, v \in k$.

For each $i \in \{2, 4, 6, 8\}$, we set

$$x_i(t) := x_i(t, 0)\,, \qquad\qquad y_i(u) := x_i(0, u)\,,$$

for all $t, u \in k$, and we set $V_i := \{x_i(t) \mid t \in k\}$; observe that $V_i = Z(U_i)$ if $|k| > 2$. Let $\mathcal{S}$ be the set consisting of the following relations:

$$\begin{aligned}
&[U_1, U_2] = 1, \qquad [U_1, V_4] = 1, \qquad [V_2, U_4] = 1,\\
&[U_1, U_3] = 1, \qquad [U_1, U_5] = 1, \qquad [U_2, U_6] = 1,\\
[x_1(t), y_4(u)] &= x_2(tu)\ ,\\
[x_1(t), x_6(u)] &= x_4(tu)\ ,\\
[x_1(t), y_6(u)^{-1}] &= x_2(t^\sigma u) \cdot x_3(tu^\sigma) \cdot x_4(tu^{\sigma+1})\ ,\\
[x_1(t), x_7(u)] &= x_3(t^\sigma u) \cdot x_5(tu^\sigma)\ ,\\
[x_1(t), x_8(u)] &= x_2(t^{\sigma+1}u) \cdot x_3(t^{\sigma+1}u^\sigma) \cdot y_4(t^\sigma u) \cdot x_5(t^{\sigma+1}u^2)\\
&\qquad \cdot y_6(tu)^{-1} \cdot x_7(tu^\sigma)\ ,\\
[x_1(t), y_8(u)^{-1}] &= y_2(tu) \cdot x_3(t^{\sigma+1}u^{\sigma+2}) \cdot y_4(t^\sigma u^{\sigma+1})^{-1} \cdot x_5(t^{\sigma+1}u^{2\sigma+2})\\
&\qquad \cdot x_6(t^{\sigma+1}u^{2\sigma+3}) \cdot x_7(tu^{\sigma+2})\ ,\\
[y_2(t), y_4(u)] &= x_3(tu)\ ,\\
[x_2(t), x_8(u)] &= x_4(t^\sigma u) \cdot x_5(tu) \cdot x_6(tu^\sigma)\ ,\\
[x_2(t), y_8(u)^{-1}] &= x_3(tu) \cdot x_4(t^\sigma u^{\sigma+1}) \cdot x_6(tu^{\sigma+2})\ ,\\
[y_2(t)^{-1}, y_8(u)^{-1}] &= x_3(t^{\sigma+1}u) \cdot y_4(t^\sigma u)^{-1} \cdot y_6(tu^\sigma) \cdot x_7(tu^{\sigma+1})\ ,
\end{aligned}$$

for all $t, u \in k$. Let $\mathcal{J} := \{1, \ldots, 8\}$. Let τ_1 be the permutation of $\mathcal{J}$ which maps each $x \in \mathcal{J}$ to the unique element $y \in \mathcal{J}$ satisfying $y \equiv x + 2 \pmod 8$; let τ_2 be the permutation of $\mathcal{J}$ which maps each $x \in \mathcal{J}$ to the unique element $y \in \mathcal{J}$ satisfying $y \equiv -x \pmod 8$. Let $N := \langle \tau_1, \tau_2 \rangle$. For each relation $r \in \mathcal{S}$ and each permutation $\rho \in N$, we define r^ρ to be the relation we get by replacing every index i occurring in r by i^ρ. We thus get a set of relations $\mathcal{S}_0 := \{r^\rho \mid r \in \mathcal{S} \text{ and } \rho \in N\}$. It is this set $\mathcal{S}_0$ that we take as set of defining commutator relations for Γ.

Now let $e_i = x_i(1)$ for $i = 1, 3, 5, 7$, and $e_i = x_i(1, 0)$ for $i = 2, 4, 6, 8$. For each element $(s, w) \in k_\sigma^{(2)}$, we write $R_{(s,w)} := s^\sigma + sw + w^{\sigma+2}$. By

