ON FAKE LENS SPACES WITH THE FUNDAMENTAL GROUP OF ORDER A POWER OF 2

TIBOR MACKO, CHRISTIAN WEGNER

Abstract. We present a classification of fake lens spaces of dimension ≥ 5 which have as fundamental group the cyclic group of order $N = 2^K$, in that we extend the results of Wall and others in the case $N = 2$.

Introduction

A *fake lens space* is the orbit space of a free action of a finite cyclic group G on a sphere S^{2d-1}. It is a generalization of the notion of a *lens space* which is the orbit space of a free action which comes from a unitary representation. The classification of lens spaces is a classical topic in algebraic topology and algebraic K-theory well explained for example in [Mil66]. For the classification of fake lens spaces in dimension ≥ 5 methods of surgery theory are especially suitable. The classification of fake lens spaces with the fundamental group of order $N = 2$ or N odd was obtained and published in the books [Wal99], [LdM71]. Since then, the problem remained open for $N \neq 2$ even. In this note we address the classification for $N = 2^K$.

An important reason why the classification for all N was not finished in [Wal99] seems to be that the so-called L-groups $L^s_n(G)$ for $G = \mathbb{Z}_N$ were unknown for N even. This is not the case anymore, see for example [HT00]. Using this additional information and the general methods of Wall from [Wal99, chapter 14] we reduce the classification question to a problem in the representation theory of G. We then develop calculational methods for solving this problem which enable us to obtain the complete solution for $N = 2^K$. In a future paper we plan to complete the classification for all $N \in \mathbb{N}$ by combining our methods for $N = 2^K$ with the methods of Wall and others for N odd.

The classification of fake lens spaces up to simple homotopy equivalence for all $N \in \mathbb{N}$ via Reidemeister torsion is described in [Wal99, chapter 14E]. The desired homeomorphism classification within a simple homotopy type can be formulated in terms of the *simple structure set* $\mathcal{S}^s(X)$ of a closed n-manifold X. An element of $\mathcal{S}^s(X)$ is represented by a simple homotopy equivalence $f\colon M \to X$ from a closed n-manifold M. Two such $f\colon M \to X$, $f'\colon M' \to X$ are equivalent if there exists a homeomorphism $h\colon M \to M'$ such that $f' \circ h \simeq f$. The simple structure set $\mathcal{S}^s(X)$ is a priori just a pointed set with the base point $\mathrm{id}\colon X \to X$. However, it can also be endowed with a preferred structure (in some sense) of an abelian group (see [Ran92, chapter 18]).

In general the simple structure set of an n-manifold for $n \geq 5$ can be determined by examining the surgery exact sequence which is recalled below as (3.1). Besides determining $\mathcal{S}^s(X)$ it is also important to determine the invariants that distinguish the elements of $\mathcal{S}^s(X)$. In the case of fake lens spaces it follows from the calculations in [Wal99, chapter 14E] that the simple structure set of a fake lens space is detected

2000 *Mathematics Subject Classification.* Primary: 57R65, 57S25.
Key words and phrases. lens space, structures set, ρ-invariant, normal invariants, surgery.
Both authors are supported by SFB 478 Geometrische Strukturen in der Mathematik, Münster.

by the ρ-invariant of [AS68] and [Wal99, chapter 14B], and by the normal invariants from surgery theory. Our results describe in the case $N = 2^K$ the redundancy among those invariants. Under a certain choice we can replace normal invariants by another collection of invariants obtaining a one-to-one correspondence. However, better geometric interpretation of the new invariants would still be desirable.

1. Statement of results

Definition 1.1. A *fake lens space* $L^{2d-1}(\alpha)$ is a manifold obtained as the orbit space of a free action α of the group $G = \mathbb{Z}_N$ on S^{2d-1}.

The fake lens space $L^{2d-1}(\alpha)$ is a $(2d-1)$-dimensional manifold with $\pi_1(L^{2d-1}(\alpha)) \cong G = \mathbb{Z}_N$ and universal cover S^{2d-1}. The main theorem in this paper is:

Theorem 1.2. *Let $L^{2d-1}(\alpha)$ be a fake lens space with $\pi_1(L^{2d-1}(\alpha)) \cong \mathbb{Z}_N$ where $N = 2^K$ and $d \geq 3$. Then we have*

$$\mathcal{S}^s(L^{2d-1}(\alpha)) \cong \bar{\Sigma} \oplus \bar{T} \cong \bar{\Sigma} \oplus \bigoplus_{i=1}^{c} \mathbb{Z}_2 \oplus \bigoplus_{i=1}^{c} \mathbb{Z}_{2^{\min\{K,2i+2\}}} \tag{1.1}$$

where $\bar{\Sigma}$ is a free abelian group of rank $N/2-1$ if $d = 2e+1$ and $N/2$ if $d = 2e$ and $c = \lfloor (d-1)/2 \rfloor$.

Unfortunately we are unable to give a direct geometric description of this isomorphism. Nevertheless, the proof of the theorem gives an indirect interpretation in terms of known geometric invariants of manifold structures. These are the *reduced ρ-invariant* and the *normal invariants* from surgery theory.

The reduced ρ-invariant is a homomorphism

$$\widetilde{\rho}\colon \mathcal{S}^s(L^{2d-1}(\alpha)) \longrightarrow \mathbb{Q}R_{\widehat{G}}^{(-1)^d}$$

where the target is the underlying abelian group of the $(-1)^d$-eigenspace of the rationalized complex representation ring of G modulo the ideal generated by the regular representation. The group $\bar{\Sigma}$ is defined as the image of $\widetilde{\rho}$.

The normal invariant is a map $\eta\colon \mathcal{S}^s(L^{2d-1}(\alpha)) \to \mathcal{N}(L^{2d-1}(\alpha))$ with the target the group of normal invariants from surgery theory, which is easily calculable. The reduced ρ-invariant induces the homomorphism

$$[\widetilde{\rho}]\colon \widetilde{\mathcal{N}}(L^{2d-1}(\alpha)) \longrightarrow \mathbb{Q}R_{\widehat{G}}^{(-1)^d}/4 \cdot R_{\widehat{G}}^{(-1)^d}.$$

Here the source is the subgroup of $\mathcal{N}(L^{2d-1}(\alpha))$ given by the image of η and in the target we have the quotient group modulo the subgroup of elements in the $(-1)^d$-eigenspace of the representation ring which are divisible by 4. We use formulas of Wall to show relations between the invariants $\widetilde{\rho}$ and η in Proposition 4.12.

The group $\bar{T}$ is defined as the kernel of $[\widetilde{\rho}]$. Our main technical result is the calculation of $\bar{T}$ in section 5. In the proof of Proposition 5.1 we describe a map $\lambda\colon \bar{T} \to \mathcal{S}^s(L^{2d-1}(\alpha))$ which fits into a short exact sequence

$$0 \longrightarrow \bar{T} \xrightarrow{\lambda} \mathcal{S}^s(L^{2d-1}(\alpha)) \xrightarrow{\widetilde{\rho}} \bar{\Sigma} \longrightarrow 0.$$

Since $\bar{\Sigma}$ is a free abelian group the sequence splits and we obtain the isomorphism of Theorem 1.2. Such a splitting s of $\widetilde{\rho}$ induces a map $\mathbf{r}\colon \mathcal{S}^s(L^{2d-1}(\alpha)) \to \bar{T}$ with $\mathbf{r} \circ \lambda = \mathrm{id}$. The calculation of $\bar{T}$ from Theorems 5.2, 5.3 tells us that it is a direct sum of copies of 2-primary cyclic groups which are indexed by $1 \leq i \leq 2c$. We denote the projection on the i-th summand by $\mathbf{r}_{2i}$. Putting these considerations together we obtain the following corollary.

Corollary 1.3. *Let $L^{2d-1}(\alpha)$ be a fake lens space with $\pi_1(L^{2d-1}(\alpha)) \cong \mathbb{Z}_N$ where $N = 2^K$ and $d \geq 3$. There exists a collection of invariants $\mathbf{r}_{4i}\colon \mathcal{S}^s(L^{2d-1}(\alpha)) \longrightarrow \mathbb{Z}_{2^{\min\{K,2i+2\}}}$ and $\mathbf{r}_{4i-2}\colon \mathcal{S}^s(L^{2d-1}(\alpha)) \longrightarrow \mathbb{Z}_2$ for $1 \leq i \leq c$ such that an element $a \in \mathcal{S}^s(L^{2d-1}(\alpha))$ is uniquely determined by*

(1) $\widetilde{\rho}(a) \in \bar{\Sigma} \subset \mathbb{Q}R_{\widehat{G}}^{(-1)^d}$
(2) $\mathbf{r}_{2i}(a) \in \mathbb{Z}_2, \mathbb{Z}_{2^{\min\{K,2i+2\}}}$

Moreover, any collection $z \in \bar{\Sigma}$, $\mathbf{r}_{2i} \in \mathbb{Z}_2, \mathbb{Z}_{2^{\min\{K,2i+2\}}}$ for $1 \leq i \leq c$ can be realized by an element of $\mathcal{S}^s(L^{2d-1}(\alpha))$.

The invariants $\mathbf{r}_{4i-2}$ are the usual normal invariants from surgery theory. Clearly, a better geometric description of the invariants $\mathbf{r}_{4i}$ would be desirable. We suspect that this might be achieved by studying their relationship to codimension 2 surgery obstructions which are elements of the so-called LS-groups of [Wal99, chapter 11], [Ran81, chapter 7].

As stated in the introduction the simple homotopy types of fake lens spaces can be distinguished by the Reidemeister torsion which is a unit in $\mathbb{Q}R_G$, the rational group ring of G modulo the ideal generated by the norm element. The Reidemeister torsion has to be added to the invariants of Corollary 1.3 in order to obtain the homeomorphism classification of all fake lens spaces for a given dimension and with the fundamental group isomorphic to $G = \mathbb{Z}_N$ for $N = 2^K$.

The paper is organized as follows. In section 2 we briefly recall the homotopy classification of fake lens spaces. In section 3 we recall the general machinery of surgery theory and we describe the known terms in the surgery exact sequence of the fake lens spaces. In section 4 we recall the definition and properties of the ρ-invariant. Finally, in section 5 we prove our main technical result which is the calculation of the group $\bar{T}$. Sections 2, 3 and most of section 4 contain no new results. Our contribution is concentrated in a part of section 4 and in the last section 5.

We thank Diarmuid Crowley, Ian Hambleton and Andrew Ranicki for useful comments.

2. Homotopy classification

In this section we briefly recall without proofs the statements of the homotopy and simple homotopy classification of fake lens spaces from [Wal99, chapter 14E]. Apart from definitions only Corollary 2.4 is of importance for the rest of the paper.

We start by introducing some notation for *lens spaces* which are a special sort of fake lens spaces. Let $N \in \mathbb{N}$, $\bar{k} = (k_1, \ldots k_d)$, where $k_i \in \mathbb{Z}$ are such that $(k_i, N) = 1$. When $G = \mathbb{Z}_N$ define a representation $\alpha_{\bar{k}}$ of G on $\mathbb{C}^d$ by $(z_1 \ldots, z_n) \mapsto (z_1 e^{2\pi i k_1/N}, \ldots, z_n e^{2\pi i k_d/N})$. Any free representation of G on a d-dimensional complex vector space is isomorphic to some $\alpha_{\bar{k}}$. The representation $\alpha_{\bar{k}}$ induces a free action of G on S^{2d-1} which we still denote $\alpha_{\bar{k}}$.

Definition 2.1. A *lens space* $L^{2d-1}(\alpha_{\bar{k}})$ is a manifold obtained as the orbit pace of a free action $\alpha_{\bar{k}}$ of the group $G = \mathbb{Z}_N$ on S^{2d-1} for some $\bar{k} = (k_1, \ldots k_d)$ as above.[1]

The lens space $L^{2d-1}(\alpha_{\bar{k}})$ is a $(2d-1)$-dimensional manifold with $\pi_1(L^{2d-1}(\alpha_{\bar{k}})) \cong \mathbb{Z}_N$. Its universal cover is S^{2d-1}, hence $\pi_i(L^{2d-1}(\alpha_{\bar{k}})) \cong \pi_i(S^{2d-1})$ for $i \geq 2$. There exists a convenient choice of a CW-structure for $L^{2d-1}(\alpha_{\bar{k}})$ with one cell e_i in each dimension $0 \leq i \leq 2d-1$. Moreover, we have $H_i(L^{2d-1}(\alpha_{\bar{k}})) \cong \mathbb{Z}$ when $i = 0, 2d-1$, $H_i(L^{2d-1}(\alpha_{\bar{k}})) \cong \mathbb{Z}_N$ when $0 < i < 2d-1$ is odd and $H_i(L^{2d-1}(\alpha_{\bar{k}})) \cong 0$ when $i \neq 0$ is even.

[1] In the notation of [Wal99, chapter 14E] we have $L(\alpha_{\bar{k}}) = L(N, k_1, \ldots, k_n)$.

The classification of the lens spaces up to homotopy equivalence and simple homotopy equivalence is presented for example in [Mil66]. The simple homotopy classification is stated in terms of Reidemeister torsion which is a unit in $\mathbb{Q}R_G$. This ring is defined as $\mathbb{Q}R_G = \mathbb{Q} \otimes R_G$ with $R_G = \mathbb{Z}G/\langle Z \rangle$ where $\mathbb{Z}G$ be the group ring of G and $\langle Z \rangle$ is the ideal generated by the norm element Z of G. We also suppose that a generator T of G is chosen. There is also an augmentation map $\varepsilon' \colon R_G \to \mathbb{Z}_N$ [Wal99, page 214]. The homotopy classification is stated in terms of a certain unit in $\mathbb{Z}_N$. These invariants also suffice for the homotopy and simple homotopy classification of finite CW-complexes L with $\pi_1(L) \cong \mathbb{Z}_N$ and with the universal cover homotopy equivalent to S^{2d-1} of which fake lens spaces are obviously a special case. It is convenient to make the following definition.

Definition 2.2. A *polarization* of a CW-complex L as above is a pair (T, e) where T is a choice of a generator of $\pi_1(L)$ and e is a choice of a homotopy equivalence $e \colon \widetilde{L} \to S^{2d-1}$.

Denote further by $L^{2d-1}(\alpha_k)$ the lens space $L^{2d-1}(\alpha_{\bar{k}})$ with $\bar{k} = (1, \ldots, 1, k)$. By $L^i(\alpha_1)$ is denoted the i-skeleton of the lens space $L^{2d-1}(\alpha_1)$. If i is odd this is a lens space, if i is even this is a CW-complex obtained by attaching an i-cell to the lens space of dimension $i - 1$.

Proposition 2.3. [Wal99, Theorem 14E.3] *Let L be a finite CW-complex as above polarized by (T, e). Then there exists a simple homotopy equivalence*

$$h \colon L \longrightarrow L^{2d-2}(\alpha_1) \cup_\phi e^{2d-1}$$

preserving the polarization. It is unique up to homotopy and the action of G. The chain complex differential on the right hand side is given by $\partial_{2d-1} e^{2d-1} = e_{2d-2}(T-1)U$ for some $U \in \mathbb{Z}G$ which maps to a unit $u \in R_G$. Furthermore, the complex L is a Poincaré complex.

1. *The polarized homotopy types of such L are in one-to-one correspondence with the units in $\mathbb{Z}_N$. The correspondence is given by $\varepsilon'(u) \in \mathbb{Z}_N$.*
2. *The polarized simple homotopy types of such L are in one-to-one correspondence with the units in R_G. The correspondence is given by $u \in R_G$.*

The existence of a fake lens space in the homotopy type of such L is addressed in [Wal99, Theorem 14E.4]. Since the units $\varepsilon'(u) \in \mathbb{Z}_N$ are exhausted by the lens spaces $L^{2d-1}(\alpha_k)$ we obtain the following corollary.

Corollary 2.4. *For any fake lens space $L^{2d-1}(\alpha)$ there exists $k \in \mathbb{N}$ and a homotopy equivalence*

$$h \colon L^{2d-1}(\alpha) \longrightarrow L^{2d-1}(\alpha_k).$$

3. The surgery exact sequence

We proceed to the homeomorphism classification within a simple homotopy type. This is the standard task of surgery theory whose main tool is the surgery exact sequence computing the structure set $\mathcal{S}^s(X)$ for a given n-manifold X with $n \geq 5$:

$$\cdots \to \mathcal{N}_\partial(X \times I) \xrightarrow{\theta} L^s_{n+1}(G) \xrightarrow{\partial} \mathcal{S}^s(X) \xrightarrow{\eta} \mathcal{N}(X) \xrightarrow{\theta} L^s_n(G), \tag{3.1}$$

where $G = \pi_1(X)$. The other terms in the sequence are reviewed below. We note that, since $\mathcal{S}^s(X)$ is a priori only a pointed set, the 'exactness' is to be understood as described in [Wal99, chapter 10] or [Lüc02, chapter 5]. However, the sequence can also be made into an exact sequence of abelian groups by the identification with the algebraic surgery exact sequence of Ranicki as explained in [Ran92, chapter 18]. We will make use of this structure since it makes certain statements and proofs easier.

However, our results can be also formulated without this identification, in a less neat way though.

By $\mathcal{N}(X)$ in (3.1) is denoted the set of *normal invariants* of X. An element of $\mathcal{N}(X)$ is represented by a *degree one normal map* $(f,b)\colon M \to X$ which consists of a map $f\colon M \to X$ of oriented closed n-manifolds of degree 1 and a stable bundle map $b\colon \nu_M \to \xi$ from the stable normal bundle of M to a stable topological reduction ξ of the stable Spivak normal fibration of X. Two such degree one normal maps $(f,b)\colon M \to X$, $(f',b')\colon M' \to X$ are equivalent in $\mathcal{N}(X)$ if there exists a degree one normal map $(F,B)\colon (W,M,M') \to (X \times I, X \times 0, X \times 1)$ of manifolds with boundary which restricts on the two ends to (f,b), (f',b') respectively. Again this is a priori a set, with a base point $(\mathrm{id},\mathrm{id})\colon X \to X$. However, the Pontrjagin-Thom construction gives a bijection

$$\mathcal{N}(X) \xrightarrow{\cong} [X; \mathrm{G/TOP}] \tag{3.2}$$

where $[—,—]$ denotes the homotopy classes of maps and G/TOP is the classifying space for topological reductions of spherical fibrations. The H-space structure on G/TOP coming from Sullivan characteristic variety theorem [MM79, chapter 4] (also called 'disjoint union H-space structure' in [Ran08]) makes $\mathcal{N}(X)$ into an abelian group. This H-space structure extends to an infinite loop space structure which expresses $\mathcal{N}(X)$ via localization in terms of familiar cohomology theories.

