COHOMOLOGY OF THE STABLE MAPPING CLASS GROUP

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The stable mapping class group is the group of isotopy classes of automorphisms of a connected oriented surface of "large" genus. The Mumford conjecture postulates that its rational cohomology is a polynomial ring generated by certain classes κ_i of dimension $2i$, for $i > 0$. Tillmann's insight [38] that the plus construction makes the classifying space of the stable mapping class group into an infinite loop space led to a stable homotopy theory version of Mumford's conjecture, stronger than the original [25]. This stronger form of the conjecture was recently proved by Ib Madsen and myself [26]. I will describe some of the ideas which led to the proof, and some retrospective thoughts, rather than trying to condense large portions of [26].

1. The stable mapping class group and stable homotopy theory

Let $F_{g,b}$ be a connected, compact, oriented smooth surface of genus g with b boundary circles (and no "punctures"). The topological group of smooth orientation preserving automorphisms of $F_{g,b}$ which restrict to the identity on $\partial F_{g,b}$ will be denoted by $\mathrm{Diff}(F_{g,b};\partial)$. The *mapping class group* of $F_{g,b}$ is

$$\Gamma_{g,b} = \pi_0 \mathrm{Diff}(F_{g,b};\partial)\,.$$

A fundamental result of Earle, Eells and Schatz [8], [9] states that the discrete group $\Gamma_{g,b}$ is homotopy equivalent to $\mathrm{Diff}(F_{g,b};\partial)$ in most cases. More precisely:

Theorem 1.1. *If $g > 1$ or $b > 0$, then the identity component of $\mathrm{Diff}(F_{g,b};\partial)$ is contractible.*

It is often convenient to assume that each boundary circle of $F_{g,b}$ comes equipped with a diffeomorphism to the standard circle S^1. Where this is orientation preserving, the boundary circle is considered to be *outgoing*, otherwise *incoming*. It is customary to write

$$b_1 + b_2$$

instead of b, to indicate that there are b_1 incoming and b_2 outgoing boundary circles. A particularly important case is $F_{g,1+1}$. By gluing outgoing to incoming boundary circles, we obtain homomorphisms

$$\Gamma_{g,1+1} \times \Gamma_{h,1+1} \longrightarrow \Gamma_{g+h,1+1}\,. \tag{1}$$

They determine a multiplication on the disjoint union of the classifying spaces $B\Gamma_{g,1+1}$ for $g \geq 0$, so that the group completion

$$\Omega B\Big(\coprod_g B\Gamma_{g,1+1}\Big)$$

is defined. As is often the case, the group completion process can be replaced by a plus construction [1]. Namely, taking $h = 1$ in display (1) and using only the neutral element of $\Gamma_{h,1+1}$ leads to stabilization homomorphisms $\Gamma_{g,1+1} \to \Gamma_{g+1,1+1}$. We write $\Gamma_{\infty,1+1} = \mathrm{colim}_g\, \Gamma_{g,1+1}$. This is the stable mapping class group of the title. It is a perfect group; in fact $\Gamma_{g,b}$ is perfect for $g \geq 3$. Let $B\Gamma_\infty^+$ be the result of a plus construction on $B\Gamma_{\infty,1+1}$.

Proposition 1.2. $\Omega B\left(\coprod_g B\Gamma_{g,1+1}\right) \;\simeq\; \mathbb{Z}\times B\Gamma_\infty^+$.

The proof uses the group completion theorem, see [1], which concerns the effect of a group completion on homology. As the referee pointed out to me, the verification of the hypotheses in the group completion theorem is not a trivial matter in the present case. It relies on the homological stability theorem of Harer [17] which we state next, with the improvements due to Ivanov [20], [21].

Theorem 1.3. *Let S be an oriented surface, $S = S_1 \cup S_2$ where $S_1 \cap S_2$ is a union of finitely many smooth circles in the interior of S. If $S_1 \cong F_{g,b}$ and $S \cong F_{h,c}$, then the inclusion-induced homomorphism $H_*(B\Gamma_{g,b};\mathbb{Z}) \to H_*(B\Gamma_{h,c};\mathbb{Z})$ is an isomorphism for $* < g/2 - 1$.*

The homological stability theorem is a very deep theorem with impressive applications, some of them much more surprising than proposition 1.2. A particularly surprising application is Tillmann's theorem [38]:

Theorem 1.4. $\mathbb{Z}\times B\Gamma_\infty^+$ *is an infinite loop space.*

Theorems 1.1 and 1.3 imply that the cohomology of $B\Gamma_\infty^+$ is a receptacle for characteristic classes of surface bundles, with fibers of "large" genus. Following Mumford, Miller and Morita we now use this point of view to construct elements in the cohomology of $B\Gamma_\infty^+$.

With the hypotheses of theorem 1.1. let $E \to B$ be any $F_{g,b}$–bundle with oriented fibers and trivialized boundary bundle $\partial E \to B$, that is, each fiber of $\partial E \to B$ is identified with a disjoint union of b standard circles. The vertical tangent bundle $T_B E$ of E is a two-dimensional oriented vector bundle, trivialized near ∂E, with Euler class $e = e(T_B E) \in H^2(E, \partial E;\mathbb{Z})$. Let

$$\kappa_i \in H^{2i}(B;\mathbb{Z})$$

be the image of $e^{i+1} \in H^{2i+2}(E,\partial E;\mathbb{Z})$ under the Gysin transfer map, also known as integration along the fibers,

$$H^{2i+2}(E,\partial E;\mathbb{Z}) \longrightarrow H^{2i}(B;\mathbb{Z})\,. \tag{2}$$

The κ_i are, up to a sign, Mumford's characteristic classes [31] in the description of Miller [27] and Morita [28], [29]. By theorem 1.1, the universal choice of B is $B\Gamma_{g,b}$ and we may therefore regard the κ_i as classes in the cohomology of $B\Gamma_{g,b}$. For $i > 0$, they are compatible with respect to homomorphisms $\Gamma_{g,b} \to \Gamma_{h,c}$ of the type considered in theorem 1.3 and we may therefore write

$$\kappa_i \in H^{2i}(B\Gamma_{\infty,1+1};\mathbb{Z})\,.$$

1.5. Mumford's conjecture [31], now a theorem:

$$H^*(B\Gamma_{\infty,1+1};\mathbb{Q}) = \mathbb{Q}[\kappa_1,\kappa_2,\dots\,]\,,$$

i.e., the classes $\kappa_i \in H^{2i}(B\Gamma_{\infty,1+1};\mathbb{Q})$ are algebraically independent and generate $H^*(B\Gamma_{\infty,1+1};\mathbb{Q})$ as a $\mathbb{Q}$-algebra.

The algebraic independence part was very soon established by Miller [27] and Morita [28], [29]. About fifteen years later, after Tillmann had proved theorem 1.4, it was noticed by Madsen and Tillmann [25] that the Miller-Morita construction of the Mumford classes κ_i provides an important clue as to “which” infinite loop space $\mathbb{Z} \times B\Gamma_\infty^+$ might be. Assume for simplicity that $b = 0$ in the above and that B is finite dimensional. A choice of a fiberwise smooth embedding $E \to B \times \mathbb{R}^{2+n}$ over B, with $n \gg 0$, leads to a Thom-Pontryagin collapse map of Thom spaces,

$$\mathrm{Th}\,(B \times \mathbb{R}^{2+n}) \longrightarrow \mathrm{Th}\,(T_B^\perp E)\,, \tag{3}$$

where $T_B^\perp E$ is the fiberwise normal bundle of E in $B \times \mathbb{R}^{2+n}$. It is well known that (3) induces the Gysin transfer (2), modulo the appropriate Thom isomorphisms. Let now $\mathrm{Gr}_2(\mathbb{R}^{2+n})$ be the Grassmannian of oriented 2-planes in $\mathbb{R}^{2+n}$ and let L_n, $L_n^\perp$ be the canonical vector bundles of dimension 2 and n on $\mathrm{Gr}_2(\mathbb{R}^{2+n})$, respectively. Composing (3) with the tautological map $\mathrm{Th}\,(T_B^\perp E) \to \mathrm{Th}\,(L_n^\perp)$ gives $\mathrm{Th}\,(B \times \mathbb{R}^{2+n}) \longrightarrow \mathrm{Th}\,(L_n^\perp)$ and hence by adjunction $B \to \Omega^{2+n}\mathrm{Th}\,(L_n^\perp)$, and finally in the limit

$$B \longrightarrow \Omega^{2+\infty}\mathrm{Th}\,(L_\infty^\perp) \tag{4}$$

where $\Omega^{2+\infty}\mathrm{Th}\,(L_\infty^\perp) = \mathrm{colim}_n\, \Omega^{2+n}\mathrm{Th}\,(L_n^\perp)$. At this stage we can also allow an infinite dimensional B, in particular $B = B\Gamma_g$ with the universal F_g–bundle. The case $B\Gamma_{g,b}$ can be dealt with by using a homomorphism $\Gamma_{g,b} \to \Gamma_g$ of the type considered in 1.3. In this way, (4) leads to a map

$$\alpha_\infty \colon \mathbb{Z} \times B\Gamma_\infty^+ \longrightarrow \Omega^{2+\infty}\mathrm{Th}\,(L_\infty^\perp). \tag{5}$$

