CALCULUS OF EMBEDDINGS

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Abstract. Let M and N be smooth manifolds, where $M \subset N$ and $\dim(N) - \dim(M) \geq 3$. A disjunction lemma for embeddings proved recently by Goodwillie leads to a calculation *up to extension problems* of the base point component of the space of smooth embeddings of M in N. This is mostly in terms of $\mathbf{imm}(M, N)$, the space of smooth immersions, which is well understood, and embedding spaces $\mathbf{emb}(S, N)$ for finite subsets S of M with few elements. The meaning of *few* depends on the precision desired.

0. Introduction: Immersions vs. Embeddings

Let M^m and N^n be smooth manifolds without boundary. Suppose that $m \leq n$, and if $m = n$ suppose that M has no compact component. Write $\mathbf{mono}(TM, TN)$ for the space of vector bundle monomorphisms $TM \to TN$. Such a vector bundle monomorphism consists of a continuous map $f : M \to N$ and, for each $x \in M$, a linear monomorphism $T_xM \to T_{f(x)}N$ which depends continuously on x. For example, an immersion $f : M \to N$ has a differential $df : TM \to TN$ which belongs to $\mathbf{mono}(TM, TN)$. In this way,

$$\mathbf{imm}(M, N) \subset \mathbf{mono}(TM, TN) \tag{*}$$

where $\mathbf{imm}(M, N)$ is the space of smooth immersions from M to N. The main theorem of immersion theory states that the inclusion (*) is a homotopy equivalence. The clearest references for this are perhaps [HaePo], for the PL analog, and [Hae1], but the theorem goes back to Smale [Sm] and Hirsch [Hi]. Smale's stunning discovery that the immersions $\mathbb{S}^2 \to \mathbb{R}^3$ given by $x \mapsto x$ and $x \mapsto -x$ respectively are *regularly homotopic* (homotopic through immersions) is a direct consequence of the main theorem.

There is reformulation of the theorem which can only be decoded with a little homotopy theory. It goes like this. The rule $V \mapsto \mathbf{imm}(V, N)$ is a contravariant functor (cofunctor for short) from the poset $\mathcal{O}$ of open subsets $V \subset M$ to spaces. That is, an inclusion $V \subset W$ determines a restriction map $\mathbf{imm}(W, N) \to \mathbf{imm}(V, N)$. The cofunctor is clearly a sheaf, so that

$$\begin{array}{ccc} \mathbf{imm}(V_1 \cup V_2, N) & \longrightarrow & \mathbf{imm}(V_1, N) \\ \downarrow & & \downarrow \\ \mathbf{imm}(V_2, N) & \longrightarrow & \mathbf{imm}(V_1 \cap V_2, N) \end{array} \tag{**}$$

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is a pullback square for arbitrary open subsets V_1, V_2 of M. Much less obvious is the fact that (**) is also a *homotopy pullback square* (see end of §1). This fact constitutes the reformulated main theorem. It means for example that the homotopy groups of the four spaces in (**) appear in a long exact Mayer–Vietoris sequence, provided suitable base points have been selected. A homotopy theorist would put it like this: the cofunctor under consideration is *excisive*. Or, if he/she has been exposed to Goodwillie calculus: the cofunctor is *(polynomial) of degree* ≤ 1.

To see that the reformulation implies the original statement we let $E(V) = \mathbf{imm}(V, N)$ and $F(V) = \mathbf{mono}(TV, TN)$ for $V \in \mathcal{O}$. There is a natural inclusion $E(V) \subset F(V)$ as in (*). We want to show that it is a homotopy equivalence for all V, in particular for $V = M$. But E and F are both excisive cofunctors (in the case of F, this is obvious). Therefore, by an inductive argument which uses triangulations or handle decompositions, it is enough to check that $E(V) \hookrightarrow F(V)$ is a homotopy equivalence when V is diffeomorphic to $\mathbb{R}^m$ and when V is empty. This is again obvious.

We see that the excision alias degree ≤ 1 property of the cofunctor $V \mapsto \mathbf{imm}(V, N)$ leads directly to "calculations" of spaces of immersions. The key property of cofunctors of degree ≤ 1 from $\mathcal{O}$ to spaces is that they are in a homotopy theoretic sense *locally determined*, determined by their behaviour on standard (=tubular) neighborhoods of single points (and their value on $\emptyset$).