Theorem 3.1 ([MM79])**.** *There are compatible homotopy equivalences*

$$\begin{aligned} \mathrm{G/TOP}_{(2)} &\simeq \Pi_{i\geq 1} K(\mathbb{Z}_{(2)}, 4i) \times K(\mathbb{Z}_2, 4i-2), \\ \mathrm{G/TOP}_{(\mathrm{odd})} &\simeq \mathrm{BO}_{(\mathrm{odd})}, \\ \mathrm{G/TOP}_{(0)} &\simeq \mathrm{BO}_{(0)} \simeq \Pi_{i\geq 1} K(\mathbb{Q}, 4i). \end{aligned}$$

Corollary 3.2. *If X is rationally trivial we have an isomorphism of abelian groups*

$$\mathcal{N}(X) \cong \bigoplus_{i\geq 1} H^{4i}(X;\mathbb{Z}_{(2)}) \oplus H^{4i-2}(X;\mathbb{Z}_2) \oplus KO(X) \otimes \mathbb{Z}[\frac{1}{2}]$$

Given $n \in \mathbb{Z}$ and G a group there is defined an abelian group $L^s_n(G)$ [Wal99, chapter 5,6]. For $n = 2k$ it is the Witt group of based $(-1)^k$-quadratic forms over the group ring $\mathbb{Z}G$, for $n = 2k+1$ it is a certain group of automorphisms of based $(-1)^k$-quadratic forms over $\mathbb{Z}G$. An alternative description of [Ran92] gives these groups uniformly for all n as cobordism groups of bounded chain complexes of based $\mathbb{Z}G$-modules with an n-dimensional Poincaré duality. The precise definition is not that important for us. We are mainly interested in the invariants which detect these groups for $G \cong \mathbb{Z}_N$.

Theorem 3.3. *For $G = 1$ we have*

$$L^s_n(1) \cong \begin{cases} 8\cdot\mathbb{Z} & n \equiv 0 \pmod 4 \text{ (signature)} \\ 0 & n \equiv 1 \pmod 4 \\ \mathbb{Z}_2 & n \equiv 2 \pmod 4 \text{ (Arf)} \\ 0 & n \equiv 3 \pmod 4 \end{cases}$$

Here 'signature' in the last column means that $L^s_{4k}(1) \cong 8\cdot\mathbb{Z}$ is given by the signature of a quadratic form over $\mathbb{Z}$, and 'Arf' means that $L^s_{4k+2}(1) \cong \mathbb{Z}_2$ is given by the Arf invariant of a quadratic form over $\mathbb{Z}_2$. For $G \neq 1$ functoriality gives maps $L^s_n(1) \to L^s_n(G)$ and $L^s_n(G) \to L^s_n(1)$ yielding the splitting

$$L^s_n(G) \cong L^s_n(1) \oplus \widetilde{L}^s_n(G). \tag{3.3}$$

Further information about the L-groups of finite groups is obtained using representation theory. For a finite group G complex conjugation induces an involution on the complex representation ring $R_{\mathbb{C}}(G)$. One can define (± 1)-eigenspaces denoted $R_{\mathbb{C}}^{(\pm 1)}(G)$. In terms of characters the $(+1)$-eigenspace corresponds to real characters, the (-1)-eigenspace corresponds to purely imaginary characters.

A non-degenerate $(-1)^k$-quadratic form over $\mathbb{Z}G$ can be complexified. One can take its associated non-degenerate $(-1)^k$-symmetric bilinear form and consider the positive and negative definite $\mathbb{C}$-vector subspaces. These become G-representations and hence can be subtracted in $R_{\mathbb{C}}(G)$. This process defines the G-signature homomorphism (see [Wal99, chapter 13] or [Ran92, chapter 22])

$$\text{G-sign}\colon L^s_{2k}(G) \to R_{\mathbb{C}}^{(-1)^k}(G).$$

Its image is $4 \cdot R_{\mathbb{C}}^{(-1)^k}(G)$. In case $G = \mathbb{Z}_N$ for $N = 2^K$ the L-groups are completely calculated (see [HT00]):[2]

Theorem 3.4. *For $G = \mathbb{Z}_N$ we have that*

$$L^s_n(G) \cong \begin{cases} 4 \cdot R_{\mathbb{C}}^{(+1)}(G) & n \equiv 0 \pmod 4 \text{ (G-sign, purely real)} \\ 0 & n \equiv 1 \pmod 4 \\ 4 \cdot R_{\mathbb{C}}^{(-1)}(G) \oplus \mathbb{Z}_2 & n \equiv 2 \pmod 4 \text{ (G-sign, purely imaginary, Arf)} \\ \mathbb{Z}_2 & n \equiv 3 \pmod 4 \text{ (codimension 1 Arf)} \end{cases}$$

$\widetilde{L}^s_{2k}(G) \cong 4 \cdot R_{\widehat{G}}^{(-1)^k}$ *where* $R_{\widehat{G}}^{(-1)^k}$ *is* $R_{\mathbb{C}}^{(-1)^k}(G)$ *modulo the regular representation.*

Next we describe briefly the maps in (3.1). If $n = 2k$ the map θ is given by first making the degree one normal map $(f, b)\colon M \to X$ k-connected and then taking the quadratic refinement of the $(-1)^k$-symmetric bilinear form over $\mathbb{Z}[G]$ on the kernel of $f_*\colon H_k(\widetilde{M}) \to H_k(\widetilde{X})$. The exactness at $\mathcal{N}(X)$ means that there is a degree one normal map $(f', b')\colon M' \to X$ with f' a homotopy equivalence in the normal cobordism class of (f, b) if and only if $\theta(f, b) = 0$.

The map η is given by taking the stable normal bundle ν_M of $f\colon M \xrightarrow{\simeq_s} X$ and associating to it $(f, b)\colon M \to X$ with $b\colon \nu_M \to (f^{-1})^* \nu_M$ induced by f.

To describe ∂ we need the realization theorem for elements of $L^s_n(G)$. It says that if M^{n-1} is a manifold and $x \in L^s_n(G)$ there exists a degree one normal map $(F, B)\colon (W, \partial_0 W, \partial_1 W) \to (M \times I, M \times 0, M \times 1)$, where $I = [0, 1]$, such that $\partial_0 F\colon \partial_0 W \to M \times 0$ is a homeomorphism, $\partial_1 F\colon \partial_1 W \to M \times 1$ is a simple homotopy equivalence and $\theta(F, B) = x$. The 'map' ∂ in fact means that there is an action of $L^s_n(G)$ on $\mathcal{S}^s(X)$ given as follows. Let $f\colon M \to X \in \mathcal{S}^s(X)$ and $x \in L^s_n(G)$, then $\partial(x, f)$ is given by $\partial_1 F_1 \circ f\colon \partial_1 W \to X$ where $(F, B)\colon W \to M \times I$ realizes x. When the abelian group structure of [Ran92, chapter 18] is imposed on $\mathcal{S}^s(X)$ the action ∂ corresponds to the group action of the subgroup generated by the image of ∂ on $\mathcal{S}^s(X)$.

Hence the problem of determining $\mathcal{S}^s(X)$ in general consists of determining firstly $\mathcal{N}(X)$, which is tractable via standard algebraic topology, secondly $L^s_n(G)$ which we know in our case, thirdly determining the maps ∂, η, θ and finally solving an extension problem which is left over.

Remark 3.5. One can also define the *structure set* $\mathcal{S}^h(X)$ of an n-manifold X. Here, in comparison with the definition of $\mathcal{S}^s(X)$, one replaces simple homotopy equivalences by homotopy equivalences and the homeomorphism relation by the h-cobordism relation. There is a version of the sequence (3.1) in this situation and again the theory of [Ran92, chapter 18] makes it into a long exact sequence of abelian groups. The obvious map $\mathcal{S}^s(X) \to \mathcal{S}^h(X)$ is a homomorphism.

[2] The choice of the notation in the last line is explained later in section 4.

3.1. **Complex projective spaces.** We also need the discussion of the classification problem for the complex projective spaces. This is useful also since the discussion is simpler in this case and will give us a simple example of the strategy we will need later.

The complex projective space $\mathbb{C}P^{d-1}$ is defined as the quotient of the diagonal S^1-action on $S^{2d-1} = S^1 * \cdots * S^1$ (d-factors). As a real manifold it has dimension $2d-2$ and $\pi_1(\mathbb{C}P^n) = 1$. Hence from (3.3) we have that the surgery exact sequence for $\mathbb{C}P^{d-1}$ becomes the short exact sequence

$$0 \to \mathcal{S}^s(\mathbb{C}P^{d-1}) \to \mathcal{N}(\mathbb{C}P^{d-1}) \xrightarrow{\theta} L^s_{2d-2}(1) \to 0. \tag{3.4}$$

For the normal invariants we have

$$\mathcal{N}(\mathbb{C}P^{d-1}) \cong \bigoplus_{i=1}^{\lfloor (d-1)/2 \rfloor} H^{4i}(\mathbb{C}P^{d-1};\mathbb{Z}) \oplus \bigoplus_{i=1}^{\lfloor d/2 \rfloor} H^{4i-2}(\mathbb{C}P^{d-1};\mathbb{Z}_2). \tag{3.5}$$

Further we can identify the factors

$$\mathbf{s}_{4i} \colon \mathcal{N}(\mathbb{C}P^{d-1}) \to H^{4i}(\mathbb{C}P^{d-1};\mathbb{Z}) \cong \mathbb{Z} \cong L_{4i}(1) \tag{3.6}$$

$$\mathbf{s}_{4i-2} \colon \mathcal{N}(\mathbb{C}P^{d-1}) \to H^{4i-2}(\mathbb{C}P^{d-1};\mathbb{Z}_2) \cong \mathbb{Z}_2 \cong L_{4i-2}(1) \tag{3.7}$$

as surgery obstructions of degree one normal maps obtained from $(f,b)\colon M \to \mathbb{C}P^{d-1}$ by first making f transverse to $\mathbb{C}P^{k-1}$ (for s_{2i} where $i = k-1$) and then taking the surgery obstruction of the degree one map obtained by restricting to the preimage of $\mathbb{C}P^i$. The maps $\mathbf{s}_{2i}$ are called the *splitting invariants*. We will sometimes use (3.5) to identify the elements of $\mathcal{N}(\mathbb{C}P^{d-1})$ by $s = (s_{2i})_i$.

The surgery obstruction map θ takes the top summand of $\mathcal{N}(\mathbb{C}P^{d-1})$ isomorphically onto $L^s_{2d-2}(1)$. Hence the short exact sequence (3.4) splits and we obtain the bijection of $\mathcal{S}^s(\mathbb{C}P^{d-1})$ given by the splitting invariants $\mathbf{s}_{2i}$ for $0 < i < d-1$:

$$\bigoplus_{0<i<d-1} \mathbf{s}_{2i} \colon \mathcal{S}^s(\mathbb{C}P^{d-1}) \xrightarrow{\cong} \bigoplus_{0<i<d-1} L^s_{2i}(1). \tag{3.8}$$

If we think of $\mathcal{S}^s(\mathbb{C}P^{d-1})$ as of an abelian group via Ranicki's identification [Ran92, chapter 18], then the map (3.8) is an isomorphism.

3.2. **Preliminaries for lens spaces.** When X is a fake lens space $L^{2d-1}(\alpha)$ with $\pi_1(L^{2d-1}(\alpha)) \cong G = \mathbb{Z}_N$ for $N = 2^K$ we obtain some information about the surgery exact sequence for $L^{2d-1}(\alpha)$ from Corollary 3.2 and Theorem 3.4. In more detail

$$\mathcal{N}(L^{2d-1}(\alpha)) \cong \bigoplus_{i=1}^{\lfloor (d-1)/2 \rfloor} H^{4i}(L^{2d-1}(\alpha);\mathbb{Z}) \oplus \bigoplus_{i=1}^{\lfloor d/2 \rfloor} H^{4i-2}(L^{2d-1}(\alpha);\mathbb{Z}_2) \tag{3.9}$$

We denote the factors

$$\mathbf{t}_{4i} \colon \mathcal{N}(L^{2d-1}(\alpha)) \to H^{4i}(L^{2d-1}(\alpha);\mathbb{Z}) \cong \mathbb{Z}_{2^K} \tag{3.10}$$

$$\mathbf{t}_{4i-2} \colon \mathcal{N}(L^{2d-1}(\alpha)) \to H^{4i-2}(L^{2d-1}(\alpha);\mathbb{Z}_2) \cong \mathbb{Z}_2 \tag{3.11}$$

and similarly as above we will sometimes use (3.9) to identify the elements of $\mathcal{N}(L^{2d-1}(\alpha))$ by $t = (t_{2i})_i$. More information is obtained from the following

Theorem 3.6 ([Wal99])**.**

(1) *If $d = 2e$, then the map*
$$\theta \colon \mathcal{N}(L^{2d-1}(\alpha)) \to L^s_{2d-1}(G) = L^s_{4e-1}(G) = \mathbb{Z}_2$$
is given by $\theta(x) = \mathbf{t}_{4e-2}(x) \in \mathbb{Z}_2$.

(2) *The map*
$$\theta \colon \mathcal{N}_\partial(L^{2d-1}(\alpha) \times I) \to L^s_{2d}(G)$$
maps onto the summand $L^s_{2d}(1)$.

Hence we obtain the short exact sequence

$$0 \to \widetilde{L}^s_{2d}(G) \xrightarrow{\partial} \mathcal{S}^s(L^{2d-1}(\alpha)) \xrightarrow{\eta} \widetilde{\mathcal{N}}(L^{2d-1}(\alpha)) \to 0 \tag{3.12}$$

where

$$\widetilde{\mathcal{N}}(L^{4e-1}(\alpha)) = \ker \big(\mathbf{t}_{4e-2} \colon \mathcal{N}(L^{4e-1}(\alpha)) \to H^{4e-2}(L^{4e-1}(\alpha);\mathbb{Z}_2) \cong \mathbb{Z}_2\big),$$
$$\widetilde{\mathcal{N}}(L^{4e+1}(\alpha)) = \mathcal{N}(L^{4e+1}(\alpha)).$$

in other words

$$\widetilde{\mathcal{N}}(L^{2d-1}(\alpha)) \cong \bigoplus_{i=1}^{c} \mathbb{Z}_N \oplus \bigoplus_{i=1}^{c} \mathbb{Z}_2 \tag{3.13}$$

where $c = \lfloor (d-1)/2 \rfloor$. The first term in the sequence (3.12) is understood by Theorem 3.4, the third term is understood by (3.13). Hence we are left with an extension problem.

3.3. **The Join Construction.**

We will make use of the join construction from [Wal99, chapter 14A]. It can be explained as follows. Let G be a group (in our case $G \leq S^1$) acting freely on the spheres S^m and S^n. Then the two actions extend to the join $S^{m+n+1} \cong S^m * S^n$ and the resulting action remains free. When we are given two lens spaces (complex projective spaces) L and L', we can pass to universal covers (S^1-bundles), form the join and then pass to the quotient again. The resulting space is again a fake lens space (a fake complex projective space). This operation will be denoted $L * L'$ and it will be called the *join*. When $L' = L^1(\alpha_1)$ we call this operation a *suspension*.

The join with $L^1(\alpha_k)$ defines a map $\Sigma_k \colon \mathcal{S}^s(L^{2d-1}(\alpha_1)) \longrightarrow \mathcal{S}^s(L^{2d+1}(\alpha_k))$. The inclusion $L^{2d-1}(\alpha_1) \subset L^{2d+1}(\alpha_k)$ induces a restriction map on the normal invariants $\mathrm{res} \colon \mathcal{N}(L^{2d+1}(\alpha_k)) \longrightarrow \mathcal{N}(L^{2d-1}(\alpha_1))$ and we have a commutative diagram [Wal99, Lemma 14A.3]:

$$\begin{CD} \mathcal{S}^s(L^{2d-1}(\alpha_1)) @>{\eta}>> \mathcal{N}(L^{2d-1}(\alpha_1)) \\ @V{\Sigma_k}VV @AA{\mathrm{res}}A \\ \mathcal{S}^s(L^{2d+1}(\alpha_k)) @>{\eta}>> \mathcal{N}(L^{2d+1}(\alpha_k)) \end{CD} \tag{3.14}$$

Note that we have $t_{2i} = \mathrm{res}(t_{2i})$. Hence the map

$$\mathrm{res} \colon \widetilde{\mathcal{N}}(L^{2d+1}(\alpha_1)) \longrightarrow \widetilde{\mathcal{N}}(L^{2d-1}(\alpha_1)) \tag{3.15}$$

is an isomorphism when $d = 2e+1$ and it is onto when $d = 2e$ with the kernel equal to $\mathbb{Z}_N(t_{4e})$. A similar diagram exists for the situation $\mathbb{C}P^d = \mathbb{C}P^{d-1} * \mathrm{pt}$.

The map Σ_k is a homomorphism when the structure sets are equipped with the abelian groups structure from [Ran92, chapter 18]. To see this notice that

$$L^{2d+1}(\alpha_k) = E(\nu) \cup_{S(\nu)} C$$

where $E(\nu)$ is the total space of the normal disk-bundle of $L^{2d-1}(\alpha_1)$ in $L^{2d+1}(\alpha_k)$, $S(\nu)$ is the associated sphere-bundle and C is the complement (it is the total space of a disk-bundle over $L^1(\alpha_k)$). Then there is a commutative diagram

$$\begin{array}{ccc} \mathcal{S}^s(L^{2d-1}(\alpha_1)) & & \\ \Big\downarrow {\scriptstyle \Sigma_k} & \searrow^{\nu^!} & \\ \mathcal{S}^s(L^{2d+1}(\alpha_k)) & \xrightarrow[\cong]{} & \mathcal{S}^s(E(\nu), S(\nu)) \end{array}$$

The map in the bottom row is obtained using [Wal99, Theorem 12.1]. It follows from the calculation $\mathcal{S}^s_\partial(C) = 0$ that it is an isomorphism. The map $\nu^!$ is the transfer map obtained via pullback. This coincides with the algebraic surgery transfer map from [Ran92, chapter 21].[3]

4. The ρ-invariant

We review the definition of the ρ-invariant for odd-dimensional manifolds and some of its properties from [AS68] and [Wal99]. It will provide us with a map from the short exact sequence (3.12) to a certain short exact sequence coming from representation theory of G. Studying this map will enable us to solve the extension problem we are left with in the next section.

4.1. **Definitions.**

Let G be a compact Lie group acting smoothly on a smooth manifold Y^{2d}. The middle intersection form becomes a non-degenerate $(-1)^d$-symmetric bilinear form on which G acts. As explained earlier, such a form yields an element in the representation ring $R(G)$ which we denote by G-sign(Y). The discussion in section 3 also tells us that we have G-sign$(Y) \in R^{(-1)^d}(G)$ which in terms of characters means that we obtain a real (purely imaginary) character, which will be denoted as G-sign$(-, Y)\colon g \in G \mapsto$ G-sign$(g, Y) \in \mathbb{C}$. The (cohomological version of the) Atiyah-Singer G-index theorem [AS68, Theorem (6.12)] tells us that if Y is closed then for all $g \in G$

$$\text{G-sign}(g, Y) = L(g, Y) \in \mathbb{C}, \tag{4.1}$$

where $L(g, Y)$ is an expression obtained by evaluating certain cohomological classes on the fundamental classes of the g-fixed point submanifolds Y^g of Y. In particular if the action is free then G-sign$(g, Y) = 0$ if $g \neq 1$. This means that G-sign(Y) is a multiple of the regular representation. This theorem was generalized by Wall to topological semifree actions on topological manifolds, which is the case we will need in this paper [Wal99, chapter 14B]. The assumption that Y is closed is essential here, and motivates the definition of the ρ-invariant. In fact, Atiyah and Singer provide two definitions. For the first one one also needs the result of Conner and Floyd [CF64] that for an odd-dimensional manifold X with a finite fundamental group there always exists a $k \in \mathbb{N}$ and a manifold with boundary $(Y, \partial Y)$ such that $\pi_1(Y) \cong \pi_1(X)$ and $\partial Y = k \cdot X$.