It is easy to recover the MMM characteristic classes κ_i by applying (5) to certain classes $\bar\kappa_i$ in the cohomology of $\Omega^{2+\infty}\mathrm{Th}\,(L_\infty^\perp)$. Namely, choose $n \gg i$ and let $\bar\kappa_i$ be the image of $(e(L_n))^{i+1}$ under the composition

$$H^{2i+2}(\mathrm{Gr}_2(\mathbb{R}^{2+n});\mathbb{Z}) \xrightarrow{\;u\;} H^{2i+2+n}(\mathrm{Th}\,(L_n^\perp);\mathbb{Z}) \xrightarrow{\;\Omega^{2+n}\;} H^{2i}(\Omega^{2+n}\mathrm{Th}\,(L_n^\perp);\mathbb{Z})$$

where u is the Thom isomorphism. Since $n \gg i$, we have

$$H^{2i}(\Omega^{2+n}\mathrm{Th}\,(L_n^\perp);\mathbb{Z}) \cong H^{2i}(\Omega^{2+\infty}\mathrm{Th}\,(L_\infty^\perp);\mathbb{Z}).$$

1.6. Madsen's integral Mumford conjecture [25], now a theorem: *The map α_∞ is a homotopy equivalence.*

Tillmann and Madsen noted in [25] that this would imply statement 1.5. They showed that α_∞ is a map of infinite loop spaces, with the Ω^∞ structure on $\mathbb{Z}\times B\Gamma_\infty^+$ from theorem 1.4, and used this fact to prove a p–local refinement of the Miller–Morita result on the rational independence of the classes κ_i, for any prime p. In the meantime Galatius [11] made a very elegant calculation of $H^*(\Omega^{2+\infty}\mathrm{Th}\,(L_\infty^\perp);\mathbb{Z}/p)$.

2. Submersion theory and the first desingularization procedure

Let X be any smooth manifold. By Thom–Pontryagin theory, homotopy classes of maps $X \to \Omega^{2+\infty}\mathrm{Th}\,(L_\infty^\perp)$ are in bijective correspondence with bordism classes of triples $(M, q, \hat q)$ where M is smooth, $q \colon M \to X$ is proper, and

$$\hat q \colon TM \times \mathbb{R}^n \to q^*TX \times \mathbb{R}^n$$

is a vector bundle surjection with a 2-dimensional oriented kernel bundle, for some n. (The correspondence is obtained by making pointed maps from $\mathrm{Th}\,(X \times \mathbb{R}^n)$ to

$\mathrm{Th}\,(L_n^\perp)$ transverse to the zero section of $L_n^\perp$; the inverse image of the zero section is a smooth M equipped with data q and $\hat{q}$ as above.) The triples $(M,q,\hat{q})$ are best memorized as commutative squares

$$\begin{tikzcd} TM \times \mathbb{R}^n \arrow[r, "\hat{q}"] \arrow[d] & TX \times \mathbb{R}^n \arrow[d] \\ M \arrow[r, "q"] & X \end{tikzcd} \tag{6}$$

with $\hat{q}$ written in adjoint form.

In particular, any bundle of closed oriented surfaces $q\colon M \to X$ determines a triple $(M,q,\hat{q})$ with $\hat{q}$ equal to the differential of q, hence a homotopy class of maps from X to $\Omega^{2+\infty}\mathrm{Th}\,(L_\infty^\perp)$. This is the fundamental idea behind (5). From this angle, statement 1.6 is a "desingularization" statement. More precisely, it is equivalent to the following:

For fixed i and sufficiently large g, every oriented i–dimensional bordism class of the degree g component of $\Omega^{2+\infty}\mathrm{Th}\,(L_\infty^\perp)$ can be represented by an F_g–bundle on a closed smooth oriented i-manifold; such a representative is unique up to an oriented bordism of F_g-bundles.

The translation uses theorem 1.3 and the fact that a map between simply connected spaces is a homotopy equivalence if and only if it induces an isomorphism in the generalized homology theory "oriented bordism".

Let $(M,q,\hat{q})$ be a triple as above, so that $q\colon M \to X$ is a proper smooth map and $\hat{q}\colon TM \to q^*TX$ is a stable vector bundle surjection with 2–dimensional oriented kernel. If $\hat{q}$ happens to agree with the differential dq of q, then q is a proper submersion, hence a surface bundle by Ehresmann's fibration theorem [4]. In general it is not possible to arrange this by deforming the pair $(q,\hat{q})$. One must settle for less. The approach taken in [26] is as follows.

Suppose for simplicity that X is closed. Let $E = M \times \mathbb{R}$ and let $\pi_E\colon E \to X$ be the composition $E \to M \to X$. By obstruction theory, $\hat{q}$ deforms to an honest surjection

$$\hat{\pi}_E\colon TE \longrightarrow \pi_E^*TX$$

of vector bundles on E, with kernel of the form $V \times \mathbb{R}$, where V is a 2–dimensional oriented vector bundle on E. Writing $\hat{\pi}_E$ in adjoint form, we can describe the situation by a commutative square

$$\begin{tikzcd} TE \arrow[r, "\hat{\pi}_E"] \arrow[d] & TX \arrow[d] \\ E \arrow[r, "q"] & X\,. \end{tikzcd} \tag{7}$$

By submersion theory [32], which is applicable here because E is an open manifold, the pair $(\pi_E, \hat{\pi}_E)$ deforms to a pair $(\pi,\hat{\pi})$ where $\pi\colon E \to X$ is a smooth submersion with differential $d\pi = \hat{\pi}$. See also section 3 below. The kernel of $d\pi\colon TE \to \pi^*TX$ is still of the form $V \times \mathbb{R}$ with 2–dimensional oriented V. In addition, we have a proper map $f\colon E \to \mathbb{R}$, the projection.

The "first desingularization" procedure $(M,q,\hat{q}) \rightsquigarrow (E,\pi,f)$ is an important conceptual step. If we forget or ignore the product structure $E \cong M \times \mathbb{R}$, we can still recover $(M,q,\hat{q})$ from (E,π,f) up to bordism by forming $(N, \pi|N, d\pi|\dots)$, where

$N = f^{-1}(c)$ for a regular value c of f. Let us now see how this reverse procedure reconstitutes the singularities.

Lemma 2.1. *For $z \in N$ with $\pi(z) = x$, the following are equivalent:*

- *$\pi|N$ is nonsingular at z;*
- *$f|E_x$ is nonsingular at z, where $E_x = \pi^{-1}(x)$.*

The following are also equivalent:

- *$\pi|N$ has a fold singularity at z;*
- *$f|E_x$ has a Morse singularity at z.*

Proof. Let T, V and H be the (total spaces of the) tangent bundle of E, the vertical subbundle (kernel of $d\pi$) and the horizontal quotient bundle, respectively, so that $H = T/V$. Let K be the tangent bundle of N. We are assuming that $df\colon T_z \to \mathbb{R}$ is onto, since $f(z) = c$ is a regular value. Hence $df|V_z$ is nonzero if and only if K_z is transverse to V_z in T_z, which means that the projection $K_z \to H_z$ is onto. This proves the first equivalence.
Suppose now that $df|V_z$ is zero. By definition, $\pi|N$ has a fold singularity at z if the differential $K_z \to H_z$ has corank 1 and the "second derivative" of $\pi|N$, as a well defined symmetric bilinear map Q from $\ker(K_z \to H_z)$ to $\operatorname{coker}(K_z \to H_z)$, is nondegenerate. In our situation, $\ker(K_z \to H_z) = V_z$ and $\operatorname{coker}(K_z \to H_z)$ is canonically identified, via $d\pi$, with T_z/K_z and hence via df with $\mathbb{R}$. Using local coordinates near z, it is not difficult to see that the second derivative of $f|E_x$ at z, regarded as a well defined symmetric bilinear map from V_z to $\mathbb{R}$, is equal to $-Q$. Hence z is a nondegenerate critical point for $f|E_x$ if and only if $\pi|N$ has a fold singularity at z. $\square$

These ideas also steer us away from a bordism theoretic approach and towards a description of $\Omega^{2+\infty}\mathrm{Th}\,(L_\infty^\perp)$ in terms of "families" of 3–manifolds.

Proposition 2.2. *The space $\Omega^{2+\infty}\mathrm{Th}\,(L_\infty^\perp)$ is a classifying space for "families" of oriented 3–manifolds without boundary, equipped with a proper smooth map to $\mathbb{R}$ and an everywhere nonzero 1–form.*

To be more precise, the families in question are parametrized by a smooth manifold without boundary, say X. They are smooth *submersions* $\pi\colon E \to X$ with oriented 3–dimensional fibers. The additional data are: a smooth $f\colon E \to \mathbb{R}$ such that $(\pi, f)\colon E \to X \times \mathbb{R}$ is proper, and a vector bundle surjection from $\ker(d\pi)$, the vertical tangent bundle of E, to a trivial line bundle on E.
Two such families on X are *concordant* if their disjoint union, regarded as a family on $X \times \{0, 1\}$, extends to a family of the same type on $X \times \mathbb{R}$. The content of proposition 2.2 is that the set of concordance classes is in natural bijection with the set of homotopy classes of maps from X to $\Omega^{2+\infty}\mathrm{Th}\,(L_\infty^\perp)$. Note that both sets depend contravariantly on X.