Here we want to investigate the cofunctor taking an open $V \subset M$ to the space of embeddings $\mathbf{emb}(V, N)$. This cofunctor is not polynomial of degree ≤ 1 as a rule. However, there is the more general concept of a cofunctor (from $\mathcal{O}$ to spaces) which is *polynomial of degree* $\leq k$ for some $k \geq 0$. Such a cofunctor is still multi–locally determined in a homotopy theoretic sense, determined by its behaviour on tubular neighborhoods of subsets S of M with not more than k elements. See §2 for details. As a rule the embedding cofunctor $V \mapsto \mathbf{emb}(V, N)$ is not polynomial of degree $\leq k$ for any k, but if $n - m \geq 3$ it is *analytic*—it can be nicely approximated by polynomial cofunctors. In particular, its Taylor series converges to it. (There is a small problem with components, and it would be more accurate to speak of componentwise convergence.) The "terms" of the Taylor series of the embedding cofunctor are very easily described (see 4.2, 4.3, 3.3 below). They are also cofunctors—of a very special type—from the poset of open subsets of M to spaces. Taking $V = M$, and assuming $M \subset N$, the convergence theorem then amounts to a calculation up to extension problems of the base point component of $\mathbf{emb}(M, N)$ if $n - m \geq 3$. Haefliger's theory [Hae2] of embeddings in the stable range (some would say: metastable range) is closely related and appears as a calculation by second order Taylor approximation.

I emphasize that the only hard theorem in this report, Theorem 4.5, is due to Goodwillie. The so–called main theorem, 4.4, is really a corollary, and this was also pointed out by Goodwillie. The calculus outlined here is a way to exploit theorem 4.5 for the calculation of spaces of embeddings. It has many features in common with Goodwillie's calculus of homotopy functors, but it is a much smaller machine. Large parts of it were known to Goodwillie before he invented the calculus of homotopy functors. More historical comments can be found at the end of the paper, §6.

1. Homotopy Limits and Homotopy Colimits

Any functor F from a small category $\mathcal{A}$ to the category of spaces has a *colimit* (alias direct limit), which is a quotient space $\operatorname{colim} F$ of the coproduct (alias disjoint union) $\coprod_a F(a)$; and a *limit*, which is a subspace $\lim F$ of the product $\prod_a F(a)$. See [K]. Suppose that $w : E \to F$ is a natural transformation between such functors ; suppose also that $w_a : E(a) \to F(a)$ is a homotopy equivalence for any object a in $\mathcal{A}$. We call such a w an *equivalence*. One might hope that the maps induced by w from $\operatorname{colim} E$ to $\operatorname{colim} F$ and from $\lim E$ to $\lim F$ are again homotopy equivalences, but this is not always the case. For example, the diagram $\{0\} \hookrightarrow [0,1] \hookleftarrow \{1\}$ can be viewed as a functor from a certain category with three objects to spaces ; it has empty limit, but there is an equivalence from it to $\{0\} \longrightarrow \{0\} \longleftarrow \{0\}$, which has nonempty limit. Homotopy colimits and homotopy limits were invented to repair deficiencies of this sort. Actually, the current view seems to be that the concepts of *colimit* and *limit* are good enough, but the functors to which we want to apply them are not always of the best quality. Following are some quality criteria. All functors in sight are from $\mathcal{A}$ to the category of spaces.

For the purposes of this discussion a functor E is *cofibrant* if, for any diagram of functors and natural transformations

$$E \xrightarrow{v} D \xleftarrow{w} D'$$

where w is an equivalence, there exists a natural transformation $v' : E \to D'$ and a *natural* homotopy $E(a) \times [0,1] \to D(a)$ (for all a) connecting wv' and v. Dually, a functor G is *fibrant* if, for any diagram of functors and natural transformations

$$G \xleftarrow{p} H \xrightarrow{q} H'$$

where q is an equivalence, there exists a natural transformation $p' : H' \to G$ and a *natural* homotopy connecting $p'q$ and p. It is an exercise to show the following: If $v : E_1 \to E_2$ is an equivalence, and both E_1 and E_2 are cofibrant, then v has a natural homotopy inverse (with *natural* homotopies) and therefore $v_* : \operatorname{colim} E_1 \to \operatorname{colim} E_2$ is a homotopy equivalence. Similarly, if $v : E_1 \to E_2$ is an equivalence, where both E_1 and E_2 are fibrant, then $v_* : \lim E_1 \to \lim E_2$ is a homotopy equivalence.