[11, (32.13)], we have

$$\mu_8(s,w) = x_0\left((s+w^{\sigma+1})R^{-\sigma}, sR^{-1}\right)\cdot x_8(s,w)\cdot x_0\left(wR^{-1},(s+w^{\sigma+1})R^{-1}\right),$$

where $R = R_{(s,w)}$, and we have $\mu_1(t) = x_9(t^{-1})x_1(t)x_9(t^{-1})$. It turns out (see [11, (33.17)]) that $h_8(s,w)$ depends only on $t = R_{(s,w)}$, so we will write $h_8(t)$ instead. We obtain

$$\begin{aligned}
x_1(r)^{h_1(t)} &= x_1\left(rt^2\right); & x_1(r)^{h_8(t)} &= x_1\left(rt^{-1}\right);\\
x_2(s,w)^{h_1(t)} &= x_2\left(st^{\sigma+1}, wt\right); & x_2(s,w)^{h_8(t)} &= x_2\left(st^{-1}, wt^{-\sigma+1}\right);\\
x_3(r)^{h_1(t)} &= x_3\left(rt^{\sigma}\right); & x_3(r)^{h_8(t)} &= x_3\left(rt^{-\sigma+1}\right);\\
x_4(s,w)^{h_1(t)} &= x_4\left(st, wt^{\sigma-1}\right); & x_4(s,w)^{h_8(t)} &= x_4\left(s,w\right);\\
x_5(r)^{h_1(t)} &= x_5\left(r\right); & x_5(r)^{h_8(t)} &= x_5\left(rt^{\sigma-1}\right);\\
x_6(s,w)^{h_1(t)} &= x_6\left(st^{-1}, wt^{-\sigma+1}\right); & x_6(s,w)^{h_8(t)} &= x_6\left(st, wt^{\sigma-1}\right);\\
x_7(r)^{h_1(t)} &= x_7\left(rt^{-\sigma}\right); & x_7(r)^{h_8(t)} &= x_7\left(rt\right);\\
x_8(s,w)^{h_1(t)} &= x_8\left(st^{-\sigma-1}, wt^{-1}\right); & x_8(s,w)^{h_8(t)} &= x_8\left(st^{\sigma}, wt^{-\sigma+2}\right);
\end{aligned}$$

for all $r \in k$, $t \in k^*$, $(s,w) \in k_\sigma^{(2)}$. Note that H is abelian in this case. We will again assume from now on that k has at least five elements (and this is the only assumption we will make). In fact, this only excludes the smallest Moufang octagon defined over $\mathsf{GF}(2)$ (for which the little projective group is not even perfect), since the field $\mathsf{GF}(4)$ does not admit a Tits endomorphism.

Choose a fixed element $t \in k \setminus \{0,1\}$, and note that $1 + t^{\sigma-1} \neq 0$. Define a lifting λ on the root groups U_3 and U_4 by the commutator relations

$$\tilde{x}_3\big(r(1+t^{\sigma})\big) = [\tilde{x}_3(r), \tilde{h}_1(t)]; \tag{6.1}$$

$$\tilde{x}_4\big(s(1+t) + w^{\sigma+1}(1+t^{\sigma-1}), w(1+t^{\sigma-1})\big) = [\tilde{x}_4(s,w), \tilde{h}_1(t)]; \tag{6.2}$$

for all $r, s, w \in k$.

As in the previous cases, conjugating the commutator $[\tilde{x}_3(r), \tilde{x}_3(s)]$ by $\tilde{h}_1(t)$ yields, by the standard Steinberg argument, that $\tilde{U}_3$ is an abelian subgroup of E, and that $\tilde{x}_3$ is an isomorphism from U_3 to $\tilde{U}_3$.

The commutator relations of the lifting of the non-abelian group U_4 are again somewhat more involved. Let

$$g\big((s,w),(u,v)\big) := [\tilde{x}_4(s,w), \tilde{x}_4(u,v)]\cdot\lambda\big([x_4(s,w), x_4(u,v)]\big)^{-1}$$

as usual. Conjugating by $h_1(t)$ yields

$$g\big((s,w),(u,v)\big) = g\big((ts, wt^{\sigma-1}), (ut, vt^{\sigma-1})\big) \tag{6.3}$$

for all $s, w, u, v \in k$, $t \in k^*$. Moreover, g satisfies the "bi-additivity relations"

$$\begin{aligned}
g(a \boxplus b, c) &= g(a,c)\, g(b,c)\, g([a,c], b),\\
g(a, b \boxplus c) &= g(a,b)\, g(a,c)\, g([a,b], c),
\end{aligned}$$

for all $a, b, c \in k_\sigma^{(2)}$; note that $[(s,w),(u,v)] = [w^\sigma v + v^\sigma w, 0]$ for all elements $(s,w),(u,v) \in k_\sigma^{(2)}$. Let $Z := \{(s,0) \mid s \in k\}$, and observe that $Z\big(k_\sigma^{(2)}\big) = \big[k_\sigma^{(2)}, k_\sigma^{(2)}\big] = Z$ since $|k| \neq 2$.