Remark 2.3. When using proposition 2.2, beware that most smooth submersions are not bundles. For example, the inclusion of $\mathbb{R} \smallsetminus \{0\}$ in $\mathbb{R}$ and the first coordinate projection from $\mathbb{R}^2 \smallsetminus \{0\}$ to $\mathbb{R}$ are smooth submersions. Proposition 2.2 is therefore still rather far from being a description of $\Omega^{2+\infty}\mathrm{Th}\,(L_\infty^\perp)$ in terms of manifold bundles. But it is a start, and we will complement in the following sections with methods for improving submersions to bundles or decomposing submersions into bundles.

Remark 2.4. There exists another formulation of proposition 2.2 in which all 3–manifolds in sight have a prescribed boundary equal to $\{0,1\} \times \mathbb{R} \times S^1$. This is more suitable where concatenation as in (1) matters. But since the equivalence of the two formulations is easy to prove, there is much to be said for working with boundariless manifolds until the concatenation issues need to be addressed.

3. More h–principles and the second desingularization procedure

Let M, N be smooth manifolds without boundary, $z \in M$. A *k–jet* from M to N at z is an equivalence class of smooth maps $f\colon M \to N$, where two such maps are considered equivalent if they agree to k–th order at z. Let $J^k(M,N)_z$ be the set of equivalence classes and let

$$J^k(M,N) = \bigcup_z J^k(M,N)_z.$$

This has the structure of a differentiable manifold. The projection $J^k(M,N) \to M$ is a smooth bundle. Every smooth function $f\colon M \to N$ determines a smooth section $j^k f$ of the jet bundle $J^k(M,N) \to M$, the *k–jet prolongation* of f. The value of $j^k f$ at $z \in M$ is the k–jet of f at z.
A smooth section of $J^k(M,N) \to M$ is *integrable* or *holonomic* if it has the form $j^k f$ for some smooth $f\colon M \to N$. Most smooth sections of $J^k(M,N) \to M$ are not integrable. Nevertheless there exists a highly developed culture of integrability theorems up to homotopy, so–called h–principles [14], [10]. Such a theorem typically begins with the description of an open subbundle $A \to M$ of $J^k(M,N) \to M$, and states that the inclusion of the space of integrable sections of $A \to M$ into the space of all sections of $A \to M$ is a homotopy equivalence. (For us the cases where $k = 1$ or $k = 2$ are the most important.)

The relevance of these notions to the Mumford–Madsen project is clear if we adopt the bordism–free point of view developed in section 2. Consider a single oriented smooth 3–manifold E with a proper smooth $f\colon E \to \mathbb{R}$ and an everywhere nonvanishing 1–form, as in proposition 2.2. The map f and the 1–form together define a section of the jet bundle $J^1(E,\mathbb{R})$. If this is integrable, then f is a proper submersion. Hence $f\colon E \to \mathbb{R}$ is a bundle of oriented surfaces, again by Ehresmann's fibration theorem. The argument goes through in a parametrized setting: a family as in 2.2, parametrized by X, is a surface bundle on $X \times \mathbb{R}$ provided it satisfies the additional condition of integrability. From this point of view, statement 1.6 is roughly an h–principle "up to group completion". (It is unusual in that the source manifolds are allowed to vary.)

Examples 3.1 and 3.2 below are established h–principles. The h–principle of theorem 3.4 is closely related to a special case of 3.2 and at the same time rather similar to statement 1.6.

Example 3.1. An element in $J^1(M,N)$ can be regarded as a triple (x,y,g) where $(x,y) \in M \times N$ and g is a linear map from the tangent space of M at x to the tangent space of N at y. Let $U_1 \subset J^1(M,N)$ consist of the triples (x,y,g) where g is injective and let $U_2 \subset J^1(M,N)$ consist of the triples (x,y,g) where g is surjective. Let $\Gamma(U_1)$, $\Gamma(U_2)$ be the section spaces of the bundles $U_1 \to M$ and $U_2 \to M$, respectively. Let Γ_{itg} be the space of integrable (alias holonomic) sections of $J^1(M,N)\to M$. Note that $\Gamma_{\mathrm{itg}}\cap\Gamma(U_1)$ is identified with the space of smooth immersions from M to N, and $\Gamma_{\mathrm{itg}}\cap\Gamma(U_2)$ is identified with the space of smooth submersions from M to N. One of the main results of immersion theory [36], [18] is the statement that the inclusion

$$\Gamma_{\mathrm{itg}}\cap\Gamma(U_1)\longrightarrow\Gamma(U_1)$$

is a homotopy equivalence if $\dim(M)<\dim(N)$. The main result of submersion theory [32] is that the inclusion

$$\Gamma_{\mathrm{itg}}\cap\Gamma(U_2)\longrightarrow\Gamma(U_2)$$

is a homotopy equivalence if M is an open manifold. Gromov's 1969 thesis, outlined in [15], develops a general method for proving these and related h–principles using sheaf–theoretic arguments. This has become the standard. Much of it is reproduced in [14, §2.2]. See also [16] and [10].

Example 3.2. Fix positive integers m,n,k. Let $\mathfrak{A}$ be a closed semialgebraic subset [3] of the vector space $J^k(\mathbb{R}^m,\mathbb{R}^n)$. Suppose that $\mathfrak{A}$ is invariant under the right action of the group of diffeomorphisms $\mathbb{R}^m\to\mathbb{R}^m$, and of codimension $\geq m+2$ in $J^k(\mathbb{R}^m,\mathbb{R}^n)$. Fix a smooth m–manifold M and let $\mathfrak{A}(M)\subset J^k(M,\mathbb{R}^n)$ consist of the jets which, in local coordinates about their source, belong to $\mathfrak{A}$. Let Γ be the space of smooth sections of $J^k(M,\mathbb{R}^n)\to M$, let $\Gamma_{\mathrm{itg}}\subset\Gamma$ consist of the integrable sections, and let $\Gamma_{\neg\mathfrak{A}}\subset\Gamma$ consist of the sections which avoid $\mathfrak{A}(M)$. Note that $\Gamma_{\mathrm{itg}}\cap\Gamma_{\neg\mathfrak{A}}$ is identified with the space of smooth maps from M to $\mathbb{R}^n$ having no singularities of type $\mathfrak{A}$. Vassiliev's h–principle [40, Thm 0.A], [39, III,1.1] states among other things that the inclusion

$$\Gamma_{\mathrm{itg}}\cap\Gamma_{\neg\mathfrak{A}}\quad\longrightarrow\quad\Gamma_{\neg\mathfrak{A}}$$

induces an isomorphism in integral cohomology. (There is also a relative version in which M is compact with boundary.) If the codimension of $\mathfrak{A}$ is at least $m+3$, then both $\Gamma_{\mathrm{itg}}\cap\Gamma_{\neg\mathfrak{A}}$ and $\Gamma_{\neg\mathfrak{A}}$ are simply connected; it follows that in this case the inclusion map is a homotopy equivalence.
Vassiliev's proof of this h–principle is meticulously and admirably organized, but still not easy to read. As far as I can see, it is totally different from anything described in [14] or [10]. An overview is given in section A.

In theorem 3.4 below, we will need an analogue of proposition 2.2. Let $\mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+n})$ be the space of 3–dimensional oriented linear subspaces $V\subset\mathbb{R}^{3+n}$ equipped with a certain type of map $q+\ell\colon V\to\mathbb{R}$. Here q is a quadratic form, ℓ is a linear form, and we require that q be nondegenerate if $\ell=0$. Denote by

$$U_{\mathcal{W},n}\,,\;U_{\mathcal{W},n}^{\perp}$$

the tautological 3–dimensional vector bundle on $\mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+n})$ and its n–dimensional complement, respectively, so that $U_{\mathcal{W},n}\oplus U_{\mathcal{W},n}^{\perp}$ is a trivial vector bundle with fiber $\mathbb{R}^{3+n}$. Let

$$\Omega^{2+\infty}\mathrm{Th}\,(U_{\mathcal{W},\infty}^{\perp})\;=\;\mathrm{colim}_n\,\Omega^{2+n}\mathrm{Th}\,(U_{\mathcal{W},n}^{\perp})\,.$$

Proposition 3.3. *The space* $\Omega^{2+\infty}\mathrm{Th}\,(U^{\perp}_{\mathcal{W},\infty})$ *is a classifying space for "families" of oriented smooth 3–manifolds* E_x *without boundary, equipped with a section of* $J^2(E_x,\mathbb{R}) \to E_x$ *whose values are all of Morse type, and whose underlying map* $f_x\colon E_x \to \mathbb{R}$ *is proper.*

Some details: An element of $J^2(E_x,\mathbb{R})$ is, in local coordinates about its source $z \in E_x$, uniquely represented by a function of the form $q+\ell+c\colon \mathbb{R}^3 \to \mathbb{R}$ where q is a quadratic form, ℓ is a linear form and c is a constant. It is *of Morse type* if either $\ell \neq 0$ or q is nondegenerate.
The families in question are smooth submersions $\pi\colon E \to X$ where each fiber E_x is a 3–manifold with the structure and properties described in proposition 3.3. The content of proposition 3.3 is that the set of concordance classes of such families on X is in natural bijection with the set of homotopy classes of maps from X to $\Omega^{2+\infty}\mathrm{Th}\,(U^{\perp}_{\infty})$. The proof mainly uses the Thom–Pontryagin construction and submersion theory, just like the proof of proposition 2.2 sketched in section 2.