This suggests the following procedure for making good limits out of bad functors. Suppose that F from $\mathcal{A}$ to spaces is any functor. Try to find an equivalence $F^\flat \to F$ where $F^\flat$ is cofibrant, and try to find another equivalence $F \to F^\sharp$ where $F^\sharp$ is fibrant. Then define

$$\operatorname{hocolim} F := \operatorname{colim} F^\flat , \qquad \operatorname{holim} F := \lim F^\sharp .$$

If it can be done, $\operatorname{hocolim} F$ and $\operatorname{holim} F$ are at least well defined up to homotopy equivalence. [BK] and [Dr] show that it can indeed be done and their constructions of $F^\flat$ and $F^\sharp$ are natural in the variable F. Beware however that *space* means *simplicial set* in [BK]. See [Ma]. Incidentally: A cofunctor from $\mathcal{A}$ to spaces is a functor from $\mathcal{A}^{\mathrm{op}}$ to spaces, and as such it has again a homotopy colimit and a homotopy limit.

It is clear from the constructions that $\operatorname{hocolim} F$ comes with a canonical map to $\operatorname{colim} F$, and $\lim F$ comes with a canonical map to $\operatorname{holim} F$. *Notation:* It is often convenient to write $\operatorname{holim}_{\mathcal{A}} F$ or $\operatorname{holim}_a F(a)$ etc., the latter especially when the category (here $\mathcal{A}$) is nameless but the objects (here a) have familiar names.

Here are a few examples for illustration. A diagram of spaces and maps $X \leftarrow Y \rightarrow Z$ can be viewed as a functor F from a certain category with three objects to spaces. To define $F^\flat$ in this case, leave Y unchanged and replace X and Z by the mapping cylinders of $X \leftarrow Y$ and $Y \rightarrow Z$, respectively. Then $\operatorname{hocolim} F = \operatorname{colim} F^\flat$ is the *homotopy pushout* of $X \leftarrow Y \rightarrow Z$. Next, a diagram of spaces and maps $U \rightarrow V \leftarrow W$ can be viewed as a functor G from a certain category with three objects to spaces. To define $G^\sharp$ in this case, leave V unchanged and change U and W by converting $U \rightarrow V$ and $V \leftarrow W$ into fibrations (Serre's method). Then $\operatorname{holim} G = \lim G^\sharp$ is the *homotopy pullback* of $U \rightarrow V \leftarrow W$. When W consists of a single point, with image x in V, then $\operatorname{holim} G$ is also called the *homotopy fiber* of $U \rightarrow V$ over x.

Let S be a finite set with k elements. A covariant functor $\mathcal{X}$ from the poset of all subsets of S to spaces is called a *k–cube* of spaces. (It can be visualized as a cubical diagram, especially for $k = 3$.) The k–cube $\mathcal{X}$ is *homotopy cartesian* or just *cartesian* if the map from $\mathcal{X}(\emptyset)$ to $\operatorname{holim}_{R\neq\emptyset} \mathcal{X}(R)$ which it determines (via $\lim_{R\neq\emptyset} \mathcal{X}(R)$) is a homotopy equivalence. When $k = 2$ it is more common to speak of a *homotopy pullback square.*

2. Polynomial Cofunctors

M and $\mathcal{O}$ are as in the introduction. We want to study certain cofunctors from $\mathcal{O}$ to spaces. Readers who are concerned about technical points should substitute *simplicial set* for *space.* (There is a standard procedure for converting arbitrary spaces into simplicial sets.)

2.1. Examples. For any smooth manifold N of dimension $n \geq m$, we have cofunctors from $\mathcal{O}$ to spaces given by $V \mapsto \mathbf{emb}(V, N)$ (space of smooth embeddings), and $V \mapsto \mathbf{imm}(V, N)$ (space of smooth immersions).

2.2. Definition. A cofunctor F from $\mathcal{O}$ to spaces is *good* if (like the examples in 2.1)

(a) it takes isotopy equivalences (explanation just below) to homotopy equivalences ;

(b) for any sequence $\{V_i \mid i \geq 0\}$ of objects in $\mathcal{O}$ with $V_i \subset V_{i+1}$ for all $i \geq 0$, the canonical map $F(\cup_i V_i) \longrightarrow \operatorname{holim}_i F(V_i)$ is a homotopy equivalence.

A smooth codimension zero embedding $i_1 : V \rightarrow W$ of smooth manifolds is an *isotopy equivalence* if there exists a smooth embedding $i_2 : W \rightarrow V$ such that $i_1 i_2$ and $i_2 i_1$ are smoothly isotopic to id_W and id_V, respectively.