Definition 4.1. [AS68, Remark after Corollary 7.5] Let X^{2d-1} be a closed manifold with $\pi_1(X) \cong G$ a finite group. Define

$$\rho(X) = \frac{1}{k} \cdot \text{G-sign}(\widetilde{Y}) \in \mathbb{Q}R^{(-1)^d}(G)/\langle \text{reg} \rangle \tag{4.2}$$

for some $k \in \mathbb{N}$ and $(Y, \partial Y)$ such that $\pi_1(Y) \cong \pi_1(X)$ and $\partial Y = k \cdot X$. The symbol $\langle \text{reg} \rangle$ denotes the ideal generated by the regular representation.

By the Atiyah-Singer G-index theorem [AS68, Theorem (6.12)] is ρ well defined.

Definition 4.2. Let G be a compact Lie group acting freely on a manifold $\widetilde{X}^{2d-1}$. Suppose in addition that there is a manifold with boundary $(Y, \partial Y)$ on which G acts (not necessarily freely) and such that $\partial Y = \widetilde{X}$. Define

$$\rho_G(\widetilde{X})\colon g \in G \mapsto \text{G-sign}(g, Y) - L(g, Y) \in \mathbb{C}.$$

[3] We thank A. Ranicki for informing us about the last claim.

In this definition we think about the ρ-invariant as about a function $G \smallsetminus \{1\} \to \mathbb{C}$. When both definitions apply (that means when G is a finite group), then they coincide, that means $\rho(X) = \rho_G(\widetilde{X})$.

For finite $G < S^1$ we will use special notation following [Wal99, Proof of Proposition 14E.6 on page 222]. By $\widehat{G}$ is denoted the Pontrjagin dual of G, the group $\mathrm{Hom}_{\mathbb{Z}}(G, S^1)$. Recall that for a finite cyclic G the representation ring $R(G)$ can be canonically identified with the group ring $\mathbb{Z}\widehat{G}$. Then we also have $\mathbb{Q}R(G) = \mathbb{Q} \otimes R(G) = \mathbb{Q}\widehat{G}$. Dividing out the regular representation corresponds to dividing out the norm element, denoted by Z, hence $R(G)/\langle \mathrm{reg} \rangle = R_{\widehat{G}} = \mathbb{Z}\widehat{G}/\langle Z \rangle$ and $\mathbb{Q}R(G)/\langle \mathrm{reg} \rangle = \mathbb{Q}R_{\widehat{G}} = \mathbb{Q}\widehat{G}/\langle Z \rangle$. Choosing a generator $\widehat{G} = \langle \chi \rangle$ gives the identifications $\mathbb{Q}R_{\widehat{G}} = \mathbb{Q}[\chi]/\langle 1 + \chi + \cdots + \chi^{N-1} \rangle$ where N is the order of G. In order to save space we also use the following notation $I\langle K \rangle = \langle 1 + \chi + \cdots + \chi^{N-1} \rangle$.

4.2. **Properties.**

The ρ-invariant is an h-cobordism invariant [AS68, Corollary 7.5]. For X^{2d-1} with $\pi_1(X) \cong G$ it defines a function of $\mathcal{S}^s(X)$ by sending $a = [h \colon M \longrightarrow X]$ to $\widetilde{\rho}(a) = \rho(M) - \rho(X)$. If we put on $\mathcal{S}^s(X)$ the abelian group structure from [Ran92, chapter 18] it is not clear whether $\widetilde{\rho}$ is a homomorphism in general.[4] Still the following property holds always.

Proposition 4.3. *For X^{2d-1} with $\pi_1(X) \cong G$ there is a commutative diagram*

$$\begin{array}{ccc} L^s_{2d}(G) & \xrightarrow{\quad \partial \quad} & \mathcal{S}^s(X) \\ \big\downarrow \text{G-sign} & & \big\downarrow \widetilde{\rho} \\ 4 \cdot R^{(-1)^d}_{\mathbb{C}}(G) & \longrightarrow & \mathbb{Q}R^{(-1)^d}_{\mathbb{C}}(G)/\langle \mathrm{reg} \rangle. \end{array}$$

Moreover, for $z \in L^s_{2d}(G)$ and $x \in \mathcal{S}^s(X)$ we have

$$\widetilde{\rho}(x + \partial z) = \widetilde{\rho}(x) + \widetilde{\rho}(\partial z).$$

Proof. See [Pet70, Theorem 2.3]. It essentially follows from definitions. We also use the identification of the geometrically given action of $L^s_{2d}(G)$ on $\mathcal{S}^s(X)$ with the action coming from the abelian group structure on $\mathcal{S}^s(X)$ of [Ran92, chapter 18]. ☐

Remark 4.4. The map $\widetilde{\rho}$ also obviously factors through the map $\mathcal{S}^s(X) \to \mathcal{S}^h(X)$ of Remark 3.5.

When $X = L^{2d-1}(\alpha_k)$, it follows from the above diagram, the exactness of the surgery exact sequence, the Atiyah-Singer G-index theorem and the calculation of the groups $L^s_{2d}(G)$ that the action of $\widetilde{L}^s_{2k}(G)$ on $\mathcal{S}^s(L^{2k-1})$ is free. In fact we have

Proposition 4.5. *There is the following commutative diagram of abelian groups and homomorphisms with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{L}^s_{2d}(G) & \xrightarrow{\partial} & \mathcal{S}^s(L^{2d-1}(\alpha)) & \xrightarrow{\eta} & \widetilde{\mathcal{N}}(L^{2d-1}(\alpha)) & \longrightarrow & 0 \\ & & \cong \big\downarrow \text{G-sign} & & \big\downarrow \widetilde{\rho} & & \big\downarrow [\widetilde{\rho}] & & \\ 0 & \longrightarrow & 4 \cdot R^{(-1)^d}_{\widehat{G}} & \longrightarrow & \mathbb{Q}R^{(-1)^d}_{\widehat{G}} & \longrightarrow & \mathbb{Q}R^{(-1)^d}_{\widehat{G}}/4 \cdot R^{(-1)^d}_{\widehat{G}} & \longrightarrow & 0 \end{array}$$

where $[\widetilde{\rho}]$ is the homomorphism induced by $\widetilde{\rho}$.

[4] We will see that it is a homomorphism when $X = L^{2d-1}(\alpha)$ below.

All the statements follow from the previous discussion except the claim that $\widetilde{\rho}$ and $[\widetilde{\rho}]$ are homomorphisms. This will be proved in this section, first for α_1, then for α_k, and finally for general α. To this end we need some way to calculate the ρ-invariant for fake lens spaces. The formulas we obtain will give us first a good understanding of the map $[\widetilde{\rho}]$. Using certain naturality properties we will obtain also the claim about $\widetilde{\rho}$.

Recall the join $L * L'$ of the lens spaces L and L' from section 3.3. We have [Wal99, chapter 14A]

$$\rho(L * L') = \rho(L) \cdot \rho(L'). \tag{4.3}$$

For $L^1(\alpha_k)$ we have [Wal99, Proof of Theorem 14C.4]

$$\rho(L^1(\alpha_k)) = f_k \in \mathbb{Q}R^-_{\widehat{G}} \tag{4.4}$$

where f_k is defined as follows.

Definition 4.6. For odd $k \in \mathbb{N}$ we set

$$f_k := \frac{1+\chi^k}{1-\chi^k} \qquad \text{and} \qquad f'_k := \frac{1-\chi+\chi^2-\cdots+\chi^{k-2}-\chi^{k-1}}{1+\chi+\chi^2+\cdots+\chi^{k-2}+\chi^{k-1}}.$$

We abbreviate $f := f_1$.

Lemma 4.7. *Let $G = \mathbb{Z}_N$ with $N = 2^K$. For odd $k \in \mathbb{N}$ we have*

$$f_k \in \mathbb{Q}R^-_{\widehat{G}}, \qquad f_k = f \cdot f'_k, \qquad f'_k \in R_{\widehat{G}}.$$

Proof. Notice that $1-\chi^k$ is invertible in $\mathbb{Q}R_{\widehat{G}}$ because

$$(1-\chi^k)^{-1} = -\frac{1}{N}(1 + 2\cdot\chi^k + 3\cdot\chi^{2k} + \cdots + N\cdot\chi^{(N-1)k}) \in \mathbb{Q}R_{\widehat{G}}.$$

Therefore $f_k \in \mathbb{Q}R_{\widehat{G}}$ and the identity

$$\frac{1+\chi^{-k}}{1-\chi^{-k}} = -\frac{1+\chi^k}{1-\chi^k} = -f_k$$

implies $f_k \in \mathbb{Q}R^-_{\widehat{G}}$. An easy calculation shows $f_k = f \cdot f'_k$. That $f'_k \in R_{\widehat{G}}$ follows from the fact that $1+\chi+\chi^2+\cdots+\chi^{k-1}$ is invertible in $R_{\widehat{G}}$. The inverse is given by $1+\chi^k+\chi^{2k}+\cdots+\chi^{(r-1)k}$ where r denotes a natural number such that $r \cdot k$ is a multiple of $N = 2^K$. □

Also a formula of Wall which calculates the ρ-invariant for fake complex projective spaces will be useful. Let $a = [h\colon Q \longrightarrow \mathbb{C}P^{d-1}]$ be an element of $\mathcal{S}^s(\mathbb{C}P^{d-1})$ and let $\widetilde{h}\colon \widetilde{Q} \longrightarrow S^{2d-1}$ be the associated map of S^1-manifolds. Denote $\widetilde{\rho}_{S^1}(a) := \widetilde{\rho}_{S^1}(\widetilde{Q}) - \widetilde{\rho}_{S^1}(S^{2d-1})$ defining a function of $\mathcal{S}^s(\mathbb{C}P^{d-1})$.

Theorem 4.8. [Wal99, Theorem 14C.4] *Let $a = [h\colon Q \to \mathbb{C}P^{d-1}]$ be an element in $\mathcal{S}^s(\mathbb{C}P^{d-1})$. Then for $t \in S^1$*

$$\widetilde{\rho}_{S^1}(t,a) = \sum_{1 \le i \le \lfloor d/2 \rfloor - 1} 8 \cdot \mathbf{s}_{4i}(\eta(a)) \cdot (f^{d-2i} - f^{d-2i-2}) \in \mathbb{C},$$

where $f = (1+t)/(1-t)$.

Among other things this also shows that $\widetilde{\rho}_{S^1}$ is a homomorphism of $\mathcal{S}^s(\mathbb{C}P^{d-1})$. Coming back to lens spaces recall that there is an S^1-bundle (better $L^1(\alpha_1)$-bundle) $p\colon L^{2d-1}(\alpha_1) \longrightarrow \mathbb{C}P^{d-1}$. Via pullback it induces a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{S}^s(\mathbb{C}P^{d-1}) & \xrightarrow{\ \eta\ } & \mathcal{N}(\mathbb{C}P^{d-1}) & \longrightarrow & L_{2(d-1)}(1) \\
 & & \downarrow{\scriptstyle p^!} & & \downarrow{\scriptstyle p^!} & & \\
\widetilde{L}^s_{2d}(G) & \longrightarrow & \mathcal{S}^s(L^{2d-1}(\alpha_1)) & \xrightarrow{\ \eta\ } & \widetilde{\mathcal{N}}(L^{2d-1}(\alpha_1)) & \longrightarrow & 0
\end{array} \tag{4.5}$$

With the abelian group structure of [Ran92, chapter 18] the maps $p^!$ are homomorphisms by the identification of geometric and algebraic transfers. Another way of thinking about $p^!$ is that it is given by passing to the subgroup $G < S^1$. Since the ρ-invariant is natural for passing to subgroups we obtain

Corollary 4.9. [Wal99, Theorem 14E.8] *Let $a \in \mathcal{S}^s(L^{2d-1}(\alpha_1))$ such that $a = p^!(b)$ for some $b \in \mathcal{S}^s(\mathbb{C}P^{d-1})$. Then*

$$\widetilde{\rho}(a) = \sum_{1 \le i \le \lfloor d/2 \rfloor - 1} 8 \cdot \mathbf{s}_{4i}(\eta(b)) \cdot (f^{d-2i} - f^{d-2i-2}) \in \mathbb{Q}R_{\widehat{G}}^{(-1)^d},$$

where $f = (1+\chi)/(1-\chi)$.

For the map $p^! \colon \mathcal{N}(\mathbb{C}P^{d-1}) \longrightarrow \widetilde{\mathcal{N}}(L^{2d-1}(\alpha_1))$ we have

$$p^!(s_{4i-2}) = t_{4i-2} \quad p^!(s_{4i}) = t_{4i} \tag{4.6}$$

and hence it is surjective. If the map $\mathcal{S}^s(\mathbb{C}P^{d-1}) \to \mathcal{N}(\mathbb{C}P^{d-1}) \to \mathcal{N}(L^{2d-1}(\alpha_1))$ were surjective we could use Corollary 4.9 to give a formula for the function $[\widetilde{\rho}]$. This is the case when $d = 2e$. In the case $d = 2e+1$ all the summands but the $\mathbb{Z}_N(t_{4e})$ from $\mathcal{N}(L^{2d-1}(\alpha_1))$ are hit from $\mathcal{S}(\mathbb{C}P^{d-1})$. We need the following

Lemma 4.10. *Let $d = 2e+1$ and let $a \in \mathcal{S}(L^{2d-1}(\alpha_1))$ be such that $a \mapsto \mathbf{t}(\eta(a)) = (0, \ldots, 1) \in \mathcal{N}(L^{2d-1}(\alpha_1))$, i.e. $\mathbf{t}(\eta(a))_{4i} = 0$ for $i \le e-1$ and $\mathbf{t}(\eta(a))_{4e} = 1$. Then*

$$\widetilde{\rho}(a) = 8f + z \quad \in \quad \mathbb{Q}R_{\widehat{G}}^-$$

for some $z \in 4 \cdot R_{\widehat{G}}^-$.

Proof. We will use the suspension map Σ_1 from section 3.3. Our assumptions mean that $\mathbf{t}(\eta(a))$ is not in the image of the composition $\mathcal{S}(\mathbb{C}P^{d-1}) \to \mathcal{N}(\mathbb{C}P^{d-1}) \to \mathcal{N}(L^{2d-1}(\alpha_1))$. However, diagram (3.14) tells us that $\mathbf{t}(\eta(\Sigma_1(a)))$ is in the image of $\mathcal{S}(\mathbb{C}P^{(d+1)-1}) \to \mathcal{N}(\mathbb{C}P^{(d+1)-1}) \to \mathcal{N}(L^{2(d+1)-1}(\alpha_1))$ and hence we have

$$f \cdot \widetilde{\rho}(a) + y = 8 \cdot 1 \cdot (f^2 - 1) \quad \in \quad \mathbb{Q}R_{\widehat{G}}^+$$

for some $y \in 4 \cdot R_{\widehat{G}}^+$. We obtain the desired identity by the following calculation. Let $\widehat{\rho} \in \mathbb{Q}[\chi]$ and $\widehat{y} \in 4 \cdot \mathbb{Z}[\chi]$ be representatives for $\widetilde{\rho}(a)$ and y. Then

$$(1+\chi)(1-\chi)\widehat{\rho} + (1-\chi)^2\widehat{y} \equiv 8 \cdot (4\chi) \qquad \text{mod } I\langle K\rangle$$

$$(1+\chi)(1-\chi)\widehat{\rho} + (1-\chi)^2(\widehat{y}+8) \equiv 8 \cdot (1+\chi)^2 \qquad \text{mod } I\langle K\rangle$$

$$(1+\chi)(1-\chi)\widehat{\rho} + (1-\chi)^2(\widehat{y}+8) = 8 \cdot (1+\chi)^2 + g(\chi)(1+\chi+\cdots+\chi^{N-1}) \in \mathbb{Q}[\chi]$$

for some $g(\chi) \in \mathbb{Q}[\chi]$. Hence $\widehat{y} + 8 = (1+\chi)w(\chi)$ for some $w(\chi) \in \mathbb{Q}[\chi]$. Since $(\widehat{y}+8) \in 4 \cdot \mathbb{Z}[\chi]$, we obtain $w(\chi) \in 4 \cdot \mathbb{Z}[\chi]$. Further write $g(\chi) = 2r + (1+\chi)g'(\chi) = r(1-\chi) + (1+\chi)(r + g'(\chi))$ for $r \in \mathbb{Q}$, $g'(\chi) \in \mathbb{Q}[\chi]$. We have

$$(1-\chi)\widehat{\rho} + (1-\chi)^2 w(\chi) = 8 \cdot (1+\chi) + g(\chi)(1+\chi^2+\cdots\chi^{N-2}) \in \mathbb{Q}[\chi]$$

and further modulo $I\langle K\rangle$

$$(1-\chi)\widehat{\rho} + (1-\chi)^2 w(\chi) \equiv 8 \cdot (1+\chi) + r(1-\chi)(1+\chi^2+\cdots\chi^{N-2})$$

$$\widehat{\rho} + (1-\chi)w(\chi) \equiv 8 \cdot f + r(1+\chi^2+\cdots\chi^{N-2})$$

Now $(1-\chi)w(\chi) = (2-(1+\chi))w(\chi) = 2w(\chi) - (\widehat{y}+8)$. Further $2w(\chi) = w^+(\chi) + w^-(\chi)$, where $w^\pm(\chi) \in 4 \cdot R_{\widehat{G}}^{(\pm 1)}$. Hence

$$\widetilde{\rho}(a) - 8 \cdot f + w^-(\chi) = (\widehat{y}+8) - w^+(\chi) + r(1+\chi^2+\cdots+\chi^{N-2})$$

in $\mathbb{Q}[\chi]/I\langle K\rangle$, while the left hand side of the equation lies in the (-1)-eigenspace and the right-hand side lies in the $(+1)$-eigenspace and hence both are equal to 0. It follows that

$$\widetilde{\rho}(a) = 8 \cdot f - w^{-}(\chi).$$

Putting $z = -w^{-}(\chi)$ yields is the desired formula. □

Lemma 4.11. *Let $d = 2e+1$ and let $a \in \mathcal{S}(L^{2d-1}(\alpha_1))$. Then*

$$\widetilde{\rho}(a) = 8 \cdot \mathbf{t}_{4e}(\eta(a)) \cdot f + \sum_{1 \leq i \leq \lfloor d/2 \rfloor - 1} 8 \cdot \mathbf{t}_{4i}(\eta(a)) \cdot (f^{d-2i} - f^{d-2i-2}) + z \quad \in \quad \mathbb{Q}R^{-}_{\widehat{G}}$$

for some $z \in 4 \cdot R^{-}_{\widehat{G}}$.