Theorem 3.4. *The space* $\Omega^{2+\infty}\mathrm{Th}\,(U^{\perp}_{\mathcal{W},\infty})$ *is also a classifying space for "families" of oriented smooth 3–manifolds without boundary, equipped with a proper smooth Morse function.*

Clearly the simultaneous validity of theorem 3.4 and proposition 3.3 implies something like an h–principle for proper Morse functions on oriented 3–manifolds without boundary — the "second desingularization procedure" which appears in the title of this section. (It can be applied to a family of 3–manifolds E_x as in proposition 2.2; the smooth function f_x and the 1–form together form a section of $J^1(E_x,\mathbb{R}) \to E_x$, which can also be regarded as a section of $J^2(E_x,\mathbb{R}) \to E_x$ after a choice of riemannian metric on E_x.) But it must be emphasized that variability of the 3–manifolds is firmly built in. No claim is made for the space of proper Morse functions on a single oriented 3–manifold without boundary.

Here is an indication of how theorem 3.4 can be deduced from Vassiliev's h–principle (example 3.2) and proposition 3.3. It is not hard to show that the concordance classification of the "families" under consideration remains unchanged if we impose the Morse condition only at level 0. This means that in theorem 3.4 we may allow families of oriented smooth 3–manifolds E_x without boundary, equipped with a proper smooth function $E_x \to \mathbb{R}$ whose critical points are nondegenerate *if the critical value is* 0. In proposition 3.3 we may allow families of oriented 3–manifolds E_x without boundary, equipped with a section $\hat f_x$ of $J^2(E_x,\mathbb{R}) \to E_x$ whose values are of Morse type *whenever their constant term is zero*, and whose underlying map $f_x : E_x \to \mathbb{R}$ is proper. Thus the elements of $J^2(E_x,\mathbb{R})$ to be avoided are those which, in local coordinates about their source, are represented by polynomial functions $\mathbb{R}^3 \to \mathbb{R}$ of degree at most two which have constant term 0, linear term 0 and degenerate quadratic term. These polynomial functions form a subset $\mathfrak{A}$ of $J^2(\mathbb{R}^3,\mathbb{R})$ which satisfies the conditions listed in 3.2; in particular, its codimension is 3+2. Unfortunately E_x is typically noncompact, and depends on x. Nevertheless, with an elaborate justification one can use Vassiliev's h–principle here, mainly on the grounds that the "integration up to homotopy" of a section

$$\hat f_x : E_x \to J(E_x,\mathbb{R})$$

satisfying the above conditions is easy to achieve outside the compact subset $f_x^{-1}(0)$ of E_x. This leads to a statement saying that two abstractly defined classifying spaces, corresponding to the two types of "families" being compared, are homology equivalent. Since the two classifying spaces come with a grouplike addition law, corresponding to the disjoint union of families, the homology equivalence is a homotopy equivalence.

In the next statement, a variation on proposition 3.3, we identify $\mathrm{Gr}_2(\mathbb{R}^{2+n})$ with the closed subspace of $\mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+n})$ consisting of the oriented 3–dimensional linear subspaces $V \subset \mathbb{R}^{3+n}$ which contain the "first" factor $\mathbb{R} \cong \{(t, 0, 0, 0, \dots)\}$, with $q + \ell \colon V \to \mathbb{R}$ equal to the corresponding projection. The restriction of $U_n^\perp$ to $\mathrm{Gr}_2(\mathbb{R}^{2+n})$ is identified with $L_n^\perp$. This leads to a cofibration

$$\mathrm{Th}\,(L_n^\perp) \longrightarrow \mathrm{Th}\,(U_n^\perp).$$

In this way $\Omega^{2+n}(\mathrm{Th}\,(U_n^\perp)/\mathrm{Th}\,(L_n^\perp))$ acquires a meaning.
For a smooth E_x and a section of $J^2(E_x, \mathbb{R}) \to E_x$, let the *formal singularity set* consist of the elements in E_x where the associated 2–jet is singular.

Proposition 3.5. *The space $\Omega^{2+\infty}(\mathrm{Th}\,(U_\infty^\perp)/\mathrm{Th}\,(L_\infty^\perp))$ is a classifying space for "families" of oriented smooth 3–manifolds E_x without boundary, equipped with a section of $J^2(E_x, \mathbb{R}) \to E_x$ whose values are all of Morse type, and whose underlying map $f_x \colon E_x \to \mathbb{R}$ is proper on the formal singularity set.*

The proof is similar to the proofs of propositions 2.2 and 3.3.

Theorem 3.6. *The space $\Omega^{2+\infty}(\mathrm{Th}\,(U_\infty^\perp)/\mathrm{Th}\,(L_\infty^\perp))$ is also a classifying space for "families" of oriented smooth 3–manifolds without boundary, equipped with a smooth Morse function which is proper on the singularity set.*

Again, the simultaneous validity of theorem 3.6 and proposition 3.5 implies something like an h–principle for Morse functions which are proper on their singularity set, and defined on oriented 3–manifolds without boundary. But this is much easier than the h–principle implicit in theorem 3.4.
Namely, let $\pi \colon E \to X$ with $f \colon E \to \mathbb{R}$ be a family of the type described in theorem 3.6. Thus π is a smooth submersion, $f|E_x$ is Morse for each $x \in X$ and $(\pi, f) \colon E \to X \times \mathbb{R}$ is proper on Σ, where $\Sigma \subset E$ is the union of the singularity sets of all $f|E_x$. The stability of Morse singularities implies that Σ is a codimension 3 smooth submanifold of E, transverse to each fiber E_x of π. Hence $\pi|\Sigma$ is an *étale* map from Σ to X, that is, a codimension zero immersion. Choose a normal bundle N of Σ in E, in such a way that each fiber of $N \to \Sigma$ is contained in a fiber of π. It is easy to show that the family given by π and f is concordant to the family given by $\pi|N$ and $f|N$. This fact leads to a very neat concordance classification for such families, and so leads directly to theorem 3.6.

4. **Strategic thoughts**

For each $k \geq 0$, the functor $\Omega^{k+\infty}$ converts homotopy cofiber sequences of spectra into homotopy fiber sequences of infinite loop spaces. Applied to our situation, this gives a homotopy fiber sequence

$$\Omega^{2+\infty}\mathrm{Th}\,(L_\infty^\perp) \longrightarrow \Omega^{2+\infty}\mathrm{Th}\,(U_\infty^\perp) \longrightarrow \Omega^{2+\infty}(\mathrm{Th}\,(U_\infty^\perp)/\mathrm{Th}\,(L_\infty^\perp)),$$

leading to a long exact sequence of homotopy groups for the three spaces. Combining this with the main results of the previous section, we obtain a homotopy fiber sequence

$$\Omega^{2+\infty}\mathrm{Th}\,(L_\infty^\perp) \longrightarrow |\mathcal{W}| \hookrightarrow |\mathcal{W}_{\mathrm{loc}}| \tag{8}$$

where $|\mathcal{W}|$ and $|\mathcal{W}_{\mathrm{loc}}|$ classify (up to concordance) certain families of oriented smooth 3–manifolds without boundary, equipped with Morse functions. In the case of $|\mathcal{W}|$, we insist on proper Morse functions; in the case of $|\mathcal{W}_{\mathrm{loc}}|$, Morse functions whose restriction to the singularity set is proper. The details are as in theorems 3.4 and 3.6. The spaces $|\mathcal{W}|$ and $|\mathcal{W}_{\mathrm{loc}}|$ can, incidentally, be constructed directly in terms of the contravariant functors $\mathcal{W}$ and $\mathcal{W}_{\mathrm{loc}}$ which to a smooth X associate the appropriate set of "families" parametrized by X.