2.3. Notation/Terminology. $\mathcal{F}$ is the category of all good cofunctors F from $\mathcal{O}$ to spaces. The morphisms in $\mathcal{F}$ are the natural transformations. Two objects in $\mathcal{F}$ are *equivalent* if they can be connected by a chain of equivalences (see §1).

We take the Taylor approach to calculus, which means that we have to say which cofunctors F in $\mathcal{F}$ qualify as polynomial cofunctors of degree $\leq k$, and how arbitrary cofunctors F in $\mathcal{F}$ can be approximated by polynomial cofunctors. See [Go1], [Go2], [Go3]. Suppose that F belongs to $\mathcal{F}$ and that V belongs to $\mathcal{O}$, and let $A_0, A_1, \dots, A_k$ be pairwise disjoint closed subsets of V. We make a $(k+1)$–cube of spaces (see end of §1) by

$$(*) \qquad S \mapsto F\big(V \smallsetminus \cup_{i\in S} A_i\big)$$

for $S \subset \{0, 1, \ldots, k\}$. Inspired by [Go2, 3.1] we decree:

2.4. Definition. The cofunctor F is *polynomial of degree* $\leq k$ if the $(k+1)$–cube (*) is cartesian for arbitrary V in $\mathcal{O}$ and pairwise disjoint closed subsets $A_0, \ldots, A_k$ of V.

2.5. Example. The cofunctor $V \mapsto \mathbf{imm}(V, N)$ (notation of 2.1) is polynomial of degree ≤ 1 if either $n > m$ or $n = m$ and M has no compact component.

2.6. Example. F is polynomial of degree 0 if and only if it is equivalent to a constant cofunctor.

2.7. Example. If F is polynomial of degree $\leq k$, then it is also polynomial of degree $\leq k+1$. This is not trivial, but it is a consequence of the fact that a $(k+2)$–cube of spaces is cartesian if two opposing $(k+1)$–dimensional faces of it are cartesian. See [Go2, 1.6].

For $k \geq 0$ let $\mathcal{O}k \subset \mathcal{O}$ be the full subcategory (sub–poset) consisting of all V which are diffeomorphic to a disjoint union of $\leq k$ copies of $\mathbb{R}^m$. There is an important relationship between $\mathcal{O}k$ and definition 2.4 which goes roughly like this: A good cofunctor F which is polynomial of degree $\leq k$ is determined by its restriction to $\mathcal{O}k$, and that restriction can be arbitrarily prescribed. The next two theorems state it with more precision.

2.8. Theorem. *Let $w : E \to F$ be a morphism in $\mathcal{F}$, where E and F are polynomial of degree $\leq k$. If $w|\mathcal{O}k : E|\mathcal{O}k \to F|\mathcal{O}k$ is an equivalence, then w is an equivalence.*

2.9. Theorem. *Suppose that E is a cofunctor from $\mathcal{O}k$ to spaces taking isotopy equivalences to homotopy equivalences. Then the cofunctor $E^!$ from $\mathcal{O}$ to spaces defined by*

$$E^!(W) := \operatorname*{holim}_{\substack{V\subset W\\ V\in\mathcal{O}k}} E(V)$$

is polynomial of degree $\leq k$, and $E^!|\mathcal{O}k$ is equivalent to E.

The formula for $E^!$, but without the prefix *ho*, is familiar to category theorists under the name *right Kan extension.*— Theorems 2.8 and 2.9 suggest the following definition. For F in $\mathcal{F}$, let T_kF in $\mathcal{F}$ be the cofunctor defined by

$$T_kF = (F|\mathcal{O}k)^! \,.$$

Then T_kF is polynomial of degree $\leq k$, and from the definitions there is a canonical natural transformation $\eta_k : F \to T_kF$. The next theorem tries to say that $\eta_k : F \to T_kF$ is the best approximation of F by a cofunctor which is polynomial of degree $\leq k$. It is mostly a restatement of 2.8 and 2.9.