Proof. Proof is by a straightforward modification of the proof of Lemma 4.10. □

Proposition 4.12. *For the map $[\widetilde{\rho}] \colon \widetilde{\mathcal{N}}(L^{2d-1}(\alpha_1)) \longrightarrow \mathbb{Q}R^{(-1)^d}_{\widehat{G}}/4 \cdot R^{(-1)^d}_{\widehat{G}}$ and an element $t = (t_{2i})_i \in \widetilde{\mathcal{N}}(L^{2d-1}(\alpha_1))$ we have that*

$$d = 2e \; : \; [\widetilde{\rho}](t) = \sum_{i=1}^{e-1} 8 \cdot t_{4i} \cdot f^{d-2i-2} \cdot (f^2 - 1)$$

$$d = 2e+1 \; : \; [\widetilde{\rho}](t) = \sum_{i=1}^{e-1} 8 \cdot t_{4i} \cdot f^{d-2i-2} \cdot (f^2 - 1) + 8 \cdot t_{4e} \cdot f.$$

Proof. It is enough to find for each $t \in \widetilde{\mathcal{N}}(L^{2d-1}(\alpha_1))$ some $a \in \mathcal{S}^s(L^{2d-1}(\alpha_1))$ with $\mathbf{t}(\eta(a)) = t$ and for which we can calculate $\widetilde{\rho}(a) \in \mathbb{Q}R^{(-1)^d}_{\widehat{G}}$. If $d = 2e$ then by discussion after Corollary 4.9 there is for each normal cobordism class a fake lens space which fibers over a fake complex projective space and hence the formula from Corollary 4.9 gives the desired formula. If $d = 2e+1$ then the same reasoning applied to Lemma 4.11 gives the desired formula. □

Corollary 4.13. *The function $\widetilde{\rho} \colon \mathcal{S}^s(L^{2d-1}(\alpha_1)) \longrightarrow \mathbb{Q}R^{(-1)^d}_{\widehat{G}}$ is a homomorphism.*

Proof. It is enough to show that for every $t,\, t' \in \widetilde{\mathcal{N}}(L^{2d-1}(\alpha_1))$ there exist elements (not necessarily unique) $a,\, a'$ in $\mathcal{S}^s(L^{2d-1}(\alpha_1))$ such that $\mathbf{t}(\eta(a)) = t$, $\mathbf{t}(\eta(a')) = t'$ and $\widetilde{\rho}(a+a') = \widetilde{\rho}(a)+\widetilde{\rho}(a')$. If this holds, then for any $x, x' \in \mathcal{S}^s(L^{2d-1}(\alpha_1))$ choose a and a' as above corresponding to the classes $\mathbf{t}(\eta(x))$, $\mathbf{t}(\eta(y)) \in \mathcal{N}(L^{2d-1}(\alpha_1))$. Then $x = a + \partial(b)$ and $x' = a' + \partial(b')$ for some $b,\, b' \in \partial\widetilde{L}^s_{2d}(G)$ and

$$\begin{aligned}\widetilde{\rho}(x + x') =& \widetilde{\rho}(a + \partial b + a' + \partial b') = \widetilde{\rho}(a + a') + \widetilde{\rho}(\partial b + \partial b') \\ =& \widetilde{\rho}(a) + \widetilde{\rho}(a') + \widetilde{\rho}(\partial b) + \widetilde{\rho}(\partial b') = \widetilde{\rho}(x) + \widetilde{\rho}(x').\end{aligned}$$

When $d = 2e$ we can associate to a given $t \in \mathcal{N}(L^{2d-1}(\alpha_1))$ an $a \in \mathcal{S}^s(L^{2d-1}(\alpha_1))$ coming from the $\mathcal{S}(\mathbb{C}P^{d-1})$, i.e. $a = p^!(b)$ where $b \in \mathcal{S}(\mathbb{C}P^{d-1})$ such that $p^!(\eta(b)) = t$. When we have $t,\, t' \in \mathcal{N}(L^{2d-1}(\alpha_1))$, then $\widetilde{\rho}(a + a') = \widetilde{\rho}(p^!(b) + p^!(b')) = \widetilde{\rho}(p^!(b+b')) = \mathrm{res}(\widetilde{\rho}_{S^1}(b+b')) = \mathrm{res}(\widetilde{\rho}_{S^1}(b)+\widetilde{\rho}_{S^1}(b')) = \mathrm{res}(\widetilde{\rho}_{S^1}(b))+\mathrm{res}(\widetilde{\rho}_{S^1}(b')) = \widetilde{\rho}(a) + \widetilde{\rho}(a')$. Here res denotes the map on the representation rings induced by the inclusion $G < S^1$.

When $d = 2e+1$ and $t \in \mathcal{N}(L^{2d-1}(\alpha_1))$ we can do the same unless $t_{4e} \neq 0$. In that case there is no fake lens space in the normal cobordism class of $\mathbf{t}$ which fibers over a fake complex projective space and we have to use a different argument. It follows from the formula in Proposition 4.12 that for $a, a' \in \mathcal{S}^s(L^{2d-1}(\alpha_1))$ we have $\widetilde{\rho}(a + a') = \widetilde{\rho}(a) + \widetilde{\rho}(a') + z$ for some $z \in 4 \cdot R^{-}_{\widehat{G}}$. If $a,\, a'$ are in the same normal cobordism class then $z = 0$. Our task is to show this for any choice of $a,\, a'$. We use the fact that Σ is a homomorphism and that we have already proved the claim for $d = 2e+2$. That implies $\widetilde{\rho}(\Sigma(a+a')) = \widetilde{\rho}(\Sigma a + \Sigma a')) = \widetilde{\rho}(\Sigma a) + \widetilde{\rho}(\Sigma a') = f \cdot \widetilde{\rho}(a) + f \cdot \widetilde{\rho}(a')$. On the other hand $\widetilde{\rho}(\Sigma(a+a')) = f \cdot \widetilde{\rho}(a+a') = f \cdot \widetilde{\rho}(a) + f \cdot \widetilde{\rho}(a') + f \cdot z$. Hence it is enough to show that for any $z \in 4 \cdot R^-_{\widehat{G}}$ such that $f \cdot z = 0$ in $\mathbb{Q}R^+_{\widehat{G}}$ we have $z = 0$ in $4 \cdot R^-_{\widehat{G}}$. This is proved below in Lemma 5.6. □

Now we proceed to the case of α_k where $k \in \mathbb{N}$ is odd.

Proposition 4.14. *For the map* $[\widetilde{\rho}] \colon \widetilde{\mathcal{N}}(L^{2d-1}(\alpha_k)) \longrightarrow \mathbb{Q}R^{(-1)^d}_{\widehat{G}}/4 \cdot R^{(-1)^d}_{\widehat{G}}$ *an element* $t = (t_{2i})_i \in \widetilde{\mathcal{N}}(L^{2d-1}(\alpha_k))$ *we have that*

$$d = 2e \ : \ [\widetilde{\rho}](t) = \sum_{i=1}^{e-1} 8 \cdot t_{4i} \cdot f'_k \cdot f^{d-2i-2} \cdot (f^2 - 1)$$

$$d = 2e+1 \ : \ [\widetilde{\rho}](t) = \sum_{i=1}^{e-1} 8 \cdot t_{4i} \cdot f'_k \cdot f^{d-2i-2} \cdot (f^2 - 1) + 8 \cdot t_{4e} \cdot f'_k \cdot f.$$

Proof. We will use the calculation for α_1 and the homeomorphisms

$$L^{2d+1}(\alpha_k) \cong L^{2d-1}(\alpha_1) * L^1(\alpha_k) \quad \text{and} \quad L^{2d+1}(\alpha_k) \cong L^{2d-1}(\alpha_k) * L^1(\alpha_1)$$

For $d = 2e$ recall the diagram

$$\begin{array}{ccccc} \mathbb{Q}R^-_{\widehat{G}} & \xleftarrow{\widetilde{\rho}} & \mathcal{S}^s(L^{4e-3}(\alpha_1)) & \xrightarrow{\eta} & \mathcal{N}(L^{4e-3}(\alpha_1)) \\ \Big\downarrow \cdot f_k & & \Big\downarrow \Sigma_k & & \cong \Big\uparrow \text{res} \\ \mathbb{Q}R^+_{\widehat{G}} & \xleftarrow{\widetilde{\rho}} & \mathcal{S}^s(L^{4e-1}(\alpha_k)) & \xrightarrow{\eta} & \mathcal{N}(L^{4e-1}(\alpha_k)) \end{array} \tag{4.7}$$

Let $t \in \mathcal{N}(L^{4e-1}(\alpha_k))$. Choose $x \in \mathcal{S}^s(L^{4e-3}(\alpha_1))$ such that $\mathbf{t}(\eta(x)) = t = \text{res}(t)$. Then we have $\mathbf{t}(\eta(\Sigma_k x)) = t$ and $[\widetilde{\rho}](\eta(\Sigma_k x)) = [\widetilde{\rho}(x) \cdot f_k]$ can be calculated using the formulas from the case $k = 1$.

For $d = 2e+1$ recall the diagram

$$\begin{array}{ccccc} \mathbb{Q}R^-_{\widehat{G}} & \xleftarrow{\widetilde{\rho}} & \mathcal{S}^s(L^{4e+1}(\alpha_k)) & \xrightarrow{\eta} & \mathcal{N}(L^{4e+1}(\alpha_k)) \\ \Big\downarrow \cdot f & & \Big\downarrow \Sigma_1 & & \cong \Big\uparrow \text{res} \\ \mathbb{Q}R^+_{\widehat{G}} & \xleftarrow{\widetilde{\rho}} & \mathcal{S}^s(L^{4e+3}(\alpha_k)) & \xrightarrow{\eta} & \mathcal{N}(L^{4e+3}(\alpha_k)) \end{array} \tag{4.8}$$

Let $t \in \mathcal{N}(L^{4e+1}(\alpha_k))$. Choose $x \in \mathcal{S}^s(L^{4e+1}(\alpha_1))$ such that $\mathbf{t}(\eta(x)) = t$. Then we have $\mathbf{t}(\eta(\Sigma_1 x)) = t$ and $\widetilde{\rho}(\Sigma_1 x) = \widetilde{\rho}(x) \cdot f$. We obtain the equation

$$f \cdot \widetilde{\rho}(x) + y = \sum_{i=1}^{e-1} 8 \cdot t_{4i} \cdot f'_k \cdot f^{d+1-2i-2} \cdot (f^2 - 1) + 8 \cdot t_{4e} \cdot f'_k \cdot (f^2 - 1) \in \mathbb{Q}R^+_{\widehat{G}}$$

for some $y \in 4 \cdot R^+_{\widehat{G}}$ using the formulas from the case $d = 2e+2$ which we have already dealt with. Now a modification of the argument from the proof of Lemma 4.10 can be used to obtain the formula for $[\widetilde{\rho}](\eta(x))$. □

Corollary 4.15. *The function* $\widetilde{\rho} \colon \mathcal{S}^s(L^{2d-1}(\alpha_k)) \longrightarrow \mathbb{Q}R^{(-1)^d}_{\widehat{G}}$ *is a homomorphism.*

Proof. Just as in the case α_1 we have to show that the addition of elements in different normal cobordism classes works. For this it is enough to find suitable representatives. In the case $d = 2e$ we can choose in each normal cobordism class an element coming from $\mathcal{S}^s(L^{4e-3}(\alpha_1))$. In the case $d = 2e+1$ there is again a problem with the summand $\mathbb{Z}_N(t_{4e})$ which can be resolved by the same reasoning as in the case α_1. □

Corollary 4.16. *The function* $\widetilde{\rho}\colon \mathcal{S}^s(L^{2d-1}(\alpha)) \longrightarrow \mathbb{Q}R_{\widehat{G}}^{(-1)^d}$ *is a homomorphism.*

Proof. From Corollary 2.4 we have that for some $k \in \mathbb{N}$ there is a homotopy equivalence $f\colon L^{2d-1}(\alpha) \to L^{2d-1}(\alpha_k)$. It induces a homomorphism $f_*\colon \mathcal{S}^s(L^{2d-1}(\alpha)) \to \mathcal{S}^s(L^{2d-1}(\alpha_k))$. We will show that $\widetilde{\rho} = \widetilde{\rho} \circ f_*$. This implies that $\widetilde{\rho}$ is a homomorphism of $\mathcal{S}^s(L^{2d-1}(\alpha))$ since it is equal to a composition of homomorphisms.

We use the observation from Remark 4.4 and the composition formula of [Ran08, Theorem 2.3]. Let $h\colon L \to L^{2d-1}(\alpha)$ represent an element $a \in \mathcal{S}^s(L^{2d-1}(\alpha))$ and note that the homotopy equivalence f represents an element in $\mathcal{S}^h(L^{2d-1}(\alpha_k))$, call it b. The composition $h \circ f$ represents another element in $\mathcal{S}^h(L^{2d-1}(\alpha_k))$, call it c. The formula of [Ran08, Theorem 2.3] says $f_*a = b - c$. Now clearly

$$\begin{aligned}\widetilde{\rho}(f_*a) &= \widetilde{\rho}(b) - \widetilde{\rho}(c) = \\ &= \rho(L) - \rho(L^{2d-1}(\alpha_k)) - \rho(L^{2d-1}(\alpha)) + \rho(L^{2d-1}(\alpha_k)) = \widetilde{\rho}(a).\end{aligned}$$

This finishes the proof. □

5. Calculations

We want to prove Theorem 1.2 by investigating the short exact sequence (3.12) using the relation to a short exact sequence from representation theory of G via the ρ-invariant as described in Proposition 4.5. Theorem 1.2 is obtained when we put together statements of Theorems 5.1, 5.2 and 5.3.

Theorem 5.1. *Let* $\bar{T} = \ker[\widetilde{\rho}]\colon \widetilde{\mathcal{N}}(L^{2d-1}(\alpha)) \longrightarrow \mathbb{Q}R_{\widehat{G}}^{(-1)^d}/4 \cdot R_{\widehat{G}}^{(-1)^d}$. *Then we have*

$$\mathcal{S}^s(L^{2d-1}(\alpha)) \cong \bar{\Sigma} \oplus \bar{T}$$

where $\bar{\Sigma} = \widetilde{\rho}(\mathcal{S}^s(L^{2d-1}(\alpha)))$ *is a free abelian group of rank* $N/2 - 1$ *if* $d = 2e+1$ *and of rank* $N/2$ *if* $d = 2e$.

Proof. Recall the commutative diagram of Proposition 4.5. Since $\widetilde{\rho}$ is a homomorphism, we have that $\bar{\Sigma}$ is a subgroup of $\mathbb{Q}R_{\widehat{G}}^{(-1)^d}$, which as an abelian group is a direct sum of $N/2-1$ copies of $\mathbb{Q}$ if $d = 2e+1$ and of $N/2$ copies of $\mathbb{Q}$ if $d = 2e$. It contains a subgroup $\widetilde{\rho}(\partial \widetilde{L}^s_{2d}(G))$ which is a free abelian group of the same rank as the theorem claims for $\bar{\Sigma}$ in the respective cases. The claim about the rank of $\bar{\Sigma}$ follows.

Now replace in the diagram of Proposition 4.5 the middle and the third term of the lower sequence by the image of $\widetilde{\rho}$ and by the image of $[\widetilde{\rho}]$ respectively. Then the right hand square becomes a pullback square. It follows that $\bar{T}$ is isomorphic to the kernel of the map $\widetilde{\rho}\colon \mathcal{S}^s(L^{2d-1}(\alpha)) \longrightarrow \bar{\Sigma}$. We obtain a short exact sequence of abelian groups

$$0 \longrightarrow \bar{T} \xrightarrow{\lambda} \mathcal{S}^s(L^{2d-1}(\alpha)) \xrightarrow{\widetilde{\rho}} \bar{\Sigma} \longrightarrow 0$$

where $\bar{\Sigma}$ is a free abelian group and hence the sequence splits. □

So our goal is to understand the subgroup $\bar{T}$ of $\widetilde{\mathcal{N}}(L^{2d-1}(\alpha_1))$, which is a group isomorphic to the direct sum $T_N(d) \oplus T_2(d)$ of an N-torsion group $T_N(d)$ and a 2-torsion group $T_2(d)$

$$T_N(d) = \bigoplus_{i=1}^{c} \mathbb{Z}_N = \bigoplus_{i=1}^{c} \mathbb{Z}_N(t_{4i}) \quad T_2(d) = \bigoplus_{i=1}^{c} \mathbb{Z}_2 = \bigoplus_{i=1}^{c} \mathbb{Z}_2(t_{4i+2}).$$

where $c = \lfloor (d-1)/2 \rfloor$. We will denote $\bar{T}_N(d) = \bar{T} \cap T_N(d)$, $\bar{T}_2(d) = \bar{T} \cap T_2(d)$ and we will determine the two subgroups separately.

Theorem 5.2. *We have*

$$\bar{T}_2(d) = T_2(d)$$

Proof. By Proposition 4.12 the formula for $[\widetilde{\rho}]$ only depends on t_{4i}. □

Theorem 5.3. *We have*

$$\bar{T}_N(d) = \bigoplus_{i=1}^{c} \mathbb{Z}_{2^{\min\{K,2i+2\}}}$$

where $c = \lfloor (d-1)/2 \rfloor$.

