DUALITY IN WALDHAUSEN CATEGORIES

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Abstract. We develop a theory of Spanier–Whitehead duality in categories with cofibrations and weak equivalences (Waldhausen categories, for short). This includes L–theory, the involution on K–theory introduced by [Vo] in a special case, and a map Ξ relating L–theory to the Tate spectrum of $\mathbb{Z}/2$ acting on K–theory. The map Ξ is a distillation of the long exact Rothenberg sequences [Sha], [Ra1], [Ra2], including analogs involving higher K–groups. It goes back to [WW2] in special cases. Among the examples covered here, but not in [WW2], are categories of retractive spaces where the notion of weak equivalence involves control.

0. Introduction

For any ring R, Quillen has defined an algebraic K-theory Ω-spectrum, $\boldsymbol{K}(R)$. His construction is in terms of the category of finitely generated projective modules over R, but it can be applied to any exact category. In order to study concordances of manifolds Waldhausen generalized Quillen's construction to apply to what Waldhausen calls categories with cofibrations and weak equivalences. (Following Thomason we will call them Waldhausen categories.) An example is given by the category of based compact CW–spaces. More generally, the category of retractive relative CW–spaces $Y \rightleftarrows X$ over a fixed space X, with compact quotient Y/X, is an example of a Waldhausen category ; its K–theory spectrum is known as $\boldsymbol{A}(X)$.

Waldhausen supplied several powerful tools along with his construction of $\boldsymbol{K}(\mathcal{C})$ for a Waldhausen category $\mathcal{C}$, such as the additivity theorem, the approximation theorem, and the generic fibration theorem. Even if one is primarily interested in the algebraic K-theory of rings these tools have important applications [Sta] , [Tho2] . Also in [CPed] excision in controlled algebraic K–theory of rings is proved using Waldhausen's machinery.

Suppose that R is a ring with involution (involutory anti–automorphism). The problem of classifying manifolds up to homeomorphism or diffeomorphism lead Wall to define algebraic L-groups $L_n(R)$ in terms of quadratic forms on finitely generated projective (or free, or based free) modules over R. The L–groups turned out to be the homotopy groups of a spectrum [Q]. Ranicki [Ra3] associates such L–theory spectra to any *additive category with chain duality.* (There is a *quadratic*

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L–theory spectrum made using quadratic forms, and a *symmetric* L–theory spectrum made using symmetric forms.) Specifically, by using additive categories of "modules parametrized by a simplicial complex" Ranicki gave an algebraic description of Quinn's L-theory assembly map which is used to classify manifolds up to homeomorphism. Also Ranicki's chain duality setup has been used to construct controlled versions of L-theory for rings [FP]. The assembly map in L-theory can then also be identified with a forget control map. This has been used to prove many cases of the Novikov conjecture ; see [FRR].

If one wants to study the spaces of homeomorphisms or diffeomorphisms of a manifold, then the L-theory of rings or even additive categories is not adequate. One is forced to consider L-theory *and* K–theory of certain Waldhausen categories equipped with duality, and a certain map Ξ relating the L–theory to the K–theory. For this reason we need a theory of duality in Waldhausen categories which, unlike Ranicki's chain duality theory, allows "nonlinear" cases ; and we need to understand, in this generality, L–theory, K–theory, and the Ξ–map. Waldhausen categories of retractive spaces where the notion of weak equivalence involves *control* are important in applications to geometry and should therefore be the central examples of such a theory.

To explain what Ξ is about, we note that many of the classical invariants for symmetric forms over a ring R take values in groups constructed from $K_i(R)$ where $i = 0, 1,$ or 2. This suggests that there should be a connection between the symmetric L–theory spectrum $\boldsymbol{L}^\bullet(R)$ and the algebraic K–theory spectrum $\boldsymbol{K}(R)$. Since the classical constructions of L–theory are modelled on the pre-Quillen definitions of "low dimensional K-theory" , the connection is hidden. In [WW2] we established the connection by constructing Ξ, a natural map from $\boldsymbol{L}^\bullet(R)$ to the Tate spectrum for $\mathbb{Z}/2$ acting on $\boldsymbol{K}(R)$. A nonlinear version of this does already appear in [WW2], but it is very limited and there is no mention of "control". But it was certainly motivated by our study of spaces of homeomorphisms.