2.10. Theorem. (Similar to [We1, 6.3].) *The functor $T_k : \mathcal{F} \to \mathcal{F}$ and the natural transformation $\eta_k : \mathrm{id}_{\mathcal{F}} \longrightarrow T_k$ have the following properties:*

- *T_k takes equivalences to equivalences.*
- *T_kF is polynomial of degree $\leq k$, for all F in $\mathcal{F}$.*
- *If F is polynomial of degree $\leq k$, then $\eta_k : F \to T_kF$ is an equivalence.*
- *For every F in $\mathcal{F}$, the map $T_k(\eta_k) : T_kF \to T_kT_kF$ is an equivalence.*

2.11. Notation/Terminology. We call T_kF the *k–th Taylor approximation* of F. From the explicit formula for T_kF, there are forgetful maps $r_k : T_kF \to T_{k-1}F$ such that $\eta_{k-1} = r_k\eta_k : F \to T_{k-1}F$.

Digression. There exists a sheaf–theoretic approach to polynomial cofunctors. We saw already in the introduction that polynomial cofunctors of degree ≤ 1 are in a homotopy theoretic sense locally determined. The word *local* indicates the presence of a Grothendieck topology [MaMoe] on the category (poset) $\mathcal{O}$. This is the usual one, denoted $\mathcal{J}_1$, where a family of morphisms $\{V_i \hookrightarrow W \mid i \in S\}$ in $\mathcal{O}$ qualifies as a covering if each element of W is contained in some V_i. Polynomial cofunctors of degree $\leq k$ in $\mathcal{F}$ are also in a homotopy theoretic sense locally determined, but here the word *local* refers to another Grothendieck topology $\mathcal{J}_k$ on $\mathcal{O}$. In $\mathcal{J}_k$, a family of morphisms $\{V_i \hookrightarrow W \mid i \in S\}$ qualifies as a covering if each subset of W of cardinality $\leq k$ is contained in some V_i. In this approach T_kF appears as the homotopy sheafification of F with respect to the Grothendieck topology $\mathcal{J}_k$. See the introduction to [We2] for more details.

3. Homogeneous Cofunctors

3.1. Definition. A cofunctor F in $\mathcal{F}$ is *homogeneous* of degree k if it is polynomial of degree $\leq k$ and $T_{k-1}F(V)$ is contractible for all V.

3.2. Example. Choose a point $x \in F(M)$. Since M is the terminal object in $\mathcal{O}$, this makes $F(V)$ into a pointed space for every V, and we let

$$L_kF(V) := \text{hofiber}\big[T_kF(V) \xrightarrow{r_k} T_{k-1}F(V)\big]$$

where *hofiber* is short for homotopy fiber (and L is short for layer, say). Then L_kF is homogeneous of degree k.

3.3. Example. Let $\binom{M}{k}$ be the space of subsets of M of cardinality k. This is the complement of the fat diagonal in the k–fold symmetric product $(M\times M \ldots \times M)/\Sigma_k$. Suppose that

$$p : Z \longrightarrow \binom{M}{k}$$

is a fibration, with a (partial) section $s : \binom{M}{k} \cap Q \to Z$ where Q is a neighborhood of the fat diagonal in the k–fold symmetric product. For V in $\mathcal{O}$ let $F(V)$ be the space of sections of p which are defined on $\binom{V}{k}$ and agree with s on $\binom{V}{k} \cap Q'$ for some neighborhood Q' of the fat diagonal, where $Q' \subset Q$. The cofunctor F is homogeneous of degree k. Informally, and in view of the next theorem, we say that F is *classified* by the fibration p and the partial section s.

3.4. Theorem. *Every F in $\mathcal{F}$ which is homogeneous of degree k is equivalent to one of the cofunctors described in* 3.3.

Remark. It can very well happen that F in $\mathcal{F}$ is homogeneous of degree k and $F(M) = \emptyset$. Of course it will not happen if F is a cofunctor from $\mathcal{O}$ to *pointed* spaces, as in example 3.2. There is a classification theorem for homogeneous cofunctors from $\mathcal{F}$ to pointed spaces which looks much like 3.4, except that the section s must be defined on all of $\binom{M}{k}$.

Returning to the situation of 3.2, we face the following question: what does the fibration $p : Z \to \binom{M}{k}$ which classifies the homogeneous cofunctor L_kF look like ?

In general, the fiber of p over some point S is easily described, and the total space of p is difficult to understand. The fiber of p over $S \in \binom{M}{k}$ is the *total homotopy fiber* (explanation follows) of the k–cube of pointed spaces

$$(*) \qquad R \mapsto F(V(R)) \qquad (R \subset S)$$

where $V(R)$ is a tubular neighbourhood of the finite set R in M. Note that $R \subset S \subset M$.