In view of Proposition 4.14 it is convenient to make the following reformulation. If $d = 2e$ then the group $T_N(d)$ can be identified with the underlying abelian group $\mathbb{Z}_N[x](d)$ of the truncated polynomial ring in the variable x:

$$(5.1)\qquad \begin{aligned} T_N(d) &\xrightarrow{\cong} \mathbb{Z}_N[x](d) := \{q(x) \in \mathbb{Z}_N[x] \mid \deg(q) \leq c-1\} \\ t = (t_{4i})_{i=1}^{c} &\mapsto q_t(x) = \sum_{i=0}^{c-1} t_{4(i+1)} x^{c-i-1}. \end{aligned}$$

The map $[\widetilde{\rho}]$ becomes

$$(5.2)\qquad q \mapsto 8 \cdot f_k' \cdot (f^2 - 1) \cdot q(f^2).$$

If $d = 2e+1$ then the group $T_N(d)$ can be identified with the underlying abelian group $\mathbb{Z}_N[x](d)$ of the truncated polynomial ring in the variable x as follows:

$$(5.3)\qquad \begin{aligned} T_N(d) &\xrightarrow{\cong} \mathbb{Z}_N[x](d) := \{q(x) \in \mathbb{Z}_N[x] \mid \deg(q) \leq c-1\} \\ t = (t_{4i})_{i=1}^{c} &\mapsto q_t(x) = \sum_{i=1}^{c-1} t_{4i} x^{c-i-2}(x-1) + t_{4c} \\ &= \sum_{i=1}^{c-1} (t_{4(i+1)} - t_{4i}) x^{c-i-1} + t_4 x^{c-1} \end{aligned}$$

The map $[\widetilde{\rho}]$ then becomes

$$(5.4)\qquad q \mapsto 8 \cdot f_k \cdot q(f^2).$$

Further it is convenient to work with the underlying abelian group of

$$\mathbb{Z}[x](d) := \{q(x) \in \mathbb{Z}[x] \mid \deg(q) \leq c-1\},$$

use the formulas (5.2), (5.4) to define a map $[\widehat{\rho}] \colon \mathbb{Z}[x](d) \longrightarrow \mathbb{Q}R_{\widehat{G}}^{(-1)^d}$ and study the preimage of $4 \cdot R_{\widehat{G}}^{(-1)^d}$. So the task becomes to find

$$\begin{aligned} A_K^k(2e) &:= \left\{ q \in \mathbb{Z}[x] \mid \deg(q) \leq c-1,\ 8 \cdot f_k' \cdot (f^2-1) \cdot q(f^2) \in 4 \cdot \mathbb{Z}[\chi]/\langle I\langle K\rangle \right\}, \\ A_K^k(2e+1) &:= \left\{ q \in \mathbb{Z}[x] \mid \deg(q) \leq c-1,\ 8 \cdot f_k \cdot q(f^2) \in 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle \right\}. \end{aligned}$$

We will show that $A_K^k(d) = B_K(d)$ where $B_K(d)$ is a subgroup of polynomials described in terms of certain polynomials $r_n^{\pm}(x)$ of degree n for all $n \in \mathbb{N}$. These are the best polynomial of degree n in a sense that

$$\begin{aligned} 8 \cdot f_k' \cdot (f^2-1) \cdot r_n^{+}(f^2) &\in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2\rangle \\ 8 \cdot f_k \cdot r_n^{-}(f^2) &\in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2\rangle \end{aligned}$$

and for all polynomials $q \in \mathbb{Z}[x]$ of degree n with leading coefficient 1 we have

$$\begin{aligned} 8 \cdot f_k' \cdot (f^2-1) \cdot q(f^2) &\notin 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+3\rangle \\ 8 \cdot f_k \cdot q(f^2) &\notin 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+3\rangle. \end{aligned}$$

We define

$$B_K(2e) := \left\{ \sum_{n=0}^{c-1} a_n \cdot 2^{\max\{K-2n-2,0\}} \cdot r_n^+ \mid a_n \in \mathbb{Z} \right\},$$

$$B_K(2e+1) := \left\{ \sum_{n=0}^{c-1} a_n \cdot 2^{\max\{K-2n-2,0\}} \cdot r_n^- \mid a_n \in \mathbb{Z} \right\}.$$

Theorem 5.4.

$$A_K^k(d) = B_K(d)$$

Proof of Theorem 5.3. It follows from Theorem 5.4 and the definition of $B_K(d)$ that $A_K^k(d)$ is a free abelian subgroup of $\mathbb{Z}[x](d)$ with a basis given by polynomials $r_n^{\pm}$. Under the homomorphism $\mathbb{Z}[x](d) \to \mathbb{Z}_N[x](d)$ the subgroup $A_K^k(d)$ is mapped onto a subgroup isomorphic to a direct sum as claimed by the theorem. □

Scheme of the proof of Theorem 5.4.

The proof requires a formidable amount of machinery and special constructions. For better orientation we offer the following scheme.

In subsection 5.1 we develop some general methods for deciding whether a given element $g \in \mathbb{Q}[\chi]/I\langle K\rangle$ is in $4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle$ or not. We use the Chinese remainder theorem and certain 'valuation' functions w_l on $\mathbb{Q}[\chi]/I\langle K\rangle$. These are effectively calculable as is shown in Lemma 5.9. The criteria for deciding whether g is in $4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle$ or not using w_l are presented in Theorem 5.12. Later we apply the criteria to decide whether a polynomial $q \in \mathbb{Z}[x](d)$ is in $A_K^k(d)$ or not, but there is a problem that we do not obtain a necessary and sufficient condition since the criteria do not apply to all $g \in \mathbb{Q}[\chi]/I\langle K\rangle$. In the sequel we therefore have to combine the w_l-technology with some ad hoc considerations.

In subsection 5.2 we construct a sequence of polynomials in $\mathbb{Z}[x](d)$ for $d = 2e+1$ which are good in a sense that they have the leading coefficient 1 and they yield elements in $4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle$ for a large K in comparison with the other polynomials of the same degree with leading coefficient 1. First we construct auxiliary polynomials p_k in Definition 5.13, which are used to define 'good' polynomials q_n in Definition 5.16. These are in turn be used to define 'the best' polynomials r_n^- in Definition 5.23. This last definition is inductive, the crucial inductive step is described in Proposition 5.22. The 'goodness' properties are summarized in Corollary 5.15 (p_k), in Proposition 5.17 (q_n) and in Corollary 5.24 (r_n^-).

The final subsection 5.3 completes the proof of Theorem 5.4. This part treats first the case $d = 2e+1$ and proceeds by induction on K. The proof in the case $d = 2e$ is short and proceeds by a reduction to the case $d = 2e+1$.

5.1. w_l technology.

For given $g \in \mathbb{Q}R_{\widehat{G}}$ we want to decide whether $g \in 4 \cdot R_{\widehat{G}}$ or not using the homomorphisms $\mathrm{pr}_l : \mathbb{Q}R_{\widehat{G}} \cong \mathbb{Q}[\chi]/I\langle K\rangle \twoheadrightarrow \mathbb{Q}[\chi]/\langle 1+\chi^{2^l}\rangle$ for $0 \leq l \leq K-1$. Obviously, $g \in 4 \cdot R_{\widehat{G}}$ implies $\mathrm{pr}_l(g) \in 4 \cdot \mathbb{Z}[\chi]/\langle 1+\chi^{2^l}\rangle$. Of more interest is the other direction. By the Chinese remainder theorem g is uniquely determined by the elements $\mathrm{pr}_l(g)$ $(0 \leq l \leq K-1)$. More precisely, we have

Lemma 5.5. *Let $g \in \mathbb{Q}R_{\widehat{G}}$. Then*

$$g = \sum_{l=0}^{K-1} 2^{l-K} \cdot g_l \cdot (1-\chi) \cdot \prod_{\substack{0 \leq r \leq K-1 \\ r \neq l}} (1+\chi^{2^r})$$

for any elements $g_l \in \mathbb{Q}R_{\widehat{G}}$ satisfying $\mathrm{pr}_l(g_l) = \mathrm{pr}_l(g)$.

If $\mathrm{pr}_l(g) \in 2^{2+K-l} \cdot \mathbb{Z}[\chi]/\langle 1+\chi^{2^l}\rangle$ we can choose $g_l \in 2^{2+K-l} \cdot \mathbb{Z}R_{\widehat{G}}$ satisfying $\mathrm{pr}_l(g_l) = \mathrm{pr}_l(g)$ and the lemma above shows $g \in 4 \cdot \mathbb{Z}R_{\widehat{G}}$. Motivated by this observation we want to analyze whether $\mathrm{pr}_l(g)$ lies in $2^m \cdot \mathbb{Z}[\chi]/\langle 1+\chi^{2^l}\rangle$ for some integer m. For this purpose we will introduce w_l-functions which are generalizations of the p-adic valuation for $p = 2$.

Before we do so, we give the proof of Lemma 5.5 and consider an application (Lemma 5.6).

Proof of Proposition 5.5. We have $\mathbb{Q}R_{\widehat{G}} \cong \mathbb{Q}[\chi]/I\langle K\rangle$ where $I\langle K\rangle$ was defined as $I\langle K\rangle := \langle 1+\chi+\cdots+\chi^{2^K-1}\rangle$. Notice that $1+\chi+\cdots+\chi^{2^K-1} = \prod_{m=0}^{K-1}(1+\chi^{2^m})$. Since the factors $1+\chi^{2^m}$ are mutually coprime in the principal ideal domain $\mathbb{Q}[\chi]$, it suffices to check the desired equality under the epimorphism pr_m for $0 \le m \le K-1$. In $\mathbb{Q}[\chi]/\langle 1+\chi^{2^m}\rangle$ we obtain

$$\mathrm{pr}_m\left(\sum_{l=0}^{K-1} 2^{l-K}\cdot g_l\cdot(1-\chi)\cdot\prod_{\substack{0\le r\le K-1\\ r\ne l}}(1+\chi^{2^r})\right) =$$

$$\sum_{l=0}^{K-1} 2^{l-K}\cdot \mathrm{pr}_m(g)\cdot(1-\chi)\cdot\prod_{\substack{0\le r\le K-1\\ r\ne l}}(1+\chi^{2^r}) =$$

$$2^{m-K}\cdot \mathrm{pr}_m(g)\cdot(1-\chi)\cdot\prod_{\substack{0\le r\le K-1\\ r\ne m}}(1+\chi^{2^r}) =$$

$$2^{m-K}\cdot \mathrm{pr}_m(g)\cdot(1-\chi)\cdot(1+\chi+\cdots+\chi^{2^m-1})\cdot\prod_{r=m+1}^{K-1}(1+(-1)^{2^{r-m}}) =$$

$$2^{m-K}\cdot \mathrm{pr}_m(g)\cdot(1-\chi^{2^m})\cdot 2^{K-1-m} = \mathrm{pr}_m(g).$$

□

As a warm-up in learning how to work with Lemma 5.5 we prove the following lemma needed in the proof of Proposition 4.13.

Lemma 5.6. *Let $z \in \mathbb{Q}R^-_{\widehat{G}}$. If $f \cdot z = 0$ in $\mathbb{Q}R^+_{\widehat{G}}$ then $z = 0$.*

Proof. It follows from Lemma 5.5 that it is sufficient to show $\mathrm{pr}_l(z) = 0$ for all $0 \le l \le K-1$. We have $\mathrm{pr}_l(f) \cdot pr_l(z) = pr_l(f \cdot z) = 0$. Notice that $\mathrm{pr}_l(f)$ is invertible for $l \ge 1$ since

$$(1+\chi)^{-1} = \frac{1}{2}\cdot(1-\chi+\chi^2-\chi^3+\cdots-\chi^{2^l-1}) \in \mathbb{Q}[\chi]/\langle 1+\chi^{2^l}\rangle.$$

This implies $pr_l(z) = 0$ for $l \ge 1$. Further recall that we can write $z \in \mathbb{Q}R^-_{\widehat{G}}$ as

$$z = \sum_{r=1}^{N/2-1} a_r\cdot(\chi^r - \chi^{N-r}).$$

with $a_r \in \mathbb{Q}$. Since $\chi^r - \chi^{N-r}$ is a multiple of $1+\chi$, we conclude $\mathrm{pr}_0(z) = 0$. □

The following lemma is needed for the definition of the w_l-functions.

Lemma 5.7. *Let $g \in \mathbb{Q}R_{\widehat{G}}$ and $l \ge 0$ such that $(1+\chi^{2^l}) \nmid g$.*

(1) *There exist $a, u \in \mathbb{Z}$, $b \in \{0, 1, \cdots, 2^l-1\}$ and $v_1, v_2 \in \mathbb{Z}[\chi]$ such that*
  - *$\mathrm{pr}_l(g)$ and $\frac{2^a}{u}\cdot\big((1-\chi)^b\cdot v_1(\chi) + 2\cdot v_2(\chi)\big)$ coincide in $\mathbb{Q}[\chi]/\langle 1+\chi^{2^l}\rangle$,*
  - *u and $v_1(1)$ are odd.*

Moreover, if $\mathrm{pr}_l(g) \in \mathbb{Z}[\chi]/\langle 1+\chi^{2^l}\rangle$ then u can be chosen to be 1.

(2) *The numbers a and b are uniquely determined by g and l.*

Proof. (1) There exists $w \in \mathbb{Z}$ such that $w \cdot \mathrm{pr}_l(g) \in \mathbb{Z}[\chi]/\langle 1+\chi^{2^l}\rangle$. We write $w = 2^{a_1} \cdot u$ with u odd. Choose $z \in \mathbb{Z}[\chi]$ of degree $\deg(z) < 2^l$ such that $w \cdot \mathrm{pr}_l(g)$ and $z(\chi)$ coincide in $\mathbb{Q}[\chi]/\langle 1+\chi^{2^l}\rangle$. We write z as $z(\chi) = \sum_{m=0}^{2^l-1} z_m \cdot (1-\chi)^m$ with $z_m \in \mathbb{Z}$. Since $(1+\chi^{2^l})$ does not divide g, we have $z \neq 0$. Hence there exists a largest number $a_2 \in \mathbb{N}_0$ such that $\frac{z}{2^{a_2}} \in \mathbb{Z}[\chi]$. Define $a := -a_1 + a_2$, $b := \min\left\{m \;\middle|\; 2 \nmid \frac{z_m}{2^{a_2}}\right\}$, $v_1(\chi) := \sum_{m=b}^{2^l-1} \frac{z_m}{2^{a_2}} \cdot (1-\chi)^{m-b}$ and $v_2(\chi) := \sum_{m=0}^{b-1} \frac{z_m}{2^{a_2+1}} \cdot (1-\chi)^m$. Then $v_1(1) = \frac{z_b}{2^{a_2}}$ is odd and $\mathrm{pr}_l(g)$ coincides with

$$\frac{2^a}{u} \cdot \big((1-\chi)^b \cdot v_1(\chi) + 2 \cdot v_2(\chi)\big)$$

in $\mathbb{Q}[\chi]/\langle 1+\chi^{2^l}\rangle$.

(2) Let $a, a', u, u' \in \mathbb{Z}$, $b, b' \in \{0, 1, \cdots, l-1\}$ and $v_1, v_1', v_2, v_2' \in \mathbb{Z}[\chi]$ such that $u, u', v_1(1), v_1'(1)$ are odd and

$$\begin{aligned} \mathrm{pr}_l(g) &= \frac{2^a}{u} \cdot \big((1-\chi)^b \cdot v_1(\chi) + 2 \cdot v_2(\chi)\big) \\ &= \frac{2^{a'}}{u'} \cdot \Big((1-\chi)^{b'} \cdot v_1'(\chi) + 2 \cdot v_2'(\chi)\Big) \end{aligned}$$

in $\mathbb{Q}[\chi]/\langle 1+\chi^{2^l}\rangle$.

We first prove $a = a'$ by contradiction. Assume that $a < a'$. In $\mathbb{Q}[\chi]/\langle 1+\chi^{2^l}\rangle$ we get

$$u' \cdot \big((1-\chi)^b \cdot v_1(\chi) + 2 \cdot v_2(\chi)\big) = 2^{a'-a} \cdot u \cdot \Big((1-\chi)^{b'} \cdot v_1'(\chi) + 2 \cdot v_2'(\chi)\Big).$$

Hence there exists $q \in \mathbb{Q}[\chi]$ with

$$\begin{aligned} &u' \cdot \big((1-\chi)^b \cdot v_1(\chi) + 2 \cdot v_2(\chi)\big) = \\ &2^{a'-a} \cdot u \cdot \Big((1-\chi)^{b'} \cdot v_1'(\chi) + 2 \cdot v_2'(\chi)\Big) + q(\chi) \cdot (1+\chi^{2^l}). \end{aligned}$$

The equation above implies $q(\chi) \cdot (1+\chi^{2^l}) \in \mathbb{Z}[\chi]$ and hence $q \in \mathbb{Z}[\chi]$. Under the epimorphism $\mathbb{Z}[\chi] \twoheadrightarrow \mathbb{Z}_2[\chi]$ this equation becomes to

$$(1+\chi)^b \cdot \overline{v_1}(\chi) = \overline{q}(\chi) \cdot (1+\chi^{2^l}).$$

Since $1+\chi^{2^l} = (1+\chi)^{2^l}$ in $\mathbb{Z}_2[\chi]$ (proof by induction), we conclude $\overline{v_1}(\chi) = \overline{q}(\chi) \cdot (1+\chi)^{2^l-b}$ and hence $\overline{v_1}(1) = 0$ in $\mathbb{Z}_2[\chi]$. This is a contradiction to $v_1(1)$ odd. Therefore, we have $a = a'$.

It remains to prove $b = b'$. We give again a proof by contradiction. Assume that $b < b'$. In $\mathbb{Q}[\chi]/\langle 1+\chi^{2^l}\rangle$ we have

$$u' \cdot \big((1-\chi)^b \cdot v_1(\chi) + 2 \cdot v_2(\chi)\big) = u \cdot \Big((1-\chi)^{b'} \cdot v_1'(\chi) + 2 \cdot v_2'(\chi)\Big).$$

Hence there exists $q \in \mathbb{Q}[\chi]$ with

$$\begin{aligned} &u' \cdot \big((1-\chi)^b \cdot v_1(\chi) + 2 \cdot v_2(\chi)\big) = \\ &u \cdot \Big((1-\chi)^{b'} \cdot v_1'(\chi) + 2 \cdot v_2'(\chi)\Big) + q(\chi) \cdot (1+\chi^{2^l}). \end{aligned}$$

We conclude $q \in \mathbb{Z}[\chi]$. Under the epimorphism $\mathbb{Z}[\chi] \twoheadrightarrow \mathbb{Z}_2[\chi]$ this equation becomes to

$$(1+\chi)^b \cdot \overline{v_1}(\chi) = (1+\chi)^{b'} \cdot \overline{v_1'}(\chi) + \overline{q}(\chi) \cdot (1+\chi^{2^l}).$$

We finally get $\overline{v_1}(\chi) = (1+\chi)^{b'-b} \cdot \overline{v_1'}(\chi) + \overline{q}(\chi) \cdot (1+\chi)^{2^l-b}$ and hence $\overline{v_1}(1) = 0$ in $\mathbb{Z}_2[\chi]$ contradicting $v_1(1)$ odd. This shows $b = b'$. $\square$

Definition 5.8. For $l \geq 0$ define

$$w_l \colon \mathbb{Q}R_{\widehat{G}} \longrightarrow \frac{1}{2^l} \cdot \mathbb{Z} \cup \{\infty\}$$
$$g \mapsto \begin{cases} a + b/2^l & \text{with } a, b \text{ as in Lemma 5.7 when } (1+\chi^{2^l}) \nmid g \\ \infty & \text{when } (1+\chi^{2^l}) \mid g \end{cases}$$

Roughly speaking, w_l counts how many factors of 2 are contained in $\mathrm{pr}_l(g)$. We have the following calculation rules.