In this paper we introduce the notion of a Spanier–Whitehead product $\odot$ on a Waldhausen category $\mathcal{C}$. This is a functor $(C, D) \mapsto C \odot D$ from $\mathcal{C} \times \mathcal{C}$ to pointed spaces. The space $C \odot D$ should be thought of as the space of *pairings* between C and D. If $(\mathcal{C}, \odot)$ satisfies certain axioms listed in §2, reminiscent of Spanier–Whitehead duality, then for every C in $\mathcal{C}$ there exists an essentially unique object C' in $\mathcal{C}$ which comes with a *nondegenerate* pairing $\eta \in C \odot C'$. We think of C' as the dual of $\mathcal{C}$. We obtain an involution on $\boldsymbol{K}(\mathcal{C})$ which, modulo technicalities, is induced by $C \mapsto C'$. With the same hypotheses on $\mathcal{C}$ we have constructions of $\boldsymbol{L}^\bullet(\mathcal{C})$ (symmetric L–theory spectrum) and $\boldsymbol{L}_\bullet(\mathcal{C})$ (quadratic L–theory spectrum) which generalize Ranicki's constructions for additive categories with chain duality. There is a natural transformation Ξ from $\boldsymbol{L}^\bullet(\mathcal{C})$ to the Tate spectrum for $\mathbb{Z}/2$ acting on $\boldsymbol{K}(\mathcal{C})$. There is also a symmetrization map from $\boldsymbol{L}_\bullet(\mathcal{C})$ to $\boldsymbol{L}^\bullet(\mathcal{C})$.

In [DWW] we used the fact that the assembly map in the algebraic K-theory of spaces can be identified with a forget control map to prove an A-theory index theorem for the A-theory Euler characteristic. Exploiting Poincaré duality and the results and examples of this paper, we shall in a future paper produce a $\mathbb{Z}/2$-equivariant version which can be viewed as an A–theory index theorem for the A-theory signature.

Section headings

Guide. We define SW products in §1. In section §2 we list some axioms on a Waldhausen category $\mathcal{C}$ with SW product which are in force most of the time in succeeding sections. The main purpose of these axioms is to ensure that every object in $\mathcal{C}$ has a sufficiently unique Spanier–Whitehead dual. In §3 and §4 we show that this is the case.

Much of the material in the remaining sections is about verifying that SW products and the axioms are *hereditary*. That is, if the Waldhausen category $\mathcal{C}$ comes with an SW product satisfying the axioms, then certain other Waldhausen categories constructed from $\mathcal{C}$ inherit an SW product, which again satisfies the axioms.

Example 1: Waldhausen constructs the K-theory spectrum $\boldsymbol{K}(\mathcal{C})$ by first constructing a simplicial Waldhausen category $\mathcal{S}_\bullet\mathcal{C}$. To show that an SW product $\odot$ on $\mathcal{C}$ satisfying the axioms of §2 determines an involution on (a new model of) $\boldsymbol{K}(\mathcal{C})$, we first have to show that $\odot$ determines an SW product on each $\mathcal{S}_m\mathcal{C}$, again satisfying the axioms. The SW product on $\mathcal{S}_m\mathcal{C}$ appears already in §1, as an example, but the axioms are first checked in §5 and §6. Only then are we ready to produce an involution on $\boldsymbol{K}(\mathcal{C})$; this is done in §7.

Example 2: A *symmetric* object in a Waldhausen category $\mathcal{C}$ with SW product is an object C together with a homotopy fixed point ϕ of the symmetry involution on $C \odot C$. If the underlying point in $C \odot C$ is nondegenerate, we call (C, ϕ) a symmetric *Poincaré* object. For us, symmetric L–theory is the bordism theory of symmetric Poincaré objects in a Waldhausen category with SW product. The most elementary type of bordism (without any symmetric self–duality to begin with) in the Waldhausen category $\mathcal{C}$ would be a functor from the poset of nonempty faces of the standard 1–simplex Δ^1 to $\mathcal{C}$. The most elementary type of higher bordism would be a functor from the poset of nonempty faces of the standard n–simplex Δ^n to $\mathcal{C}$, with $n > 1$. Of course, these notions become more interesting when self–duality conditions are imposed. In any case, to do symmetric L–theory in $\mathcal{C}$, we need to introduce a category of functors $\mathcal{C}^*(m)$ from the poset of nonempty faces of a standard m–simplex to $\mathcal{C}$, for each $m \geq 0$. This appears already in §1, as an example. It is necessary to check that each $\mathcal{C}^*(m)$ inherits an SW–product from $\mathcal{C}$, which satisfies the axioms if the original one on $\mathcal{C}$ does. We do this in §8. Only then are we ready to define the symmetric L–theory of $\mathcal{C}$ and the map Ξ ; this is done in §9.