3.5. Explanation. Suppose that $\mathcal{X}$ is a cofunctor from the poset $\mathcal{P}_S$ of subsets of S to pointed spaces. The total homotopy fiber of $\mathcal{X}$ is the homotopy fiber of the canonical map

$$\mathcal{X}(S) \longrightarrow \operatorname*{holim}_{R \neq S} \mathcal{X}(R)\,.$$

It measures how far the cube deviates from being cartesian—if it is, then the total homotopy fiber is contractible. (The converse is almost true.) Note that $\mathcal{P}_S$ is isomorphic to its own opposite, so it is permitted to use the *cube* terminology for both covariant and contravariant functors on $\mathcal{P}_S$.

The size of the tubular neighbourhoods $V(R)$ in (*) is not clearly defined, but since F takes isotopy equivalences to homotopy equivalences this is not a serious objection. Since the total homotopy fiber of a cube of pointed spaces is a pointed space, we seem to have constructed a fibration with section—but it is still not clear how the various fibers must be glued together to make up the total space Z of p. *Idea:* Suppose that $S_1, S_2 \in \binom{M}{k}$ and that S_1 is "close" to S_2. Suppose we have selected small tubular neighbourhoods V_1 and V_2 for S_1 and S_2, respectively. Then we can find an object V_3 in $\mathcal{O}$ containing $V_1 \cup V_2$, such that the inclusions $V_1 \subset V_3$ and $V_2 \subset V_3$ are isotopy equivalences. Then

$$F(V_1) \xleftarrow{\simeq} F(V_3) \xrightarrow{\simeq} F(V_2)$$

which tells us something about how to glue.

4. The Embedding Cofunctor

Here we assume that M is a smooth submanifold of N, and apply calculus to the cofunctor F from $\mathcal{O}$ to *pointed* spaces given by $F(V) = \mathbf{emb}(V, N)$. To simplify some constructions below, we also assume that M comes with a Riemannian metric.

What can we say about the homogeneous parts L_kF ? The first observation is that the difficulties we had earlier in describing the total space Z of the classifying fibration for L_kF have disappeared. Z can now be described as the space of all triples (S, ε, z) where $S \in \binom{M}{k}$ and $\varepsilon > 0$ is sufficiently small, and z is a point in the total homotopy fiber of the k–cube

$$(*) \qquad R \mapsto F(V_\varepsilon(R)) \qquad (R \subset S)$$

where $V_\varepsilon(R)$ is the ε–neighbourhood of R. (The restrictions on ε are: less than half the minimum distance between distinct points of S, and less than the injectivity radius of the exponential map at any of the points $x \in S$.) Define $p : Z \to \binom{M}{k}$ by $p(S, \varepsilon, z) := S$.

The second observation is that the fibration p is fiber homotopy equivalent to another fibration with a much simpler description, provided $k > 1$. Namely, the k–cube (*) maps by restriction to another k–cube

$$(**) \qquad R \mapsto \mathbf{emb}(R, N) \qquad (R \subset S)$$

since $F(V_\varepsilon(R)) = \mathbf{emb}(V_\varepsilon(R)\,,\, N)$. It is easily checked that the map of cubes, from (*) to (**), induces a homotopy equivalence of the total homotopy fibers. Here $k > 1$ is essential.

4.2. Summary. *For $k > 1$, the homogeneous cofunctor L_kF is classified by the fibration $p : Z \to \binom{M}{k}$ whose fiber over $S \in \binom{M}{k}$ is the total homotopy fiber of the k–cube of spaces $R \mapsto \mathbf{emb}(R, N)$, where $R \subset S$.*

4.3. Remark. If $n-m > 0$, or if M has no compact component, then the cofunctor L_1F is equivalent to $V \mapsto \mathbf{imm}(V, N)$. This is clear from the abstract description of the classifying fibration on $\binom{M}{1} = M$, just before 3.5.

4.4. Main Theorem. *If $n-m \geq 3$, then the map $\{\eta_k\} : F(V) \longrightarrow \operatorname{holim}_k T_kF(V)$ restricts to a homotopy equivalence of the base point components, for all V in $\mathcal{O}$ and* ***any*** *choice of base point in $F(V)$. In fact, $\eta_k : F(V) \to T_kF(V)$ is $(k(n-m-2)+1-m)$–connected when restricted to base point components. This holds in particular when $V = M$.*

An indication of the proof will be given below. I do not know, and Goodwillie does not know, whether the map in 4.4 is a homotopy equivalence. If $m+1 < 2n/3$, then one can use [Hae2] to check that it induces a bijection of components, so it is a homotopy equivalence.