Lemma 5.9. *Let $g_1, g_2 \in \mathbb{Q}R_{\widehat{G}}$ and $l \geq 0$.*

(1) $w_l(g_1 \cdot g_2) = w_l(g_1) + w_l(g_2)$.
(2) *If $w_l(g_1) \neq w_l(g_2)$ then $w_l(g_1 + g_2) = \min\{w_l(g_1), w_l(g_2)\}$.*
(3) *If $w_l(g_1) = w_l(g_2)$ then $w_l(g_1 + g_2) > w_l(g_1) = w_l(g_2)$.*

Proof. Parts (2) and (3) can be proven by easy calculations. We only focus on (1). As in Lemma 5.7 we write (for $i = 1, 2$):

$$\mathrm{pr}_l(g_i) = \frac{2^{a_i}}{u_i} \cdot \Big((1-\chi)^{b_i} \cdot v_1^{(i)}(\chi) + 2 \cdot v_2^{(i)}(\chi) \Big) \in \mathbb{Q}[\chi]/\langle 1+\chi^{2^l} \rangle.$$

In $\mathbb{Q}[\chi]/\langle 1+\chi^{2^l}\rangle$ we obtain

$$\mathrm{pr}_l(g_1 \cdot g_2) = \mathrm{pr}_l(g_1) \cdot \mathrm{pr}_1(g_2) =$$
$$\frac{2^{a_1+a_2}}{u_1 \cdot u_2} \cdot (1-\chi)^{b_1+b_2} \cdot v_1^{(1)}(\chi) \cdot v_1^{(2)}(\chi) +$$
$$\frac{2^{a_1+a_2+1}}{u_1 \cdot u_2} \cdot \Big((1-\chi)^{b_1} \cdot v_2^{(2)}(\chi) + (1-\chi)^{b_2} \cdot v_2^{(1)}(\chi) + 2 \cdot v_2^{(1)}(\chi) \cdot v_2^{(2)}(\chi) \Big).$$

If $b_1 + b_2 < 2^l$ then we conclude

$$w_l(g_1 \cdot g_2) = a_1 + a_2 + \frac{b_1 + b_2}{2^l} = a_1 + \frac{b_1}{2^l} + a_2 + \frac{b_2}{2^l} = w_l(g_1) + w_l(g_2).$$

It remains to study the case $b_1 + b_2 \geq 2^l$. Define $x_m \in \mathbb{Z}[\chi]$ by $x_0 := -\chi$ and $x_m := \chi^{2^{m-1}} + 2 \cdot x_{m-1}(\chi) \cdot (1 + \chi^{2^{m-1}} + x_{m-1}(\chi))$ for $m \geq 1$. A proof by induction shows $(1-\chi)^{2^k} = 2 \cdot x_k(\chi) + (1+\chi^{2^k})$ for all $k \geq 0$. In $\mathbb{Q}[\chi]/\langle 1+\chi^{2^l}\rangle$ we obtain

$$\mathrm{pr}_l(g_1 \cdot g_2) = \mathrm{pr}_l(g_1) \cdot \mathrm{pr}_1(g_2) = \frac{2^{a_1+a_2+1}}{u_1 \cdot u_2} \cdot (1-\chi)^{b_1+b_2-2^l} \cdot$$
$$\Big(x_l(\chi) \cdot v_1^{(1)}(\chi) \cdot v_1^{(2)}(\chi) + (1-\chi)^{2^l - b_2} \cdot v_2^{(2)}(\chi) + (1-\chi)^{2^l-b_1} \cdot v_2^{(1)}(\chi) \Big) +$$
$$\frac{2^{a_1+a_2+2}}{u_1 \cdot u_2} \cdot v_2^{(1)}(\chi) \cdot v_2^{(2)}(\chi).$$

We finally conclude

$$w_l(g_1 \cdot g_2) = a_1 + a_2 + 1 + \frac{b_1 + b_2 - 2^l}{2^l} = a_1 + \frac{b_1}{2^l} + a_2 + \frac{b_2}{2^l} = w_l(g_1) + w_l(g_2).$$

□

Remark 5.10. For the reader who is familiar with number fields and valuations we give the following description for the w_l-functions.

Let $\zeta_{2^{l+1}} \in \mathbb{C}$ be a primitive 2^{l+1}-th root of unity. Consider the ring of algebraic integers $\mathbb{Z}[\zeta_{2^{l+1}}]$ in the cyclotomic field $\mathbb{Q}(\zeta_{2^{l+1}})$. The ideal $\mathcal{P} := (2, 1 - \zeta_{2^{l+1}})$ in $\mathbb{Z}[\zeta_{2^{l+1}}]$ is a prime ideal satisfying $\mathcal{P}^{2^l} = (2)$. Let $\nu_{\mathcal{P}}$ be the (exponential) valuation with respect to this prime ideal $\mathcal{P}$. Then the w_l-function is given by

$$w_l(g) = \frac{1}{2^l} \cdot \nu_{\mathcal{P}}\left(\alpha\left(\mathrm{pr}_l(g)\right)\right)$$

where $\alpha : \mathbb{Q}[\chi]/\langle 1+\chi^{2^l}\rangle \to \mathbb{Q}(\zeta_{2^{l+1}})$ is the isomorphism induced by $\chi \mapsto \zeta_{2^{l+1}}$.

Example 5.11. The reader is invited to calculate the w_l for the following examples.

(1) If $q \in \mathbb{Q} \subset \mathbb{Q}R_{\widehat{G}}$ then $w_l(q)$ coincides with the p-adic valuation for $p = 2$. In particular, $w_l(2^a) = a$.

(2) $w_l(f) = \begin{cases} \infty & \text{when } l = 0 \\ 0 & \text{when } l \geq 1 \end{cases}$

(3) $w_l(f \pm 1) = 1 - 2^{-l}$

(4) $w_l(f^2 - 1) = 2 - 2^{1-l}$

(5) $w_l(f^2 + 1) = \begin{cases} 0 & \text{when } l = 0 \\ \infty & \text{when } l = 1 \\ 1 & \text{when } l \geq 2 \end{cases}$

(6) $w_l(f'_k) = 0$

Hints. (2): For $l = 0$ use $f \equiv 0 \bmod 1+\chi$. For $l \geq 1$ use $f \cdot (1-\chi) = 1+\chi$ and $1+\chi = 2-(1-\chi)$. (3): Use $(1-\chi)(f+1) = 2$ and $(1-\chi)(f-1) = 2\chi$. (4): Use (3). (5): For $l = 1$ use $f^2+1 \equiv 0 \bmod 1+\chi^2$. For $l \neq 1$ use $f^2+1 = (f^2-1)+2$ and (4). (6): Use f'_k, $f'^{-1}_k \in R_{\widehat{G}}$ and the fact $w_l(g) \geq 0$ when $g \in R_{\widehat{G}}$.

We now come back to the initial question of this subsection: For a given $g \in \mathbb{Q}R_{\widehat{G}}$ we want to decide whether g lies in $4 \cdot R_{\widehat{G}}$ or not. The following theorem can answer this question in many cases.

Theorem 5.12. *Let $g \in \mathbb{Q}R_{\widehat{G}}$. Suppose that $\mathrm{pr}_l(g) \in 4 \cdot \mathbb{Z}[\chi]/\langle 1+\chi^{2^l}\rangle$ for all $0 \leq l \leq K-1$.*

(1) If $w_l(g) \geq 2+K-l-2^{-l}$ for all $0 \leq l \leq K-1$ then $g \in 4 \cdot R_{\widehat{G}}$.

(2) If there exist $h \in R_{\widehat{G}}$ and $0 \leq l' \leq K-1$ such that

$w_l(g) + w_l(h) \geq 2+K-l-2^{-l}$ for all $l \in \{0, 1, \cdots, K-1\} - \{l'\}$ and

$w_{l'}(g) + w_{l'}(h) < 2+K-l'-2^{-l'}$

then $g \notin 4 \cdot R_{\widehat{G}}$.

Proof. (1) The assumption $w_l(g) \geq 2+K-l-2^{-l}$ implies $w_l((1-\chi)\cdot g) \geq 2+K-l$. By Lemma 5.7 and Definition 5.8 there exist $z_l \in \mathbb{Z}[\chi]$ such that

$$(1-\chi)\cdot \mathrm{pr}_l(g) = 2^{2+K-l} \cdot z_l(\chi) \in \mathbb{Q}[\chi]/\langle 1+\chi^{2^l}\rangle$$

for all $0 \leq l \leq K-1$. Using Lemma 5.5 we conclude

$$(1-\chi)\cdot g = \sum_{l=0}^{K-1} 2^{l-K} \cdot \left(2^{2+K-l} \cdot z_l(\chi)\right) \cdot (1-\chi) \cdot \prod_{\substack{0 \leq r \leq K-1 \\ r \neq l}} (1+\chi^{2^r}) \quad \text{in } \mathbb{Q}R_{\widehat{G}}$$

and hence

$$g = 4 \cdot \sum_{l=0}^{K-1} \cdot z_l(\chi) \cdot \prod_{\substack{0 \leq r \leq K-1 \\ r \neq l}} (1+\chi^{2^r}) \in 4 \cdot R_{\widehat{G}}.$$

(2) We give a proof by contradiction. Assume that $g \in 4 \cdot R_{\widehat{G}}$ and define

$$a := \min\left\{ m \in \mathbb{Z} \mid m + w_{l'}(g) + w_{l'}(h) \geq 2+K-l'-2^{-l'} \right\}.$$

Notice that $a \geq 1$. We have

$$w_l((1-\chi)\cdot 2^a \cdot g \cdot h) \geq 3+K-l \quad \text{for all } l \in \{0, 1, \cdots, K-1\} - \{l'\}$$

and

$$w_{l'}((1-\chi)\cdot 2^a \cdot g \cdot h) \geq 2+K-l'.$$

From Lemma 5.7 and Definition 5.8 we conclude that there exist $z_l \in 2 \cdot \mathbb{Z}[\chi]$ for all $l \in \{0, 1, \cdots, K-1\} - \{l'\}$ and $z_{l'} \in \mathbb{Z}[\chi]$ satisfying

$$\mathrm{pr}_l((1-\chi) \cdot 2^a \cdot g \cdot h) = 2^{2+K-l} \cdot z_l(\chi) \in \mathbb{Q}[\chi]/\langle 1 + \chi^{2^l} \rangle.$$

Lemma 5.5 implies

$$(1-\chi) \cdot 2^a \cdot g \cdot h = \sum_{l=0}^{K-1} 2^{l-K} \cdot 2^{2+K-l} \cdot z_l(\chi) \cdot (1-\chi) \cdot \prod_{\substack{0 \le r \le K-1 \\ r \ne l}} (1 + \chi^{2^r})$$

and hence

$$2^a \cdot g \cdot h = \sum_{l=0}^{K-1} 4 \cdot z_l(\chi) \cdot \prod_{\substack{0 \le r \le K-1 \\ r \ne l}} (1 + \chi^{2^r}) \quad \text{in } \mathbb{Q}\mathrm{R}_{\widehat{\mathrm{G}}}.$$

Since $g \cdot h \in 4 \cdot R_{\widehat{G}}$ there exists $y \in \mathbb{Z}[\chi]$ such that $g \cdot h$ and $4 \cdot y$ coincide in $\mathbb{Q}[\chi]/I\langle K \rangle$. We get

$$2^a \cdot y(\chi) = \sum_{l=0}^{K-1} z_l(\chi) \cdot \prod_{\substack{0 \le r \le K-1 \\ r \ne l}} (1 + \chi^{2^r}) \quad \text{in } \mathbb{Q}[\chi]/I\langle K \rangle.$$

Hence there exists $q \in \mathbb{Q}[\chi]$ with

$$2^a \cdot y(\chi) = \sum_{l=0}^{K-1} z_l(\chi) \cdot \prod_{\substack{0 \le r \le K-1 \\ r \ne l}} (1 + \chi^{2^r}) + q(\chi) \cdot (1 + \chi + \cdots + \chi^{N-1}).$$

The equation above implies $q(\chi) \cdot (1 + \chi + \cdots + \chi^{N-1}) \in \mathbb{Z}[\chi]$ and hence $q \in \mathbb{Z}[\chi]$. Under the epimorphism $\mathbb{Z}[\chi] \twoheadrightarrow \mathbb{Z}_2[\chi]$ this equation becomes

$$\begin{aligned} 0 = & \sum_{l=0}^{K-1} \overline{z_l}(\chi) \cdot \prod_{\substack{0 \le r \le K-1 \\ r \ne l}} (1 + \chi^{2^r}) + \overline{q}(\chi) \cdot (1 + \chi + \cdots + \chi^{N-1}) \\ = & \overline{z_{l'}}(\chi) \cdot \prod_{\substack{0 \le r \le K-1 \\ r \ne l'}} (1 + \chi^{2^r}) + \overline{q}(\chi) \cdot \prod_{r=0}^{K-1} (1 + \chi^{2^r}). \end{aligned}$$

Hence $\overline{z_{l'}}(\chi) = -\overline{q}(\chi) \cdot (1 + \chi^{2^{l'}}) = -\overline{q}(\chi) \cdot (1+\chi)^{2^{l'}}$ in $\mathbb{Z}_2[\chi]$. This implies $w_{l'}(z_{l'}) \ge 1$. We finally get

$$\begin{aligned} & (a-1) + w_{l'}(g) + w_{l'}(h) && = \\ & w_{l'}((1-\chi) \cdot 2^a \cdot g \cdot h) - 1 - 2^{-l'} && = \\ & w_{l'}(2^{2+K-l'} \cdot z'_l(\chi)) - 1 - 2^{-l'} && = \\ & w_{l'}(z'_l(\chi)) + 1 + K - l' - 2^{-l'} && \ge \\ & 2 + K - l' - 2^{-l'} \end{aligned}$$

which contradicts the minimality of a. □

5.2. **Good polynomials.**

Theorem 5.12 can be used to decide for a given $K \in \mathbb{N}$ whether the expression $8 \cdot f'_k \cdot f \cdot q(f^2) \in \mathbb{Q}[\chi]/I\langle K \rangle$ lands in $4 \cdot \mathbb{Z}[\chi]/I\langle K \rangle$ or not for many but not all polynomials $q(x) \in \mathbb{Q}[x]$. Here we introduce polynomials r_n^-, which are in the next subsection proved to be the best in a sense that they are polynomials with leading coefficient 1 yielding elements in $4 \cdot \mathbb{Z}[\chi]/I\langle K \rangle$ for a large K in comparison with the other polynomials of the same degree with leading coefficient 1.

We start by defining auxiliary polynomials p_k which are used to define polynomials denoted q_n whose properties are summarized in Proposition 5.17. A careful analysis gives an inductive procedure for construction of certain linear combinations of polynomials q_n, denoted $\widetilde{q}_n$, with better properties that q_n themselves. This is the content of Proposition 5.22. The assumptions of this proposition turn out to be trivially true for small n which enables us to perform the induction to define the desired polynomials r_n^- in Definition 5.23.

Notice that the for any $q(x) \in \mathbb{Q}[x]$ we get

$$w_0(8 \cdot f_k' \cdot f \cdot q(f^2)) = \infty$$

since $w_0(f) = \infty$ because of $(1+\chi) \mid f$. Further notice that for $p_1(x) := x+1$ we have

$$p_1(f^2) = f^2 + 1 = 2 \cdot \frac{1+\chi^2}{(1-\chi)^2}. \tag{5.5}$$

Hence $(1+\chi^2) \mid p_1(f^2)$ in $\mathbb{Q}[\chi]/I\langle K\rangle$ and $w_1(8 \cdot f_k' \cdot f \cdot p_1(f^2)) = \infty$. Further observe that

$$\frac{(f^2+1)^2}{4 \cdot f^2} = \frac{(1+\chi^2)^2}{(1-\chi^2)^2}. \tag{5.6}$$

Motivated by that we make the following

Definition 5.13. Let $p_1(x) := x + 1 \in \mathbb{Z}[x]$. For $k \in \mathbb{N}$ define inductively

$$p_{k+1}(x) := p_k\left(\frac{(x+1)^2}{4x}\right) \cdot (4x)^{2^{k-1}} \in \mathbb{Z}[x]. \tag{5.7}$$

Notice that $p_k(x)$ is a polynomial in $\mathbb{Z}[x]$ of degree 2^{k-1}.

Theorem 5.14. *We have*

$$p_k(f^2) = 2^{2^k-1} \cdot \frac{1+\chi^{2^k}}{(1-\chi)^{2^k}} \in \mathbb{Q}[\chi]/I\langle K\rangle \quad \text{for } k \in \mathbb{N}.$$

Proof. It suffices to prove the equality

$$p_k(f^2) = 2^{2^k-1} \cdot \frac{1+\chi^{2^k}}{(1-\chi)^{2^k}}$$

in the field of rational functions $\mathbb{Q}(\chi)$. The proof now goes by induction with respect to $k \in \mathbb{N}$. The case $k = 1$ is proved by the identity (5.5). Now the induction step. Let $\alpha \colon \mathbb{Q}(\chi) \longrightarrow \mathbb{Q}(\chi)$ be the homomorphism given by $\chi \mapsto \chi^2$. We calculate:

$$\begin{aligned}
p_{k+1}(f^2) &= p_k\left(\frac{(f^2+1)^2}{4 \cdot f^2}\right) \cdot \left(4f^2\right)^{2^{k-1}} \\
&= p_k\left(\left(\frac{1+\chi^2}{1-\chi^2}\right)^2\right) \cdot 2^{2^k} \cdot f^{2^k} && \text{by (5.6)} \\
&= p_k\big((\alpha(f))^2\big) \cdot 2^{2^k} \cdot f^{2^k} \\
&= \alpha\big(p_k(f^2)\big) \cdot 2^{2^k} \cdot f^{2^k} \\
&= \frac{1+(\chi^2)^{2^k}}{(1-\chi^2)^{2^k}} \cdot 2^{2^k-1} \cdot 2^{2^k} \cdot \frac{(1+\chi)^{2^k}}{(1-\chi)^{2^k}} \\
&= \frac{1+\chi^{2^{k+1}}}{(1-\chi)^{2^k} \cdot (1+\chi)^{2^k}} \cdot 2^{2^{k+1}-1} \cdot \frac{(1+\chi)^{2^k}}{(1-\chi)^{2^k}} \\
&= 2^{2^{k+1}-1} \cdot \frac{1+\chi^{2^{k+1}}}{(1-\chi)^{2^{k+1}}}
\end{aligned}$$

□

Corollary 5.15. *We have*

(1) $w_l(p_k(f^2)) = \infty$ *when* $l = k$
(2) $w_l(p_k(f^2)) = 2^k - 1$ *when* $l > k$

Proof. The first item is immediate from the formula of the previous theorem. For the second item note that $1+\chi^{2^k} \equiv (1-\chi)^{2^k} \bmod 2$. It follows that $w_l(1+\chi^{2^k}) = w_l((1-\chi)^{2^k})$ for $l > k$ and hence

$$w_l\left(\frac{1+\chi^{2^k}}{(1-\chi)^{2^k}}\right) = 0 \quad \text{for } l > k.$$

Finally use the formula of the previous theorem and the product formula for w_l. □

Now we are ready to introduce the polynomials q_n which will be good in the already mentioned sense. The idea is that we get good polynomials when we multiply the polynomials p_k from the previous definition.