There is an entirely different approach to $|\mathcal{W}|$ and $|\mathcal{W}_{\mathrm{loc}}|$ which eventually leads to a homotopy fiber sequence

$$\mathbb{Z}\times B\Gamma_\infty^+ \longrightarrow |\mathcal{W}| \hookrightarrow |\mathcal{W}_{\mathrm{loc}}|, \tag{9}$$

and so, in combination with (8), to a proof of (1.6). In this approach, $|\mathcal{W}|$ and $|\mathcal{W}_{\mathrm{loc}}|$ are seen as *stratified* spaces. The reasons for taking such a point of view are as follows.
Let a family of 3–manifolds E_x and proper Morse functions $f_x\colon E_x \to \mathbb{R}$ as in theorem 3.4 be given, where $x \in X$. For $x \in X$ let S_x be the finite set of critical points of f_x with critical value 0. It comes with a map $S_x \to \{0,1,2,3\}$, the Morse index map. We therefore obtain a partition of the parameter manifold X into locally closed subsets $X_{\langle S\rangle}$, indexed by the isomorphism classes of finite sets S over $\{0,1,2,3\}$. Namely, $X_{\langle S\rangle}$ consists of the $x \in X$ with $S_x \cong S$. If the family is sufficiently *generic*, the partition is a stratification (definition 5.1 below) and $X_{\langle S\rangle}$ is a smooth submanifold of X, of codimension $|S|$. At the other extreme we have the case where $X_{\langle S\rangle} = X$ for some $\langle S\rangle$; then the family is *pure of class* $\langle S\rangle$.
A careful elaboration of these matters results in a stratified model of $|\mathcal{W}|$, with strata $|\mathcal{W}_{\langle S\rangle}|$ indexed by the isomorphism classes $\langle S\rangle$ of finite sets over $\{0,1,2,3\}$, where $|\mathcal{W}_{\langle S\rangle}|$ classifies families (as above) which are pure of class $\langle S\rangle$. There is a compatibly stratified model of $|\mathcal{W}_{\mathrm{loc}}|$. It turns out, and it is not all that hard to understand, that the strata $|\mathcal{W}_{\langle S\rangle}|$ and $|\mathcal{W}_{\mathrm{loc},\langle S\rangle}|$ are also classifying spaces for certain genuine *bundle* types. More importantly, the homotopy fibers of the forgetful map $|\mathcal{W}_{\langle S\rangle}| \to |\mathcal{W}_{\mathrm{loc},\langle S\rangle}|$ are classifying spaces for bundles of compact oriented smooth surfaces with a prescribed boundary which depends on the reference point in $|\mathcal{W}_{\mathrm{loc},\langle S\rangle}|$. It is this information, coupled with the Harer stability result, which then leads to a description of the homotopy fiber of $|\mathcal{W}| \to |\mathcal{W}|_{\mathrm{loc}}$ in bundle–theoretic terms, i.e., to the homotopy fiber sequence (9).

5. Stratified spaces and homotopy colimit decompositions

This section is about a general method for extracting homotopy theoretic information from a stratification. In retrospective, the homotopy fiber sequence (9) can be regarded as an application of that general method.

Definition 5.1. A *stratification* of a space X is a locally finite partition of X into locally closed subsets, the *strata*, such that the closure of each stratum in X is a union of strata.

Example 5.2. Let Y be a nonempty Hausdorff space, S a finite set and $X = Y^S$. Then X is canonically stratified, with one stratum X_η for each equivalence relation η on S. Namely, $u \in X$ belongs to the stratum X_η if $s\eta t \Leftrightarrow (u_s = u_t)$ for $(s,t) \in S\times S$. The closure of X_η is the union of all X_ω with $\omega \supset \eta$.

Example 5.3. Let X be the space of Fredholm operators $\mathbb{H} \to \mathbb{H}$ of index 0, where $\mathbb{H}$ is a separable Hilbert space. See e.g. [2]. Then X is stratified, with one stratum X_n for each integer $n \geq 0$. Namely, X_n consists of the Fredholm operators f having $\dim(\ker(f)) = \dim(\operatorname{coker}(f)) = n$. Here the closure of X_n is the union of all X_m with $m \geq n$.

Definition 5.4. Let X be a stratified space. The set of strata of X becomes a poset, with $X_i \leq X_j$ if and only if the closure of X_i in X contains X_j. (*Warning*: This is the opposite of the obvious ordering.)

The main theme of this section is that stratifications often lead to homotopy colimit decompositions. I am therefore obliged to explain what a homotopy colimit is. Let $\mathcal{C}$ be a small category and let

$$F\colon \mathcal{C} \to \mathit{Spaces}$$

be a functor. The *colimit* of F is the quotient of the disjoint union $\coprod_c F(c)$ obtained by identifying $x \in F(c)$ with $g_*(x) \in F(d)$, for any morphism $g\colon c \to d$ in $\mathcal{C}$ and $x \in F(c)$. In general, the homotopy type of $\operatorname{colim} F$ is somewhat unpredictable. As a protection against that one may impose a condition on F.

Definition 5.5. A functor $F\colon \mathcal{C} \to \mathit{Spaces}$ is *cofibrant* if, given functors G, G' from $\mathcal{C}$ to spaces and natural transformations

$$F \xrightarrow{\;u\;} G \xleftarrow{\;e\;} G'$$

where $e\colon G'(c) \to G(c)$ is a homotopy equivalence for all c in $\mathcal{C}$, there exists a natural transformation $u'\colon F \to G'$ and a natural homotopy h from eu' to u.

If $F, G : \mathcal{C} \to \mathit{Spaces}$ are both cofibrant and $u : F \to G$ is a natural transformation such that $u : F(c) \to G(c)$ is a homotopy equivalence for each c in $\mathcal{C}$, then the induced map $\operatorname{colim} F \to \operatorname{colim} G$ is a homotopy equivalence. This follows immediately from definition 5.5. In this sense, colimits are well behaved on cofibrant functors. With standard resolution techniques, one can show that an arbitrary F from $\mathcal{C}$ to spaces admits a *cofibrant resolution*; i.e., there exist a cofibrant F' from $\mathcal{C}$ to spaces and a natural transformation $F' \to F$ such that $F'(c) \to F(c)$ is a homotopy equivalence for every c.

Definition 5.6. For $F\colon \mathcal{C} \to \mathit{Spaces}$ with a cofibrant resolution $F' \to F$, the *homotopy colimit* of F is the colimit of F'.

Definition 5.6 is unambiguous in the following sense: if $F' \to F$ and $F'' \to F$ are two cofibrant resolutions, then F' and F'' can be related by natural transformations $v\colon F' \to F''$ and $w\colon F'' \to F'$ such that vw and wv are *naturally* homotopic to the appropriate identity transformations. Hence $\operatorname{colim} F' \simeq \operatorname{colim} F''$. Of course, there is always a standard choice of a cofibrant resolution $F' \to F$, and this depends naturally on F. With the standard choice, the following holds:

Proposition 5.7. *The homotopy colimit of F is naturally homeomorphic to the classifying space of the transport category $\mathcal{C}\int F$ of F. This has object space $\coprod_c F(c)$ and morphism space*

$$\coprod_{c,d} F(c) \times \mathrm{mor}_{\mathcal{C}}(c,d)\,,$$

so that a morphism from $x \in F(c)$ to $y \in F(d)$ is an element $g \in \mathrm{mor}_{\mathcal{C}}(c,d)$ for which $g_*(x) = y$.

In particular, if $F(c)$ is a singleton for every c in $\mathcal{C}$, then the transport category determined by F is identified with $\mathcal{C}$ itself, and so the homotopy colimit of F is identified with the classifying space of $\mathcal{C}$.
Another special case worth mentioning, because it is well known, is the Borel construction. Let Y be a space with an action of a group G. The group is a category with one object, and the group action determines a functor from that category to spaces. In this case the homotopy colimit is the Borel construction alias *homotopy orbit space*, $EG \times_G Y$.

In [34], where Segal introduced classifying spaces of arbitrary (topological) categories, homotopy colimits also made their first appearance, namely as classifying spaces of transport categories. The derived functor approach in definition 5.6 was developed more thoroughly in [5], now the standard reference for homotopy colimits and homotopy limits, and later in [7].

Our theme is that most stratifications lead to homotopy colimit decompositions. Let us first note that many homotopy colimits are stratified. Compare [35].

Example 5.8. Let $\mathcal{C}$ be a small EI–category (all Endomorphisms in $\mathcal{C}$ are Isomorphisms). For each isomorphism class $[C]$ of objects in $\mathcal{C}$, we define a locally closed subset $B\mathcal{C}_{[C]}$ of the classifying space $B\mathcal{C}$, as follows. A point $x \in B\mathcal{C}$ is in $B\mathcal{C}_{[C]}$ if the unique cell of $B\mathcal{C}$ containing x corresponds to a diagram

$$C_0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_k$$

without identity arrows, where C_0 is isomorphic to C. (Remember that $B\mathcal{C}$ is a CW–space, with one cell for each diagram $C_0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_k$ as above.) Then $B\mathcal{C}$ is stratified, with one stratum $B\mathcal{C}_{[C]}$ for each isomorphism class $[C]$.

Example 5.9. Let $F\colon \mathcal{C} \to$ *Spaces* be a functor, where $\mathcal{C}$ is a small EI–category. Then $\mathcal{C}\int F$ is a topological EI–category; hence

$$B(\mathcal{C}\textstyle\int F) \;=\; \mathrm{hocolim}\, F$$

is stratified as in example 5.8, with one stratum for each isomorphism class $[C]$ of objects in $\mathcal{C}$. (This stratification can also be pulled back from the stratification of $B\mathcal{C}$ defined above, by means of the projection $B(\mathcal{C}\int F) \to B\mathcal{C}$.)

In order to show that "most" stratified spaces can be obtained by the procedure described in example 5.8, we now associate to each stratified space X a topological category.