In Goodwillie's original calculus of functors, a homotopy invariant functor from spaces to spaces is *analytic* if it has certain higher excision properties, reminiscent of the Blakers–Massey theorem. When a functor is analytic in this sense, convergence of the Taylor series (plural) to the functor is guaranteed. In the present calculus set–up, it is still true that convergence of the Taylor series of a functor to the functor can be deduced from higher excision properties. In differential topology, say in the context of embeddings, diffeomorphisms and concordance embeddings, the (higher) excision properties are sometimes called *(multiple) disjunction lemmas.* The earliest disjunction lemma, for spaces of concordance embeddings, seems to be that of Morlet [Mor], [BuLaRo]. This was made "multiple" in Goodwillie's thesis, which appeared much later as [Go4]. Goodwillie has recently extended this to spaces of diffeomorphisms, and then, automatically, to loop spaces of embedding spaces. (Loop spaces of embedding spaces can be described as homotopy fibers of certain inclusion maps between certain spaces of diffeomorphisms.) What we need in order to prove 4.4 is the following.

Let N be an n–dimensional smooth manifold, now compact and possibly with boundary. Suppose that $P, Q_1, Q_2, \ldots, Q_r$ are pairwise disjoint smooth compact proper submanifolds of N, of dimension $p, q_1, q_2, \ldots, q_r$ respectively. (*Proper* indicates that ∂P and ∂Q_i are contained in ∂N, etc. .) For $S \subset R := \{1, \ldots, r\}$ let $Q_S := \cup_{i \in S} Q_i$. Assume $p \leq n-3$ and $q_i \leq n-3$ for $1 \leq i \leq r$. All embeddings in sight are supposed to agree with the appropriate inclusions near the boundary.

4.5. Theorem [Go5]**.** *The canonical map*

$$\Omega\,\mathbf{emb}(P, N \smallsetminus Q_R) \to \operatorname*{holim}_{S \neq R} \Omega\,\mathbf{emb}(P, N \smallsetminus Q_S)$$

is $(-p + \sum_{i=1}^{r}(n-q_i-2))$*–connected.*

5. An example

Unfortunately, the really interesting examples among the simpler ones involve boundaries or at least boundary conditions. One type of boundary condition that we can easily handle now is *compact support*: Given F in $\mathcal{F}$ and a base point in $F(M)$, we define a new cofunctor F_c by

$$F_c(V) := \text{hofiber}[F(V) \to \operatorname*{hocolim}_{W} F(V \cap W)]$$

where W denotes an open subset of M with compact complement. If the Taylor tower of F converges nicely to F (with connectivity estimates as in 4.4), then it is not hard to see that

$$F_c(V) \xrightarrow{\simeq} \operatorname*{holim}_{k} (T_k F)_c(V)\,.$$

Also, the homotopy fiber of $(T_k F)_c \to (T_{k-1}F)_c$ can be identified with $(L_k F)_c$ up to equivalence ; this does not use convergence. If all this appears to be “abstract”, think of the example $F(V) = \mathbf{emb}(V, N)$, assuming that $M \subset N$ and $n - m \geq 3$. Then $F_c(M)$ is homotopy equivalent to the space of smooth embeddings $M \to N$ which agree with the inclusion outside a compact subset of M. Furthermore $(L_k F)_c(M)$ can be identified up to homotopy equivalence with the space of sections *with compact support* of the fibration described in 4.2, provided $k > 1$; if $k = 1$ it can be identified with the space of immersions $M \to N$ which agree with the inclusion outside a compact subset of M. In the following example M is $\mathbb{R}$. Beware that N does not have the usual meaning, which is *target*.

5.1. Example. Let N be smooth, connected, without boundary, of dimension $n \geq 3$, and with a base point $*$. The base point gives a *standard* embedding $\mathbb{R} \longrightarrow N \times \mathbb{R}$ sending t to $(*, t)$. Let $F(V) = \mathbf{emb}(V, N \times \mathbb{R})$ for open $V \subset \mathbb{R}$. The layers numbered 1, 2 and 3 in the Taylor tower of $F_c(\mathbb{R})$ are as follows (proof omitted):

$$\begin{aligned} (L_1 F)_c(\mathbb{R}) &\simeq \mathbf{imm}_c(\mathbb{R}, N \times \mathbb{R}) \\ (L_2 F)_c(\mathbb{R}) &\simeq \Omega^2 \Sigma^n (\Omega N_+) \\ (L_3 F)_c(\mathbb{R}) &\simeq_{3n-3} \Omega^\infty \Sigma^{\infty+2n-4} (\Omega N \times \Omega N)_+\,. \end{aligned}$$

Here $\simeq_{3n-3}$ means that the Postnikov $(3n-3)$–coskeletons (obtained by killing homotopy groups π_i for $i > 3n-3$) are homotopy equivalent. The subscript $+$ indicates an added disjoint base point.