Definition 5.16. Let $n \geq 0$. Define $a(n), b(n) \geq 0$ as the integers satisfying

$$n + 1 = 2^{a(n)} + b(n) \quad \text{with} \quad 0 \leq b(n) \leq 2^{a(n)} - 1.$$

Define

$$q_n(x) := \prod_{r=1}^{a(n)} p_r(x) \cdot (x-1)^{b(n)}.$$

Proposition 5.17. *Let* $n \geq 0$, $k \geq 1$ *and* $m \in \{1, 2\}$. *We have*

$$8 \cdot f_k' \cdot f^m \cdot q_n(f^2) \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+1\rangle,$$
$$8 \cdot f_k' \cdot f^m \cdot q_n(f^2) \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2\rangle \Longleftrightarrow b(n) = 0,$$
$$8 \cdot f_k' \cdot f^m \cdot q_n(f^2) \notin 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+3\rangle.$$

Moreover, we have

$$8 \cdot f_k' \cdot f^m \cdot q_n(f^2) \cdot (1-\chi)^{2b(n)-1} \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2\rangle \text{ if } b(n) > 0,$$
$$8 \cdot f_k' \cdot f^m \cdot q_n(f^2) \cdot (1-\chi)^{2n+1+2^{a(n)}(2^s-2)} \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2+s\rangle \text{ for all } s \geq 1,$$
$$8 \cdot f_k' \cdot f^m \cdot q_n(f^2) \cdot (1-\chi)^{2n} \notin 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+3\rangle.$$

We use the w_l-technology for which we need:

Lemma 5.18. *Let* $n \geq 0$, $k \geq 1$ *and* $m \in \{1, 2\}$. *We have*

$$w_l(8 \cdot f_k' \cdot f^m \cdot q_n(f^2)) = \begin{cases} \infty & l \leq a(n) \\ 2n+3-a(n)-\frac{b(n)}{2^{l-1}} & l \geq a(n)+1 \end{cases}$$

Proof. Use the formulas from Lemma 5.9, Example 5.11 and Corollary 5.15. □

Proof of Proposition 5.17. The desired results are obtained using the criteria from Theorem 5.12.

$$w_l\left(8 \cdot f_k' \cdot f^m \cdot q_n(f^2)\right) - \left(2+2n+1-l-2^{-l}\right) = \begin{cases} \infty & l \leq a(n) \\ l-a(n)-\frac{2b(n)-1}{2^l} \geq 0 & l \geq a(n)+1 \end{cases}$$

implies

$$8 \cdot f_k' \cdot f^m \cdot q_n(f^2) \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+1\rangle.$$

For $b(n) > 0$ we have

$$w_l\left(8\cdot f'_k\cdot f^m\cdot q_n(f^2)\right)-\left(2+2n+2-l-2^{-l}\right)=$$
$$\begin{cases}\infty & l\le a(n)\\ -\frac{2b(n)-1}{2^{a(n)+1}}<0 & l=a(n)+1\\ l-(a(n)+1)-\frac{2b(n)-1}{2^l}\ge 0 & l\ge a(n)+2\end{cases}$$

and

$$w_l\left(8\cdot f'_k\cdot f^m\cdot q_n(f^2)\cdot(1-\chi)^{2b(n)-1}\right)-\left(2+2n+2-l-2^{-l}\right)=$$
$$\begin{cases}\infty & l\le a(n)\\ l-(a(n)+1)\ge 0 & l\ge a(n)+1\end{cases}$$

which imply

$$8\cdot f'_k\cdot f^m\cdot q_n(f^2)\notin 4\cdot\mathbb{Z}[\chi]/I\langle 2n+2\rangle,$$
$$8\cdot f'_k\cdot f^m\cdot q_n(f^2)\cdot(1-\chi)^{2b(n)-1}\in 4\cdot\mathbb{Z}[\chi]/I\langle 2n+2\rangle.$$

For $b(n) = 0$ we have

$$w_l\left(8\cdot f'_k\cdot f^m\cdot q_n(f^2)\right)-\left(2+2n+2-l-2^{-l}\right)=$$
$$\begin{cases}\infty & l\le a(n)\\ l-(a(n)+1)+\frac{1}{2^l}\ge 0 & l\ge a(n)+1\end{cases}$$

which implies

$$8\cdot f'_k\cdot f^m\cdot q_n(f^2)\in 4\cdot\mathbb{Z}[\chi]/I\langle 2n+2\rangle.$$

From

$$w_l\left(8\cdot f'_k\cdot f^m\cdot q_n(f^2)\cdot(1-\chi)^{2n}\right)-\left(2+2n+3-l-2^{-l}\right)=$$
$$\begin{cases}\infty & l\le a(n)\\ -\frac{1}{2^{a(n)+1}}<0 & l=a(n)+1\\ l-(a(n)+2)+\frac{2^{a(n)+1}-1}{2^l}\ge 0 & l\ge a(n)+2\end{cases}$$

we conclude

$$8\cdot f'_k\cdot f^m\cdot q_n(f^2)\cdot(1-\chi)^{2n}\notin 4\cdot\mathbb{Z}[\chi]/I\langle 2n+3\rangle\quad\text{and hence}$$
$$8\cdot f'_k\cdot f^m\cdot q_n(f^2)\notin 4\cdot\mathbb{Z}[\chi]/I\langle 2n+3\rangle.$$

It remains to show

$$8\cdot f'_k\cdot f^m\cdot q_n(f^2)\cdot(1-\chi)^{2n+1+2^{a(n)}(2^s-2)}\in 4\cdot\mathbb{Z}[\chi]/I\langle 2n+2+s\rangle\text{ for all }s\ge 1.$$

$$w_l\left(8\cdot f'_k\cdot f^m\cdot q_n(f^2)\cdot(1-\chi)^{2n+1+2^{a(n)}(2^s-2)}\right)-\left(2+2n+2+s-l-2^{-l}\right)=$$
$$\begin{cases}\infty & l\le a(n)\\ l-a(n)-s-1+2^{a(n)+s-l} & l\ge a(n)+1\end{cases}$$

We set $c := a(n)+s-l$ and have to show $2^c \geq c+1$ for all $c \in \mathbb{Z}$. This is obviously true for $c \leq -1$. The statement for $c \geq 0$ follows by induction. □

Notice that the polynomials q_n have slightly better properties when $b(n) = 0$. This suggests that there might exist better polynomials than q_n when $b(n) > 0$. This turns out to be true, there exist polynomials r_n^- of degree n with leading coefficient 1 such that

$$8\cdot f'_k\cdot f^m\cdot r_n^-(f^2)\in 4\cdot\mathbb{Z}[\chi]/I\langle 2n+2\rangle.$$

Their construction needs some preparation.

Lemma 5.19. *Let $g \in 2 \cdot R^-_{\widehat{G}}$ where $G = \mathbb{Z}_{2^K}$. If $g \notin 4 \cdot R^-_{\widehat{G}}$ then there exists an odd natural number c such that*

$$g(\chi) \cdot (1-\chi)^c \in 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle,$$
$$g(\chi) \cdot (1-\chi)^{c-1} \notin 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle.$$

Proof. The element $g \in 2 \cdot R^-_{\widehat{G}}$ can be written as

$$g(\chi) = 2 \cdot \sum_{k=1}^{2^{K-1}} a_k \cdot (\chi^k - \chi^{-k})$$

with $a_k \in \mathbb{Z}$. We set

$$\overline{g}(\chi) := \sum_{k=1}^{2^{K-1}} a_k \cdot (\chi^k - \chi^{-k}) \in \mathbb{Z}_2[\chi]/I\langle K\rangle.$$

Now suppose that $g \notin 4 \cdot R^-_{\widehat{G}}$ i.e. $h \neq 0$. Since

$$\chi^k - \chi^{-k} = (\chi - \chi^{-1}) \cdot \left(\chi^{1-k} + \chi^{3-k} + \ldots + \chi^{k-3} + \chi^{k-1}\right),$$

any element $y \in \mathbb{Z}_2[\chi]/I\langle K\rangle$ of the shape $y(\chi) = \sum_{k=1}^{m} c_k \cdot (\chi^k - \chi^{-k})$ can be written as

$$y(\chi) = \left(\chi - \chi^{-1}\right) \cdot \left(c'_0 + \sum_{k=1}^{m-1} c'_k \cdot (\chi^k - \chi^{-k})\right).$$

Now, we transform $\overline{g}$ in this way and repeat the transformation as long as the occurring c'_0 is zero. We finally get

$$\overline{g}(\chi) = \left(\chi - \chi^{-1}\right)^n \cdot \left(1 + \sum_{k=1}^{2^{K-1}-n} b_k \cdot (\chi^k - \chi^{-k})\right).$$

We set $c := 2^K - 2n - 1$. Notice that we have in $\mathbb{Z}_2[\chi]$

$$1 + \chi + \ldots + \chi^{2^K-1} = \prod_{r=1}^{K-1} (1 + \chi^{2^r}) = \prod_{r=1}^{K-1} (1-\chi)^{2^r} = (1-\chi)^{2^K-1}$$

and

$$\left(\chi - \chi^{-1}\right)^n = \left(\chi^{-1} \cdot (1-\chi)^2\right)^n = \chi^{-n} \cdot (1-\chi)^{2n}.$$

Therefore, we calculate in $\mathbb{Z}_2[\chi]/I\langle K\rangle$

$$\left(\chi - \chi^{-1}\right)^n \cdot (1-\chi)^c = \chi^{-n}(1-\chi)^{2^K-1} = 0.$$

This implies $\overline{g}(\chi) \cdot (1-\chi)^c = 0$ in $\mathbb{Z}_2[\chi]/I\langle K\rangle$ and hence

$$g(\chi) \cdot (1-\chi)^c \in 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle.$$

It remains to show

$$g(\chi) \cdot (1-\chi)^{c-1} \notin 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle.$$

We prove this by contradiction. Suppose $g(\chi) \cdot (1-\chi)^{c-1} \in 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle$ which implies $\overline{g}(\chi) \cdot (1-\chi)^{c-1} = 0$ in $\mathbb{Z}_2[\chi]/I\langle K\rangle$. This means that there exists $q \in \mathbb{Z}_2[\chi]$ with

$$\left(\chi - \chi^{-1}\right)^n \cdot \left(1 + \sum_{k=1}^{2^{K-1}-n} b_k \cdot (\chi^k - \chi^{-k})\right) \cdot (1-\chi)^{c-1} = q(\chi) \cdot (1 + \chi + \ldots + \chi^{2^K-1}).$$

We conclude in $\mathbb{Z}_2[\chi]$

$$\chi^{-n} \cdot (1-\chi)^{2n} \cdot \left(1 + \sum_{k=1}^{2^{K-1}-n} b_k \cdot (\chi^k - \chi^{-k})\right) \cdot (1-\chi)^{c-1} = q(\chi) \cdot (1-\chi)^{2^K-1}$$

and hence

$$\chi^{-n} \cdot \left(1 + \sum_{k=1}^{2^{K-1}-n} b_k \cdot (\chi^k - \chi^{-k})\right) = q(\chi) \cdot (1-\chi).$$

This implies the desired contradiction

$$1 = 1^{-n} \cdot \left(1 + \sum_{k=1}^{2^{K-1}-n} b_k \cdot (1^k - 1^{-k})\right) = q(1) \cdot (1-1) = 0 \quad \text{in } \mathbb{Z}_2.$$

□

Lemma 5.20. *Let $g_i \in \mathbb{Q}[\chi]/I\langle K\rangle$ for $i = 1, 2$ and let $c \geq 1$ be such that*

$$2 \cdot g_i(\chi) \in 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle,$$
$$g_i(\chi) \cdot (1-\chi)^c \in 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle,$$
$$g_i(\chi) \cdot (1-\chi)^{c-1} \notin 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle.$$

Then

$$(g_1(\chi) + g_2(\chi)) \cdot (1-\chi)^{c-1} \in 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle.$$

Proof. Since $2 \cdot g_i(\chi) \cdot (1-\chi)^{c-1} \in 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle$, there exist polynomials $h_i \in \mathbb{Z}[\chi]$ $(i = 1, 2)$ such that $2 \cdot g_i(\chi) \cdot (1-\chi)^{c-1}$ and $4 \cdot h_i(\chi)$ coincide modulo $I\langle K\rangle$. We can require that $\deg(h_i) \leq 2^K - 2$. Let $\overline{h_i}$ be the image of h_i under the epimorphism $\mathbb{Z}[\chi] \twoheadrightarrow \mathbb{Z}_2[\chi]$. Notice that $1+\chi+\ldots+\chi^{2^K-1}$ divides $\overline{h_i}(\chi) \cdot (1-\chi)$ in $\mathbb{Z}_2[\chi]$ because of $g_i(\chi) \cdot (1-\chi)^c \in 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle$. In $\mathbb{Z}_2[\chi]$ we have

$$1 + \chi + \ldots + \chi^{2^K-1} = \prod_{r=0}^{K-1} (1 + \chi^{2^r}) = \prod_{r=0}^{K-1} (1+\chi)^{2^r} = (1+\chi)^{2^K-1}.$$

Therefore, $(1+\chi)^{2^K-2}$ divides $\overline{h_i}(\chi)$. Since $g_i(\chi) \cdot (1-\chi)^{c-1} \notin 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle$, we have $\overline{h_i}(\chi) \neq 0$. We conclude from $\deg(\overline{h_i}) \leq \deg(h_i) \leq 2^K - 2$ that $\overline{h_i}(\chi) = (1+\chi)^{2^K-2}$. Therefore,

$$\overline{h_1}(\chi) + \overline{h_2}(\chi) = 2 \cdot (1+\chi)^{2^K-2} = 0 \quad \text{in } \mathbb{Z}_2[\chi].$$

This implies $h_1(\chi) + h_2(\chi) \in 2 \cdot \mathbb{Z}[\chi]$. In $\mathbb{Q}[\chi]/I\langle K\rangle$ we finally conclude

$$(g_1(\chi) + g_2(\chi)) \cdot (1-\chi)^{c-1} = 2 \cdot (h_1(\chi) + h_2(\chi)) \in 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle.$$

□

Lemma 5.21. *Let $g \in \mathbb{Q}[\chi]/I\langle K+1\rangle$ such that $\mathrm{pr}_l(g) \in 4 \cdot \mathbb{Z}[\chi]/\langle 1 + \chi^{2^K}\rangle$. Then*

$$g \in 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle \iff 2g \in 4 \cdot \mathbb{Z}[\chi]/I\langle K+1\rangle.$$

Proof. Assume first $g \in 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle$. Let $h \in \mathbb{Z}[\chi]$ be such that $4h$ and g coincide in $\mathbb{Q}[\chi]/\langle 1+\chi^{2^K}\rangle$ and let $k \in \mathbb{Z}[\chi]$ such that $4k$ and g coincide in $\mathbb{Q}[\chi]/I\langle K\rangle$. Then we obtain in $\mathbb{Q}[\chi]/I\langle K+1\rangle$ the equation

$$2 \cdot g(\chi) = 4 \cdot (1 + \chi^{2^K}) \cdot k(\chi) + 4 \cdot (1 - \chi^{2^K}) \cdot h(\chi).$$

which shows $2g \in 4 \cdot \mathbb{Z}[\chi]/I\langle K+1\rangle$.

Now assume $2g \in 4 \cdot \mathbb{Z}[\chi]/I\langle K+1\rangle$. We want to show $g \in 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle$. Let $h \in \mathbb{Z}[\chi]$ be again such that $4h$ and g coincide in $\mathbb{Q}[\chi]/\langle 1+\chi^{2^K}\rangle$ and let $k \in \mathbb{Z}[\chi]$ be such that $4k$ and $2g$ (resp. $2k$ and g) coincide in $\mathbb{Q}[\chi]/I\langle K+1\rangle$. Then $2 \cdot k(\chi)$ and $k(\chi) \cdot (1 + \chi^{2^K}) + 2 \cdot h(\chi) \cdot (1 - \chi^{2^K})$ coincide in $\mathbb{Q}[\chi]/I\langle K\rangle$ and in $\mathbb{Q}[\chi]/\langle 1 + \chi^{2^K}\rangle$ and hence also in $\mathbb{Q}[\chi]/I\langle K+1\rangle$. Therefore there exists an $r \in \mathbb{Q}[\chi]$ with

$$2 \cdot k(\chi) = k(\chi) \cdot (1 + \chi^{2^K}) + 2 \cdot h(\chi) \cdot (1 - \chi^{2^K}) + r(\chi) \cdot (1 + \chi + \ldots + \chi^{2^{K+1}-1}).$$

We conclude $r \in \mathbb{Z}[\chi]$. Under the epimorphism $\mathbb{Z}[\chi] \twoheadrightarrow \mathbb{Z}_2[\chi]$ we get

$$0 = \overline{k}(\chi) \cdot (1 + \chi^{2^K}) + \overline{r}(\chi) \cdot (1 + \chi + \ldots + \chi^{2^{K+1}-1})$$

and hence

$$0 = \overline{k}(\chi) + \overline{r}(\chi) \cdot (1 + \chi + \ldots + \chi^{2^K-1}).$$

We set $s(\chi) := k(\chi) + r(\chi) \cdot (1 + \chi + \ldots + \chi^{2^K-1}) \in \mathbb{Z}[\chi]$. The vanishing of s under the epimorphism $\mathbb{Z}[\chi] \twoheadrightarrow \mathbb{Z}_2[\chi]$ implies the existence of $t \in \mathbb{Z}[\chi]$ with $2t = s$. We conclude in $\mathbb{Q}[\chi]/I\langle K\rangle$

$$g = 2k = 2s = 4t.$$

This shows $g \in 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle$. □

The desired polynomials r_n^- are obtained inductively. The crucial inductive step is based on the following proposition. The idea is motivated by the properties of q_n when $b(n) = 0$ and is based on the following observation: If we assume for a given $n \in \mathbb{N}$ with $b(n) > 0$ the existence of polynomials $\widetilde{q}_l$ for $l \le \lfloor \frac{n}{2} \rfloor - 1$ which are slightly better than q_l then we are able to conclude the existence of a $\widetilde{q}_n$ which is also better than q_n.

Proposition 5.22. *Let $n \ge 0$, $k \ge 1$ and $m \in \{1, 2\}$. Let $\widetilde{q}_l \in \mathbb{Z}[\chi]$ be polynomials for $0 \le l \le \lfloor \frac{n}{2} \rfloor - 1$ such that*

$$8 \cdot f_k' \cdot f^m \cdot \widetilde{q}_l(f^2) \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2l + 2\rangle,$$
$$8 \cdot f_k' \cdot f^m \cdot \widetilde{q}_l(f^2) \cdot (1 - \chi)^{2l} \notin 4 \cdot \mathbb{Z}[\chi]/I\langle 2l + 3\rangle,$$
$$8 \cdot f_k' \cdot f^m \cdot \widetilde{q}_l(f^2) \cdot (1 - \chi)^{2l+1+2^{a(l)}(2^s-2)} \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2l + 2 + s\rangle$$

for all $0 \le l \le \lfloor \frac{n}{2} \rfloor - 1$, $s \ge 1$. Then there exist unique $a_l \in \{0, 1\}$ for $0 \le l \le \lfloor \frac{n}{2} \rfloor - 1$ such that

$$\widetilde{q}_n := q_n + \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor - 1} a_l \cdot 2^{2(n-l)-1} \cdot \widetilde{q}_l$$

satisfies

$$8 \cdot f_k' \cdot f^m \cdot \widetilde{q}_n(f^2) \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n + 2\rangle.$$

Moreover, we get

$$8 \cdot f_k' \cdot f^m \cdot \widetilde{q}_n(f^2) \cdot (1 - \chi)^{2n} \notin 4 \cdot \mathbb{Z}[\chi]/I\langle 2n + 3\rangle,$$
$$8 \cdot f_k' \cdot f^m \cdot \widetilde{q}_n(f^2) \cdot (1 - \chi)^{2n+1+2^{a(n)}(2^s-2)} \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n + 2 + s\rangle$$

for all $s \ge 1$.