Definition 5.10. Let X be a stratified space with strata X_i. A path $\gamma\colon [0,c] \to X$, with $c \geq 0$, is *nonincreasing* if the induced map from $[0,c]$ to the poset of strata of X is nonincreasing. The *nonincreasing path category* $\mathcal{C}_X$ of a stratified space X has object set X (made discrete). The space of morphisms from $x \in X$ to $y \in X$

is the space of nonincreasing paths starting at x and ending at y. Composition of morphisms is Moore composition of paths.

Each diagram of the form $x_0 \leftarrow x_1 \leftarrow \cdots \leftarrow x_k$ in $\mathcal{C}_X$ determines real numbers $c_1, c_2, \ldots, c_k \geq 0$ and a nonincreasing path $\gamma\colon [0, c_1 + \cdots c_k] \to X$ with $\gamma(0) = x_k$ and $\gamma(c_1 + \cdots c_k) = x_0$. Composing γ with the linear map $\Delta^k \to [0, c_1 + \cdots c_k]$ taking the i–th vertex to $c_{i+1} + \cdots + c_k$, we obtain a map $\Delta^k \to X$; and by "integrating" over all such diagrams, we have a canonical map

$$B\mathcal{C}_X \longrightarrow X\,. \tag{10}$$

Definition 5.11. The stratified space X is *decomposable in the large* if (10) is a weak homotopy equivalence. It is *everywhere decomposable* if each open subset of X, with the stratification inherited from X, is decomposable in the large.

If X is decomposable in the large, we can think of (10) as a homotopy colimit decomposition of X, since $B\mathcal{C}_X = \operatorname{hocolim} F$ for the functor F given by $F(x) = *$, for all objects x. Note that $\mathcal{C}_X$ is an EI–category "up to homotopy". That is, for any object y of $\mathcal{C}_X$, the space of endomorphisms of y is a grouplike topological monoid:

$$\operatorname{mor}_{\mathcal{C}_X}(y, y) = \Omega(X_i, y)\,,$$

where X_i is the stratum containing y. More to the point, the category $\pi_0(\mathcal{C}_X)$, with the same object set as $\mathcal{C}_X$ and morphism sets

$$\operatorname{mor}_{\pi_0(\mathcal{C}_X)}(y, z) = \pi_0 \operatorname{mor}_{\mathcal{C}_X}(y, z),$$

is an EI–category.

More useful homotopy colimit decompositions of X can often be constructed from the one above by choosing a continuous functor $p\colon \mathcal{C}_X \to \mathcal{D}$, where $\mathcal{D}$ is a discrete EI–category, and using

$$\operatorname*{hocolim}_{\mathcal{C}_X} \ F \ \simeq \ \operatorname*{hocolim}_{\mathcal{D}} \ p_* F\,. \tag{11}$$

Here p_*F is the "pushforward" of F along p, also known as the left homotopy Kan extension. It associates to an object d of $\mathcal{D}$ the homotopy colimit of $F \circ \varphi_d$, where φ_d is the forgetful functor from the "over" category p/d to $\mathcal{C}_X$. An object of p/d consists of an object x in $\mathcal{C}_X$ and a morphism $p(x) \to d$ in $\mathcal{D}$. Consult the last pages of [6] for formula (11) and other useful tricks with homotopy colimits and homotopy limits.

Keeping the notation in (11), let $x \in X$ and let $g : p(x) \to d$ be a morphism in $\mathcal{D}$. A *lift* of (x, g) consists of a morphism $\gamma : x \to y$ in $\mathcal{C}_X$ and an isomorphism $u : p(y) \to d$ in $\mathcal{D}$ such that $u \circ p(\gamma) = g$. If X is locally 1–connected, then the set of lifts of (x, g) has a canonical topology which makes it into a covering space of a subspace of the space of all (Moore) paths in X.

Definition 5.12. Assume that X is locally 1-connected. We will say that p has *contractible chambers* if the space of lifts of (x, g) is weakly homotopy equivalent to a point, for each (x, g) as above.

Proposition 5.13. Suppose that X is locally 1–connected and p has contractible chambers. Then for d in $\mathcal{D}$, the value $(p_*F)(d)$ is weakly homotopy equivalent to a covering space of a union of strata of X; namely, the space of pairs (y,u) where $y \in X$ and $u : p(y) \to d$ is an isomorphism.

Example 5.14. Let $X = |\mathcal{W}|$ with the stratification discussed in the previous section. This is decomposable in the large. We can think of a point $x \in |\mathcal{W}|$ as a smooth oriented 3–manifold E_x without boundary, with a proper map $f_x : E_x \to \mathbb{R}$. A path from x to y in $|\mathcal{W}|$ amounts to a family of smooth oriented 3–manifolds $E_{\gamma(t)}$, each without boundary and with a proper map $f_{\gamma(t)} : E_{\gamma(t)} \to \mathbb{R}$; the parameter t runs through an interval $[0,c]$ and $\gamma(0) = x$, $\gamma(c) = y$. For each $t \in [0,c]$ let $S_{\gamma(t)}$ be the set of critical points of $f_{\gamma(t)}$ with critical value 0; this comes with a map to $\{0,1,2,3\}$, the Morse index map. If the path is nonincreasing, then it is easy to identify each $S_{\gamma(t)}$ with a subset of $S_{\gamma(0)} = S_x$, in such a way that we have a nonincreasing family of subsets $S_{\gamma(t)}$ of the finite set S_x, parametrized by $t \in [0,c]$. With every $z \in S_x = S_{\gamma(0)}$ which is not in the image of $S_y = S_{\gamma(c)}$, we can associate an element $\varepsilon(z) \in \{-1,+1\}$, as follows. There is a largest $t \in [0,c]$ such that $z \in S_{\gamma(t)}$; call it $t(z)$. The stability property of nondegenerate critical points ensures that for t just slightly larger than $t(z)$, the element z viewed as a point in $S_{\gamma(t(z))} \subset E_{\gamma(t(z))}$ is close to a unique critical point of $f_{\gamma(t)} : E_{\gamma(t)} \to \mathbb{R}$. The latter has critical value either greater than 0, in which case $\varepsilon(z) = +1$, or less than 0, in which case $\varepsilon(z) = -1$. Summarizing, a morphism $\gamma : x \to y$ in $\mathcal{C}_X$ determines an injective map $\gamma^* : S_y \to S_x$ over $\{0,1,2,3\}$ and a function from $S_x \smallsetminus u(S_y)$ to the set $\{+1,-1\}$.
These considerations lead us to a certain category $\mathcal{K}$. Its objects are the finite sets over $\{0,1,2,3\}$; a morphism from S to T in $\mathcal{K}$ is an injective map u from S to T over $\{0,1,2,3\}$, together with a function ε from $T \smallsetminus u(S)$ to $\{-1,+1\}$. The composition of two composable morphisms $(u_1,\varepsilon_1) : R \to S$ and $(u_2,\varepsilon_2) : S \to T$ in $\mathcal{K}$ is (u_2u_1,ε_3), where ε_3 agrees with ε_2 on $T \smallsetminus u_2(S)$ and with $\varepsilon_1 {u_2}^{-1}$ on $u_2(S \smallsetminus u_1(R))$. The rule $x \mapsto S_x$ described above is a functor p from $\mathcal{C}_X$ to $\mathcal{K}^{\mathrm{op}}$. This functor has contractible chambers. For S in $\mathcal{K}$, the space $(p_*F)(S)$ is therefore, by proposition 5.13, a classifying space for families of 3–manifolds E_x equipped with $f_x : E_x \to \mathbb{R}$ as in theorem 3.4 and with a *specified* isomorphism $S_x \to S$ in $\mathcal{K}$. Consequently, it can be identified with a finite–sheeted covering space of the stratum $|\mathcal{W}_{\langle S\rangle}|$ of $|\mathcal{W}|$; see section 4. It is convenient to write $|\mathcal{W}_S|$ for $(p_*F)(S)$.
Proposition 5.13 does not say anything very explicit about the map $|\mathcal{W}_S| \to |\mathcal{W}_R|$ induced by a morphism $(u,\varepsilon) : R \to S$ in $\mathcal{K}$, but this is easily described up to homotopy. Namely, suppose given a family of smooth 3–manifolds E_x with Morse functions f_x and isomorphisms $a_x : S_x \to S$, as above. Now perturb each f_x slightly by adding a small smooth function $g_x : E_x \to \mathbb{R}$ with support in a small neighborhood of S_x, locally constant in a smaller neighborhood of S_x, and such that for $z \in S_x$ we have

$$g_x(z) \quad \begin{cases} = 0 & \text{if} \quad a_x(z) \in u(R) \\ > 0 & \text{if} \quad \varepsilon(a_x(z)) = +1 \\ < 0 & \text{if} \quad \varepsilon(a_x(z)) = -1\,. \end{cases}$$