Keeping N and $*$ as in 5.1, let $\mathrm{CE}(*, N)$ be the space of smooth *concordance embeddings* of $*$ in N. This is the space of smooth embeddings $e : [0,1] \to N \times [0,1]$ which are transverse to $\partial(N \times [0,1])$ and satisfy $e(0) = (*, 0)$ and $e(1) \in N \times \{1\}$. There is a fibration $p : \mathrm{CE}(*, N) \to N$ defined by $p(e) = e(1) \in N \times \{1\} \cong N$, and up to homotopy equivalence the fiber $p^{-1}(*)$ is $\mathbf{emb}_c(\mathbb{R}, N \times \mathbb{R})$, which we have just explored. Of course the other fibers can be explored similarly, and the result is a partial calculation of $\mathrm{CE}(*, N)$. It is convenient to lump the base of p together with the first layers of the fibers of p ; this gives $\mathrm{CI}(*, N)$, the space of *concordance immersions* of $*$ in N.

5.2. Corollary. *There is a* $3(n-2)$*–connected map from* $\mathrm{CE}(*, N)$ *to a 3–layer tower of fibrations with layers as follows:*

$$\begin{array}{rcl} \textit{first layer} & = & \mathrm{CI}(*, N) \quad \simeq \quad \Omega\mathbb{S}^n \\ \textit{second layer} & \simeq & \Omega^2\Sigma^n(\Omega N_+) \\ \textit{third layer} & \simeq_{3n-3} & \Omega^\infty\Sigma^{\infty+2n-4}(\Omega N \times \Omega N)_+ \,. \end{array}$$

This leaves us with an *extension problem*—finding out how the layers of a Taylor tower or partial Taylor tower are to be pieced together. In the situation of 5.2 there is at least an obvious guess for a connecting map

$$\Omega(\text{first layer}) \longrightarrow \text{second layer}\,,$$

namely, the inclusion $\Omega^2\Sigma^n\mathbb{S}^0 \to \Omega^2\Sigma^n(\Omega N_+)$. This guess is correct. We conclude that first and second layers cancel when N is contractible.

5.3. Corollary. *There is a* $(3n-6)$*–connected map* $\mathrm{CE}(*, \mathbb{R}^n) \to \Omega^\infty\Sigma^\infty\mathbb{S}^{2n-4}$*, provided* $n \geq 3$*.*

The calculations 5.2 and 5.3 are explicit or implicit in Meng's thesis [Me]. The hard part of [Me] is about spaces of embeddings (relative to the boundary) of $[0,1]^2$ in $N \times [0,1]^2$. What makes it hard is the extension problem just mentioned ; the determination of the first few layers is comparatively easy.

A pretty test case for 5.2 is the case $N = \mathbb{S}^n$, because $\mathrm{CE}(*, \mathbb{S}^n)$ is known to be contractible. (This is Hatcher's *light bulb trick* ; see [Hat, p.12] for a hint.)

6. History

Much of the calculus of good cofunctors on the poset of open subsets of a manifold was known to Goodwillie in the very early 80's, including definition 2.4 and theorem 3.4. At the time Goodwillie needed it to set up his calculus of homotopy functors. It (the calculus of good cofunctors ...) was eventually abandoned because of technical problems and because a manifold–free approach to the calculus of homotopy functors emerged. The technical problems were due to the absence of something like 2.9, which is my contribution.

I learned about Goodwillie's theorem 4.5 and its applications to the calculation of spaces of embeddings in 1993, mostly through the medium of pioneer user Guowu Meng who had just completed his Ph.D. thesis [Me] under the direction of Goodwillie. Meng used theorem 4.5 quite directly to study certain embedding spaces $\mathbf{emb}(M, N)$ where $\dim(M)$ is 1 or 2, and to answer related questions which arise in pseudo–isotopy theory (=concordance theory). He did not use theorem 3.4 explicitly, which prompted me to rediscover it. Meng could not be persuaded to be a coauthor of this report.

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