Proof. From Proposition 5.17 we know that $b(n) = 0$ implies

$$8 \cdot f_k' \cdot f^m \cdot q_n(f^2) \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n + 2\rangle.$$

In the case $b(n) > 0$ we have

$$8 \cdot f_k' \cdot f^m \cdot q_n(f^2) \cdot (1 - \chi)^{2b(n)-1} \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n + 2\rangle.$$

Now let $c \ge 0$ be the smallest number such that there exist coefficients a_l satisfying

$$(5.8)\quad 8 \cdot f_k' \cdot f^m \cdot \left(q_n(f^2) + \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor - 1} a_l \cdot 2^{2(n-l)-1} \cdot \widetilde{q}_l(f^2) \right) \cdot (1 - \chi)^c \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n + 2\rangle.$$

We have to show $c = 0$. We will give a proof by contradiction and assume that $c > 0$. We already know that $c \le 2b(n) - 1$. From Lemma 5.19 we conclude that c is odd. We set $l' := \frac{c-1}{2}$. Since $2l' + 1 = c \le 2b(n) - 1 \le b(n) + 2^{a(n)} - 2 = n - 1$, we conclude $l' \leq \lfloor \frac{n}{2} \rfloor - 1$. Let (a_l) be a choice of coefficients with the property (5.8). We set

$$g_1(\chi) := 8 \cdot f'_k \cdot f^m \cdot q_n(f^2) + 8 \cdot f'_k \cdot f^m \cdot \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor - 1} a_l \cdot 2^{2(n-l)-1} \cdot \widetilde{q}_l(f^2).$$

Notice that $2 \cdot g_1(\chi) \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2\rangle$ because all summands lie in this ring (use Lemma 5.21). Moreover, we have $g_1(\chi) \cdot (1-\chi) \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2\rangle$ and $g_1(\chi) \notin 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2\rangle$. Define

$$g_2(\chi) := 8 \cdot f'_k \cdot f^m \cdot (-1)^{a_{l'}} \cdot 2^{2(n-l')-1} \cdot \widetilde{q}_{l'}(f^2)$$

Using Lemma 5.21 we see that $2 \cdot g_2(\chi)$, $g_2(\chi) \cdot (1-\chi)^c \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2\rangle$ but $g_2(\chi) \cdot (1-\chi)^{c-1} \notin 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2\rangle$. Now, we can use Lemma 5.20 and get

$$(g_1(\chi) + g_2(\chi)) \cdot (1-\chi)^{c-1} \in 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle.$$

But this means that

$$8 \cdot f'_k \cdot f^m \cdot \left(q_n(f^2) + \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor - 1} a'_l \cdot 2^{2(n-l)-1} \cdot \widetilde{q}_l(f^2) \right) \cdot (1-\chi)^{c-1} \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2\rangle.$$

where the coefficients (a'_l) are given by

$$a'_l := \begin{cases} a_l & l \neq l' \\ a_{l'} + (-1)^{a_{l'}} & l = l' \end{cases}.$$

This is a contradiction to the minimality of c. So far we have shown that there exist $a_l \in \{0,1\}$ $(0 \leq l \leq \lfloor \frac{n}{2} \rfloor - 1)$ such that

$$\widetilde{q}_n := q_n + \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor - 1} a_l \cdot 2^{2(n-l)-1} \cdot \widetilde{q}_l$$

satisfies

$$8 \cdot f'_k \cdot f^m \cdot \widetilde{q}_n(f^2) \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2\rangle.$$

Our next aim is to show the uniqueness of the coefficients. We will give a proof by contradiction. Assume that there exist two different choices of coefficients (a_l), (a'_l) such that the corresponding $\widetilde{q}_n$, $\widetilde{q}'_n$ satisfy

$$8 \cdot f'_k \cdot f^m \cdot \widetilde{q}_n(f^2),\ 8 \cdot f'_k \cdot f^m \cdot \widetilde{q}'_n(f^2) \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2\rangle.$$

We set $b_l := a_l - a'_l$ and conclude

$$8 \cdot f'_k \cdot f^m \cdot \left(\sum_{l=0}^{\lfloor \frac{n}{2} \rfloor - 1} b_l \cdot 2^{2(n-l)-1} \cdot \widetilde{q}_l(f^2) \right) \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2\rangle.$$

Let $\hat{l}$ be the largest element with $b_{\hat{l}} \neq 0$ (i.e. $b_{\hat{l}} = \pm 1$). Using Lemma 5.21, we conclude

$$8 \cdot f'_k \cdot f^m \cdot 2^{2(n-l)-1} \cdot \widetilde{q}_l(f^2) \cdot (1-\chi)^{2\hat{l}} \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2\rangle \quad \text{for } l < \hat{l}$$

and

$$8 \cdot f'_k \cdot f^m \cdot 2^{2(n-l)-1} \cdot \widetilde{q}_{\hat{l}}(f^2) \cdot (1-\chi)^{2\hat{l}} \notin 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2\rangle.$$

This implies

$$8 \cdot f'_k \cdot f^m \cdot \left(\sum_{l=0}^{\lfloor \frac{n}{2} \rfloor - 1} b_l \cdot 2^{2(n-l)-1} \cdot \widetilde{q}_l(f^2) \right) \cdot (1-\chi)^{2\hat{l}} \notin 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2\rangle$$

contradicting

$$8 \cdot f'_k \cdot f^m \cdot \left(\sum_{l=0}^{\lfloor \frac{n}{2} \rfloor - 1} b_l \cdot 2^{2(n-l)-1} \cdot \widetilde{q}_l(f^2) \right) \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2 \rangle.$$

It remains to prove

$$8 \cdot f'_k \cdot f^m \cdot \widetilde{q}_n(f^2) \cdot (1-\chi)^{2n} \notin 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+3 \rangle,$$
$$8 \cdot f'_k \cdot f^m \cdot \widetilde{q}_n(f^2) \cdot (1-\chi)^{2n+1+2^{a(n)}(2^s-2)} \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2+s \rangle$$

for all $s \geq 1$. Using Lemma 5.21 we obtain

$$8 \cdot f'_k \cdot f^m \cdot 2^{2(n-l)-1} \cdot \widetilde{q}_l(f^2) \cdot (1-\chi)^{2l+1+2^{a(l)}(2^{s'}-2)} \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+1+s' \rangle$$

for all $s' \geq 2$. Let $l \leq \lfloor \frac{n}{2} \rfloor - 1$. Setting $s' := s+1$ we conclude

$$8 \cdot f'_k \cdot f^m \cdot 2^{2(n-l)-1} \cdot \widetilde{q}_l(f^2) \cdot (1-\chi)^{2n+1+2^{a(n)}(2^s-2)} \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2+s \rangle$$

because

$$2l+1+2^{a(l)}(2^{s+1}-2) \;\leq\; 2(\frac{n}{2}-1)+1+2^{a(n)-1}(2^{s+1}-2) \;=$$
$$2n+1+2^{a(n)}(2^s-2)-b(n) \;\leq\; 2n+1+2^{a(n)}(2^s-2).$$

Setting $s' := 2$ we obtain

$$8 \cdot f'_k \cdot f^m \cdot 2^{2(n-l)-1} \cdot \widetilde{q}_l(f^2) \cdot (1-\chi)^{2n} \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+3 \rangle$$

because

$$2l+1+2^{a(l)}(2^2-2) \leq 2(\frac{n}{2}-1)+1+2^{a(n)} = 2n-b(n) \leq 2n.$$

Therefore, it suffices to show

$$8 \cdot f'_k \cdot f^m \cdot q_n(f^2) \cdot (1-\chi)^{2n} \notin 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+3 \rangle,$$
$$8 \cdot f'_k \cdot f^m \cdot q_n(f^2) \cdot (1-\chi)^{2n+1+2^{a(n)}(2^s-2)} \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2+s \rangle$$

for all $s \geq 1$. But this was proved in Proposition 5.17. □

Notice that the assumptions in Proposition 5.22 are trivially fulfilled if $n = 0, 1$.

Definition 5.23. We define $r_n^- \in \mathbb{Z}[\chi]$ as the polynomials $\widetilde{q}_n$ we obtain successively from Proposition 5.22 starting with $n = 0$ and proceeding with $n = 1, 2, 3, \ldots$.

For example, $r_0^- = q_0$, $r_1^- = q_1$, $r_2^- = q_2 + 2^3 \cdot q_0$, $r_3^- = q_3$, $r_4^- = q_4 + 2^7 \cdot r_0$.

Corollary 5.24. *The polynomial r_n^- is of degree $n \in \mathbb{N}$ with leading coefficient 1 and it satisfies*

$$8 \cdot f'_k \cdot f^m \cdot r_n^-(f^2) \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2 \rangle,$$
$$8 \cdot f'_k \cdot f^m \cdot r_n^-(f^2) \cdot (1-\chi)^{2n} \notin 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+3 \rangle,$$
$$8 \cdot f'_k \cdot f^m \cdot r_n^-(f^2) \cdot (1-\chi)^{2n+1+2^{a(n)}(2^s-2)} \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2+s \rangle$$

for all $s \geq 1$.

Are the polynomials r_n^- best possible? Or does there exist a polynomial q of degree n with leading coefficient 1 such that $8 \cdot f_k \cdot q(f^2) \in \mathbb{Z}[\chi]/I\langle 2n+3 \rangle$? In the next section we will see that any polynomial q of degree n with the property $8 \cdot f_k \cdot q(f^2) \in \mathbb{Z}[\chi]/I\langle 2n+3 \rangle$ is of the shape

$$\sum_{l=0}^{n} a_l \cdot 2^{\max\{2(n-l)+1,0\}} \cdot r_l^-$$

with $a_n \in \mathbb{Z}$. Hence such a polynomial can not have 1 as leading coefficient.

5.3. **The equation $\mathbf{A_K^k(d) = B_K(d)}$.**

In this subsection we first prove $A_K^k(d) = B_K(d)$ for $d = 2e+1$. Recall that

$$A_K^k(2e+1) := \left\{q \in \mathbb{Z}[x] \mid \deg(q) \le e-1,\ 8 \cdot f_k \cdot q(f^2) \in 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle\right\},$$

$$B_K(2e+1) := \left\{\sum_{n=0}^{e-1} a_n \cdot 2^{\max\{K-2n-2,0\}} \cdot r_n^- \mid a_n \in \mathbb{Z}\right\}.$$

We want to consider a slightly more general situation and prove $A_K^{k,m}(2e+1) = B_K(2e+1)$ where $A_K^{k,m}(2e+1)$ is defined as follows.

Definition 5.25. Let $K, k \ge 1$, $e \ge 2$, $m \in \{1,2\}$. Define

$$A_K^{k,m}(2e+1) := \left\{q \in \mathbb{Z}[x] \mid \deg(q) \le e-1,\ 8 \cdot f'_k \cdot f^m \cdot q(f^2) \in 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle\right\}$$

Notice that $A_K^{k,1}(2e+1) = A_K^k(2e+1)$.

Theorem 5.26. *Let $K, k \ge 1$, $e \ge 2$, $m \in \{1,2\}$. Then $A_K^{k,m}(2e+1) = B_K(2e+1)$. In particular,*

$$A_K^k(2e+1) = B_K(2e+1).$$

Proof. Since $8 \cdot f'_k \cdot f^m \cdot r_n^-(f^2) \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2\rangle$, Lemma 5.21 implies

$$8 \cdot f'_k \cdot f^m \cdot 2^{\max\{K-2n-2,0\}} \cdot r_n^-(f^2) \in 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle.$$

This proves $A_K^{k,m}(2e+1) \supseteq B_K(2e+1)$.

It remains to show $A_K^{k,m}(2e+1) \subseteq B_K(2e+1)$. We will give a proof by induction with respect to K. For the basis case $K=1$ we get

$$A_1^{k,m}(2e+1) = \{q \in \mathbb{Z}[x] \mid \deg(q) \le e-1\} = B_1(2e+1).$$

Inductive step: We assume that $A_{K-1}^{k,m}(2e+1) \subseteq B_{K-1}(2e+1)$ $(K \ge 2)$ and have to prove $A_K^{k,m}(2e+1) \subseteq B_K(2e+1)$. Let $q \in A_K^{k,m}(2e+1)$. Since

$$A_K^{k,m}(2e+1) \subseteq A_{K-1}^{k,m}(2e+1) \subseteq B_{K-1}(2e+1),$$

we can write q as $q = \sum_{n=0}^{e-1} a_n \cdot 2^{\max\{K-2n-3,0\}} \cdot r_n^-$ with $a_n \in \mathbb{Z}$. The polynomial q lies in $B_K(2e+1)$ if a_n is even for all n with $2n+2 \le K-1$. We set

$$M := \{0 \le n \le e-1 \mid 2n+2 \le K-1,\ a_n \text{ is odd}\}.$$

It remains to show $M = \emptyset$. We will give a proof by contradiction and assume $M \neq \emptyset$. Since $q \in A_K^{k,m}(2e+1)$ and

$$\sum_{n \notin M} a_n \cdot 2^{\max\{K-2n-3,0\}} \cdot r_n^- + \sum_{n \in M} (a_n - 1) \cdot 2^{\max\{K-2n-3,0\}} \cdot r_n^- \in B_K(2e+1) \subseteq A_K^{k,m}(2e+1),$$

we have

$$\sum_{n \in M} 2^{K-2n-3} \cdot r_n^- \in A_K^{k,m}(2e+1).$$

This implies

$$\sum_{n \in M} 2^{K-2n} \cdot f'_k \cdot f^m \cdot r_n^-(f^2) \cdot (1-\chi)^{2 \cdot \max(M)} \in 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle. \tag{5.9}$$

Using Lemma 5.21 we conclude from

$$8 \cdot f'_k \cdot f^m \cdot r_n^-(f^2) \cdot (1-\chi)^{2n+1} \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+3\rangle$$

that

$$2^{K-2n} \cdot f'_k \cdot f^m \cdot r_n^-(f^2) \cdot (1-\chi)^{2n+1} \in 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle$$

and hence, if $n < \max(M)$ then

$$2^{K-2n} \cdot f_k' \cdot f^m \cdot r_n^-(f^2) \cdot (1-\chi)^{2\cdot\max(M)} \in 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle. \tag{5.10}$$

The property

$$8 \cdot f_k' \cdot f^m \cdot r_{\max(M)}^-(f^2) \cdot (1-\chi)^{2\cdot\max(M)} \notin 4 \cdot \mathbb{Z}[\chi]/I\langle 2\cdot\max(M)+3\rangle$$

and Lemma 5.21 imply

$$2^{K-2\cdot\max(M)} \cdot f_k' \cdot f^m \cdot r_{\max(M)}^-(f^2) \cdot (1-\chi)^{2\cdot\max(M)} \notin 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle. \tag{5.11}$$

Combining (5.9), (5.10) and (5.11) we obtain the desired contradiction. □

We now come to the case $d = 2e$.

Definition 5.27. Define $\beta : \mathbb{Z}[x] \to \mathbb{Z}[x]$ by

$$\beta(q)(x) := \frac{x \cdot q(x) - q(1)}{x-1}$$

and set

$$r_n^+ := \beta(r_n^-) \text{ for } n \geq 0.$$

Notice that β is an isomorphism of $\mathbb{Z}$-modules and preserves the degree of the polynomial. The inverse is given by

$$\beta^{-1}(q)(x) = \frac{(x-1)\cdot q(x) + q(0)}{x}.$$

r_n^+ is a polynomial of degree n with leading coefficient 1.

Theorem 5.28. *Let $K, k \geq 1$, $e \geq 3$. Then*

$$A_K^k(2e) = B_K(2e).$$

Proof. Recall that

$$A_K^k(2e) := \left\{q \in \mathbb{Z}[x] \mid \deg(q) \leq e-2,\ 8 \cdot f_k' \cdot (f^2-1) \cdot q(f^2) \in 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle\right\},$$

$$B_K(2e) := \left\{\sum_{n=0}^{e-2} a_n \cdot 2^{\max\{K-2n-2,0\}} \cdot r_n^+ \mid a_n \in \mathbb{Z}\right\}.$$

For $q \in \mathbb{Z}[x]$ with $\deg(q) \leq e-2$ we conclude

$$\begin{aligned}
&q \in A_K^k(2e) \iff \\
&\iff 8 \cdot f_k' \cdot (f^2-1) \cdot q(f^2) \in 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle \\
&\iff 8 \cdot f_k' \cdot \big((f^2-1)\cdot q(f^2) + q(0)\big) \in 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle \quad (\text{since } f_k' \in R_{\widehat{G}}) \\
&\iff 8 \cdot f_k' \cdot f^2 \cdot \beta^{-1}(q)(f^2) \in 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle \\
&\iff \beta^{-1}(q) \in A_K^{k,2}(2e-1) \\
&\iff \beta^{-1}(q) \in B_K(2e-1) \quad (\text{see Theorem 5.26}) \\
&\iff q \in B_K(2e).
\end{aligned}$$

□

References

[AS68] M. F. Atiyah and I. M. Singer. The index of elliptic operators. III. *Ann. of Math. (2)*, 87:546–604, 1968.

[CF64] P. E. Conner and E. E. Floyd. *Differentiable periodic maps*. Ergebnisse der Mathematik und ihrer Grenzgebiete, N. F., Band 33. Academic Press Inc., Publishers, New York, 1964.

[HT00] Ian Hambleton and Laurence R. Taylor. A guide to the calculation of the surgery obstruction groups for finite groups. In *Surveys on surgery theory, Vol. 1*, volume 145 of *Ann. of Math. Stud.*, pages 225–274. Princeton Univ. Press, Princeton, NJ, 2000.

[LdM71] S. López de Medrano. *Involutions on manifolds*. Springer-Verlag, New York, 1971. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 59.

[Lüc02] Wolfgang Lück. A basic introduction to surgery theory. In *Topology of high-dimensional manifolds, No. 1, 2 (Trieste, 2001)*, volume 9 of *ICTP Lect. Notes*, pages 1–224. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2002.

[Mil66] J. Milnor. Whitehead torsion. *Bull. Amer. Math. Soc.*, 72:358–426, 1966.

[MM79] Ib Madsen and R. James Milgram. *The classifying spaces for surgery and cobordism of manifolds*, volume 92 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, N.J., 1979.

[Pet70] Ted Petrie. The Atiyah-Singer invariant, the Wall groups $L_n(\pi, 1)$, and the function $(te^x + 1)/(te^x - 1)$. *Ann. of Math. (2)*, 92:174–187, 1970.

[Ran81] Andrew Ranicki. *Exact sequences in the algebraic theory of surgery*, volume 26 of *Mathematical Notes*. Princeton University Press, Princeton, N.J., 1981.

[Ran92] A. A. Ranicki. *Algebraic L-theory and topological manifolds*, volume 102 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1992.

[Ran08] Andrew Ranicki. A composition formula for manifold structures. *Preprint, arXiv: math.AT.0608705*, 2008.

[Wal99] C. T. C. Wall. *Surgery on compact manifolds*, volume 69 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 1999. Edited and with a foreword by A. A. Ranicki.

Mathematisches Institut, Universität Münster, Einsteinstrasse 62, Münster, D-48149, Germany, and Matematický Ústav SAV, Štefánikova 49, Bratislava, SK-81473, Slovakia

E-mail address: macko@math.uni-muenster.de

Mathematisches Institut, Universität Münster, Einsteinstrasse 62, Münster, D-48149, Germany

E-mail address: c.wegner@uni-muenster.de