(The g_x should also depend smoothly on the parameter x, like the f_x.) The set of critical points of $f_x + g_x$ having critical value 0 is then identified with $u(R) \cong R$. Therefore, by keeping the E_x and substituting the $f_x + g_x$ for the f_x, we obtain

a family of the type which is classified by maps to $|\mathcal{W}_R|$. Letting the parameter manifold approximate $|\mathcal{W}_S|$, we obtain a well defined homotopy class of maps $|\mathcal{W}_S| \to |\mathcal{W}_R|$.
For fixed S, the characterization of $|\mathcal{W}_S|$ as a classifying space for families of oriented 3–manifolds E_x with proper Morse functions $f_x : E_x \to \mathbb{R}$ and isomorphisms $S_x \cong S$ can be simplified. It turns out that we need only allow Morse functions f_x having no other critical points than those in S_x; that is, no critical values other than, possibly, 0. When this extra condition is imposed, the families considered are automatically *bundles* of 3–manifolds over the parameter space — not just submersions with 3–dimensional fibers. This is an easy consequence of Ehresmann's fibration theorem. Equally important is the fact that each of the 3–manifolds E_x in such a bundle can be reconstructed from the closed oriented surface ${f_x}^{-1}(-1)$ and certain surgery data. These data are instructions for disjoint oriented surgeries [41, §1] on the surface, one for each element of $S_x \cong S$.
In this way, we end up with a description of $|\mathcal{W}_S|$ as a classifying space for bundles of closed oriented surfaces, where each surface comes with data for disjoint oriented surgeries labelled by elements of S. (*End of example.*)

The stratification of $|\mathcal{W}_{\mathrm{loc}}|$ sketched in the previous section can be taken to pieces in a similar fashion. The result is a homotopy colimit decomposition

$$|\mathcal{W}_{\mathrm{loc}}| \quad \simeq \quad \operatorname*{hocolim}_{S} \; |\mathcal{W}_{\mathrm{loc},S}|$$

where S runs through $\mathcal{K}$. Here $|\mathcal{W}_{\mathrm{loc},S}|$ should be thought of as the space of S–tuples of oriented surgery instructions on an oriented surface — but without a specified surface! See [26] for details. The homotopy colimit decompositions for $|\mathcal{W}|$ and $|\mathcal{W}_{\mathrm{loc}}|$ are related via obvious forgetful maps.

Now, in order to obtain information about the homotopy fibers of $|\mathcal{W}| \to |\mathcal{W}_{\mathrm{loc}}|$, one can ask what the homotopy fibers of

$$|\mathcal{W}_S| \to |\mathcal{W}_{\mathrm{loc},S}|$$

are, for each S in $\mathcal{K}$, and then how they vary with S. The first question is easy to answer: the homotopy fibers of $|\mathcal{W}_S| \to |\mathcal{W}_{\mathrm{loc},S}|$ are classifying spaces for bundles of compact oriented surfaces with a prescribed boundary depending on the chosen base point in $|\mathcal{W}_{\mathrm{loc},S}|$. The dependence on S can be seen in commutative squares of the form

$$\begin{array}{ccc} |\mathcal{W}_S| & \longrightarrow & |\mathcal{W}_{\mathrm{loc},S}| \\ \big\downarrow & & \big\downarrow \\ |\mathcal{W}_R| & \longrightarrow & |\mathcal{W}_{\mathrm{loc},R}| \end{array}$$

where the vertical arrows are induced by a morphism $R \to S$ in $\mathcal{K}$. With the above geometric description, the induced maps from the homotopy fibers of the top horizontal arrow to the homotopy fibers of the bottom horizontal arrow are maps of the kind considered in Harer's stability theorem 1.3. Unfortunately the stability theorem cannot be used here without further preparation: there is no reason to suppose that all surfaces in sight are connected and of large genus. Fortunately, however, the homotopy colimit decomposition of $|\mathcal{W}|$ described in 5.14 can be rearranged and modified in such a way that this objection can no longer be made. (At this point, concatenation matters must be taken seriously and consequently some of the main results obtained so far must be reworded, as explained in remark 2.4.) The stability theorem 1.3 can then be applied and the homotopy fiber sequence (9) is a formal consequence.

To conclude, it seems worthwhile to stress that the Harer stability theorem 1.3 is an enormously important ingredient in the proof of Madsen's conjecture 1.6. But in contrast to Vassiliev's h–principle (example 3.2), which is an equally important ingredient, the stability theorem only makes a very brief and decisive appearance at the end of the proof. There it is used almost exactly as in Tillmann's proof of 1.4.

A. **Vassiliev's h–principle: An outline of the proof**

This outline covers only the case where the manifold M is closed. It follows [40] in all essentials. I have made some minor rearrangements in the overall presentation, emphasizing the way in which transversality theory and interpolation theory shape the proof. I am indebted to Thomas Huettemann for suggesting this change in emphasis. Any errors and exaggerations which may have resulted from it should nevertheless be blamed on me. Besides, it is not a big change: *plus ça change, plus c'est la même chose.*

Let Z be the topological vector space of all smooth maps $M \to \mathbb{R}^n$, with the Whitney C^∞ topology [13]. Let $Z_{\mathfrak{A}} \subset Z$ be the closed subset consisting of those $f\colon M \to \mathbb{R}^n$ which have at least one singularity of type $\mathfrak{A}$. Then $Z \smallsetminus Z_{\mathfrak{A}}$ is identified with $\Gamma_{\mathrm{itg}} \cap \Gamma_{\neg\mathfrak{A}}$. As our starting point, we take the idea to approximate $Z \smallsetminus Z_{\mathfrak{A}}$ by subspaces of the form $D \smallsetminus Z_{\mathfrak{A}}$ where D can be any finite dimensional *affine* subspace of Z; in other words, D is a translate of a finite dimensional linear subspace. To be more precise, let r be a positive integer; we will look for finite dimensional affine subspaces D of Z such that the inclusion–induced map in integer cohomology

$$H^*(Z \smallsetminus Z_{\mathfrak{A}}) \longrightarrow H^*(D \smallsetminus Z_{\mathfrak{A}}) \tag{12}$$

is an isomorphism for $* \le r$.
Vassiliev's method for solving this important approximation problem is to impose a general position condition $(\mathbf{c_1})$ and an interpolation condition $(\mathbf{c_{2,r}})$ on D. The two conditions are described just below. Further down there is a sketch of Vassiliev's argument showing that (12) is indeed an isomorphism for $* \le r$ if D satisfies both $(\mathbf{c_1})$ and $(\mathbf{c_{2,r}})$. The h–principle then "falls out" as a corollary.

Condition $(\mathbf{c_1})$ requires, roughly, that the finite dimensional affine subspace $D \subset Z$ be in general position relative to $Z_{\mathfrak{A}}$.
Vassiliev is not very precise on this point, but I understand from [23] that every semialgebraic subset S of a finite dimensional real vector space V has a preferred *regular* stratification. This is a partition of S into smooth submanifolds of V which satisfies the conditions for a stratification and, in addition, Whitney's regularity conditions [42]. In particular, the portion of $\mathfrak{A}$ lying over $0 \in \mathbb{R}^m$ has a preferred regular stratification; it follows that $\mathfrak{A}(M)$ has a preferred regular stratification as a subset of the smooth manifold $J^k(M, \mathbb{R}^n)$.

Definition A.1. Let D be a finite dimensional affine subspace of Z. We say that D satisfies condition $(\mathbf{c_1})$ if

- the map $u : D \times M \longrightarrow J^k(M, \mathbb{R}^n)$ given by $u(f, x) = j^k f(x)$ is transverse to each stratum of $\mathfrak{A}(M)$, so that $u^{-1}(\mathfrak{A}(M))$ is regularly stratified in $D \times M$;

- the projection from $u^{-1}(\mathfrak{A}(M))$ to D is *generic*.

The second item in definition A.1 amounts to a condition on the multijets [13] of the evaluation map $D \times M \to \mathbb{R}^n$ at finite subsets of $u^{-1}(\mathfrak{A}(M))$. The condition implies local injectivity of the projection from $u^{-1}(\mathfrak{A}(M))$ to D, and self-transversality in a stratified setting. More precision would take us too far.

The content of the much more striking condition $(\mathbf{c_{2,r}})$ is that D must contain at least one solution for each interpolation problem on M of a certain type depending on r.

Definition A.2. Let d_{kmn} be the dimension of the real vector space of degree $\leq k$ polynomial maps from $\mathbb{R}^m$ to $\mathbb{R}^n$. Let D be a finite dimensional affine subspace of Z. We say that D satisfies condition $(\mathbf{c_{2,r}})$ if, for every $C^\infty(M,\mathbb{R})$–submodule Y of Z with $\dim_{\mathbb{R}}(Z/Y) \leq r \cdot d_{kmn}$, the projection $D \to Z/Y$ is onto.

To see what this has to do with interpolation, fix distinct points $z_1, z_2, \ldots, z_s$ in M, with $s \leq r$, and k–jets $u_1, u_2, \ldots, u_s \in J^k(M, \mathbb{R}^n)$ so that z_i is the source of u_i. Let Y consist of the $f \in Z$ whose k–jets at $z_1, z_2, \ldots, z_s$ vanish. If D satisfies $(\mathbf{c_{2,r}})$, then $D \to Z/Y$ must be onto and so there exists $g \in D$ with $j^k g(z_i) = u_i$ for $i = 1, 2, \ldots, s$.

Let $\mathcal{L}_r$ be the collection of all finite dimensional affine subspaces D of Z satisfying $(\mathbf{c_1})$ and $(\mathbf{c_{2,r}})$. Then we have $\mathcal{L}_{r+1} \subset \mathcal{L}_r$ for all $r \geq 0$.