The standard Steinberg argument shows that $g(Z,Z) = 1$. We now consider $g\big((0,w),(u,0)\big)$. Using the bi-additivity formulas, equation (6.3), and the fact that $g(Z,Z) = 1$, we get

$$\begin{aligned}
&g\big((0,w),(u,0)\big) \\
&\quad = g\big((0,w),(u(1+t),0)\big) \cdot g\big((0,w),(ut,0)\big) \\
&\quad = g\big((0,wt^{\sigma-1}),(u(1+t)t,0)\big) \cdot g\big((0,w(1+t)^{\sigma-1}),(ut(1+t),0)\big) \\
&\quad = g\big((wt^{\sigma-1})^\sigma \cdot w(1+t)^{\sigma-1}, wt^{\sigma-1} + w(1+t)^{\sigma-1}),(u(1+t)t,0)\big) \\
&\quad = g\big((0, w\cdot(t^{\sigma-1} + (1+t)^{\sigma-1})),(u(1+t)t,0)\big)\,;
\end{aligned}$$

on the other hand, if we apply equation (6.3) with $t(t+1)$ in place of t, we obtain

$$g\big((0,w),(u,0)\big) = g\big((0, w\cdot(t^{\sigma-1}\cdot(1+t)^{\sigma-1})),(u(1+t)t,0)\big)\,.$$

Combining these two expressions, we see that it suffices to find an element $t \in k \setminus \{0,1\}$ such that $t^{\sigma-1} + (1+t)^{\sigma-1} \neq t^{\sigma-1}\cdot(1+t)^{\sigma-1}$ to be able to conclude that $g\big((0,w),(u,0)\big) = 1$ for all $w,u \in k$. Substituting $a = (1+t^{-1})^{\sigma-1}$ reduces this equation to $a + a^{-1} = 1$, and since this equation has at most two solutions in k and $|k| \geq 8$, we can indeed find such an element t. Hence $g(k_\sigma^{(2)}, Z) = g(Z, k_\sigma^{(2)}) = 1$. Note that, in particular, g is now bi-additive.

It only remains to consider the case $g\big((0,w),(0,v)\big)$ with $v,w \in k$. Again by equation (6.3), we see that $g\big((0,w),(0,v)\big) = g\big((0,wt),(0,vt)\big)$ for all $t \in k$, and we can apply the standard Steinberg argument to conclude that $g\big((0,w),(0,v)\big) = 1$ for all $v,w \in k$. Bringing everything together, we see that we have shown that $g\big(k_\sigma^{(2)}, k_\sigma^{(2)}\big) = 1$.

The proof that $\tilde{x}_3$ and $\tilde{x}_4$ are group isomorphisms is very similar to the proof of these facts in the case of the Moufang quadrangles, and we will omit the details.

To show that all the other commutator relations lift to E, we will use the same method as in the case of the Moufang hexagons, and we will only mention by which elements we have to conjugate to obtain the required result. For $\psi_{4,5}$, conjugate by $\tilde{h}_8(t^{\sigma+1})$; for $\psi_{3,5}$, conjugate by $\tilde{h}_1(t)$; for $\psi_{4,6}$, conjugate by $\tilde{h}_8(t)$; for $\psi_{4,7}$, conjugate by $\tilde{h}_8(t)$; for $\psi_{4,8}$, conjugate by $\tilde{h}_8(t)$; for $\psi_{1,5}$, conjugate by $\tilde{h}_1(t)$; for $\psi_{1,6}$, conjugate by $\tilde{h}_1(t)\tilde{h}_8(t)$; for $\psi_{1,7}$, conjugate by $\tilde{h}_1(t)\tilde{h}_8(t^2)$; for $\psi_{2,8}$, conjugate by $\tilde{h}_1(t)\tilde{h}_8(t^{\sigma+1})$; for $\psi_{1,8}$, conjugate by $\tilde{h}_1(t^{2-\sigma})\tilde{h}_8(t)$. We conclude that all the commutator relations lift to E, so we have shown:

Theorem 6.1. *Let Γ be a Moufang octagon $\mathcal{O}(k,\sigma)$ defined over an octagonal set (k,σ), let G be the little projective group of Γ, and let $\hat{G}$ be defined by the commutator relations* (2.1) *as explained above. Assume that $|k| \neq 2$. Then $\hat{G}$ is the universal central extension of G.*

This finishes the proof of the Main Theorem.

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