Lemma A.3. *There exists an increasing sequence of finite dimensional affine subspaces $D_1, D_2, D_3, D_4, \ldots$ of Z such that $D_i \in \mathcal{L}_i$ and the union $\bigcup_i D_i$ is dense in the space Z.*

The proof of this is an application of transversality theory as in [13] and interpolation theory as in [12]. The fact that the set of all $C^\infty(M,\mathbb{R})$–submodules Y of Z satisfying the conditions in definition A.2 has a canonical topology making it into a *compact* Hausdorff space is an essential ingredient.

Theorem A.4. *The map* (12) *is an isomorphism if $D \in \mathcal{L}_r$ and $* \leq r$.*

Sketch proof. Write $D_{\mathfrak{A}} = D \cap Z_{\mathfrak{A}}$. In the notation of definition A.2, this is the image of $u^{-1}(\mathfrak{A}(M))$ under the projection $D \times M \to D$. Condition $(\mathbf{c_1})$ on D ensures that $D_{\mathfrak{A}}$ is a well–behaved subset of D, so that there is an Alexander duality isomorphism

$$H^*(D \smallsetminus Z_{\mathfrak{A}}) \xrightarrow{\cong} H^{lf}_{\dim(D)-*-1}(D_{\mathfrak{A}}), \tag{13}$$

where the superscript *lf* indicates that locally finite chains are used. To investigate $D_{\mathfrak{A}}$, Vassiliev introduces a *resolution* $RD_{\mathfrak{A}}$ of $D_{\mathfrak{A}}$, as follows. Let $\Delta(M)$ be the simplex spanned by M, in other words, the set of all functions w from M to $[0,1]$ such that $\{x \in M \mid w(x) > 0\}$ is finite and $\sum_{x \in M} w(x) = 1$. The standard topology of $\Delta(M)$ as a simplicial complex is not of interest here, since it does not reflect the topology of M. Instead, we endow $\Delta(M)$ with the smallest topology such that, for each continuous $g : M \to \mathbb{R}$, the map $w \mapsto \sum_x w(x)g(x)$ is continuous on $\Delta(M)$. We write $\Delta(M)_t$ to indicate this topology. Now $RD_{\mathfrak{A}}$ can be defined as the subspace of $D_{\mathfrak{A}} \times \Delta(M)_t$ consisting of all (f, w) such that the support of w is contained in the set of $\mathfrak{A}$–singularities of f. Because D satisfies condition $(\mathbf{c_1})$, the projection

$$RD_{\mathfrak{A}} \longrightarrow D_{\mathfrak{A}}$$

is a proper map between locally compact spaces. Each of its fibers is a simplex, and it is not difficult to deduce that it induces an isomorphism in locally finite homology:

$$H_*^{lf}(RD_{\mathfrak{A}}) \xrightarrow{\ \cong\ } H_*^{lf}(D_{\mathfrak{A}}). \tag{14}$$

For an integer p, let $RD^p_{\mathfrak{A}} \subset RD_{\mathfrak{A}}$ consist of the pairs (f,w) where the support of w has at most p elements. The filtration of $RD_{\mathfrak{A}}$ by the closed subspaces $RD^p_{\mathfrak{A}}$ leads in the usual manner to a homology spectral sequence of the form

$$E^1_{p,q} = H^{lf}_{p+q+\dim(D)}(RD^p_{\mathfrak{A}}, RD^{p-1}_{\mathfrak{A}}) \implies H^{lf}_{p+q+\dim(D)}(RD_{\mathfrak{A}}) \tag{15}$$

where $p,q \in \mathbb{Z}$. There are three vanishing lines: $E^1_{p,q} = 0$ for $p < 0$ and for $p+q < -\dim(D)$ by construction, but also

$$E^1_{p,q} = 0 \text{ when } 2p+q > 0. \tag{16}$$

To understand (16), note or accept that by the general position condition $(\mathbf{c_1})$ on D, the codimension of the image of $RD^p_{\mathfrak{A}}$ in D is at least $p(\mathrm{codim}(\mathfrak{A}) - m)$; here $\mathrm{codim}(\mathfrak{A})$ denotes the codimension of $\mathfrak{A}$ in $J^k(\mathbb{R}^m, \mathbb{R}^n)$. Since the fibers of the projection $RD^p_{\mathfrak{A}} \to D$ are at most p–dimensional, it follows that the *dimension* of $RD^p_{\mathfrak{A}}$ is not greater than $p + \dim(D) - p(\mathrm{codim}(\mathfrak{A}) - m)$. With our hypothesis $\mathrm{codim}(\mathfrak{A}) \geq m+2$ this implies

$$\dim(RD^p_{\mathfrak{A}}) \leq \dim(D) - p\,,$$

and (16) follows.

Now comes the crucial observation that $E^1_{p,q}$ does not depend on our choice of $D \in \mathcal{L}_r$, as long as $p \leq r$. To see this, we use the multi–jet prolongation map

$$RD_{\mathfrak{A}} \longrightarrow \Delta(\mathfrak{A}(M))_t \tag{17}$$

which takes $(f,w) \in RD_{\mathfrak{A}}$ to $\bar{w}$ with $\bar{w}(u) = w(x)$ if $u = j^k f(x)$ and $\bar{w}(u) = 0$ otherwise. Here $\Delta(\mathfrak{A}(M))_t$ is the simplex spanned by the set $\mathfrak{A}(M) \subset J^k(M, \mathbb{R}^n)$, but again topologized so that the topology of $\mathfrak{A}(M)$ is reflected; cf. the definition of $\Delta(M)_t$. Note that each fiber of (17) is identified with an affine subspace of D; but the fiber dimensions can vary and some fibers may even be empty. But restricting (17), we have

$$RD^p_{\mathfrak{A}} \smallsetminus RD^{p-1}_{\mathfrak{A}} \quad \longrightarrow \quad \Delta(\mathfrak{A}(M))^p_t \smallsetminus \Delta(\mathfrak{A}(M))^{p-1}_t \tag{18}$$

where $\Delta(\mathfrak{A}(M))^p_t$ consists of the $w \in \Delta(\mathfrak{A}(M))_t$ whose support has at most p elements. (The indexing goes against all traditions, but it is consistent.) Now the interpolation condition $(\mathbf{c_{2,r}})$ on D and our assumption $p \leq r$ imply that the fibers of (18) are *nonempty* affine spaces, and all of the same dimension; in other words, (18) is a bundle of affine spaces. Its base space obviously does not depend on D, and it can be shown that its first Stiefel–Whitney class, too, is independent of D. Consequently the locally finite homology of the total space,

$$H^{lf}_*(RD^p_{\mathfrak{A}} \smallsetminus RD^{p-1}_{\mathfrak{A}}) \;\cong\; H^{lf}_*(RD^p_{\mathfrak{A}}, RD^{p-1}_{\mathfrak{A}}) \;=\; E^1_{p,*-p-\dim(D)}\,,$$

is identified with the locally finite homology of the base space, with twisted integer coefficients, and so is independent of D except for the obvious dimension shift. (Note the strong excision property of locally finite homology groups.) To state the

independence result more precisely, the spectral sequence (15) depends contravariantly on D, and for $C, D \in \mathcal{L}_r$ with $C \subset D$, the induced map from the D–version of $E^1_{p,q}$ to the C–version of $E^1_{p,q}$ is an isomorphism whenever $p \le r$.
Remembering (16) now, we can immediately deduce that $E^m_{p,q}$ is also independent of $D \in \mathcal{L}_r$, in the same sense, for any $m \ge 0$ and p, q with $p + q \ge -r$. Remembering the isomorphisms (14) and (13) also, we then conclude that for $C, D \in \mathcal{L}_r$ with $C \subset D$, the inclusion $C \smallsetminus Z_{\mathfrak{A}} \to D \smallsetminus Z_{\mathfrak{A}}$ induces an isomorphism

$$H^*(D \smallsetminus Z_{\mathfrak{A}}) \quad \longrightarrow \quad H^*(C \smallsetminus Z_{\mathfrak{A}})$$

for $* \le r$. With lemma A.3, this leads us finally to the statement that (12) is an isomorphism for $* \le r$ and $D \in \mathcal{L}_r$. (First suppose that D is one of the D_i in lemma A.3; then for the general case, approximate D by affine subspaces of the D_i for $i \gg 0$.) $\square$

But we have achieved much more. Letting r tend to ∞, we have a well defined spectral sequence converging to

$$H^*(Z \smallsetminus Z_{\mathfrak{A}}) = H^*(\Gamma_{\mathrm{itg}} \cap \Gamma_{\neg\mathfrak{A}}),$$

independent of r. (Convergence is a consequence of (16), and again lemma A.3 is needed to show that the spectral sequence is independent of all choices.) Similar but easier reasoning leads to an analogous spectral sequence converging to the cohomology of $\Gamma_{\neg\mathfrak{A}}$. By a straightforward inspection, the inclusion of $\Gamma_{\mathrm{itg}} \cap \Gamma_{\neg\mathfrak{A}}$ in $\Gamma_{\neg\mathfrak{A}}$ induces an isomorphism of the E^1–pages. This establishes Vassiliev's h–principle in the case where M is closed.

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