The discussion in §9 takes place at the space level. For example, in §9 we define the symmetric L–theory of $\mathcal{C}$ as a space, not as a spectrum. It turns out that in order to raise the discussion to spectrum level one needs a *spectrum*–valued SW product on $\mathcal{C}$. This is also needed to define *quadratic objects* in $\mathcal{C}$. Namely, suppose that $\mathcal{C}$ comes with a spectrum–valued SW product $(C, D) \mapsto C \odot_\bullet D$. Then we can (re–)define symmetric objects in $\mathcal{C}$ as pairs (C, ϕ) with $\phi \in \Omega^\infty((C \odot_\bullet C)^{h\mathbb{Z}/2})$ and we can define quadratic objects in $\mathcal{C}$ as pairs (C, ψ) with $\psi \in \Omega^\infty((C \odot_\bullet C)_{h\mathbb{Z}/2})$, where the subscript $h\mathbb{Z}/2$ indicates a homotopy orbit spectrum. A quadratic object (C, ψ) determines a symmetric object $(C, \mathcal{N}(\phi))$ where $\mathcal{N}$ is the *norm* map. We call (C, ψ) a quadratic *Poincaré* object if $(C, \mathcal{N}(\psi))$ is a symmetric Poincaré object. For us, quadratic L–theory is the bordism theory of quadratic Poincaré objects in a Waldhausen category with spectrum–valued SW product.

We show in §10 that a space–valued SW product on $\mathcal{C}$ is always the 0–th term of an Ω–spectrum–valued SW product. In §11, we assume that $\mathcal{C}$ comes with an Ω–spectrum–valued SW product satisfying the axioms. We then introduce the symmetric L–theory *spectrum* of $\mathcal{C}$, improve Ξ to a map of spectra, and define, at last, the quadratic L–theory spectrum of $\mathcal{C}$ and the symmetrization map from quadratic L–theory to symmetric L–theory.

In §12 we explain how these constructions depend functorially on the Waldhausen category $\mathcal{C}$ with SW product (satisfying the axioms).

The example sections §1.A, §2.A, §5.A and §12.A can be read in the order in which they come, after the theoretical part. Another possibility is to read §n.A after §n.

0.1. Conventions. A *space* is a compactly generated Hausdorff space, unless we say otherwise. Products and mapping spaces are formed in this category in the usual way [MaL]. Base points are nondegenerate unless we say otherwise. These conventions are taken from [Go1].

The geometric realization of a simplicial set X is a space $|X|$. The *nerve* of a category $\mathcal{C}$ is a certain simplicial set $\mathcal{N}\mathcal{C}$, and the *classifying space* of $\mathcal{C}$ is $|\mathcal{N}\mathcal{C}|$, which we usually shorten to $|\mathcal{C}|$.

Homotopy limits and homotopy colimits of diagrams of spaces are defined as in [Go1] (which is to say that they are defined as in [BK], except that we work with diagrams of topological spaces rather than diagrams of simplicial sets). We distinguish between *reduced* homotopy colimits (of diagrams of based spaces) and *unreduced* homotopy colimits (of diagrams of unbased spaces). A reduced homotopy colimit is the quotient of the corresponding unreduced homotopy colimit by the classifying space of the indexing category.

1. Spanier–Whitehead products

Let $\mathcal{C}$ be a *Waldhausen category*, that is, a category with cofibrations and weak equivalences [Wald2]. Let $*$ denote the zero object in $\mathcal{C}$.

1.1. Definition. By an *SW product* on $\mathcal{C}$ we shall mean a functor

$$(C, D) \mapsto C \odot D$$

from $\mathcal{C} \times \mathcal{C}$ to the category of based spaces (see 0.1) which is *w–invariant*, *symmetric* and *bilinear* (explanations follow).

- *w–Invariance* means that the functor takes pairs of weak equivalences to homotopy equivalences.
- *Symmetry* means that the functor comes with a natural isomorphism $\tau : C \odot D \cong D \odot C$, whose square is the identity on $C \odot D$.
- *Bilinearity* means (in the presence of symmetry) that, for fixed but arbitrary D, the functor $C \mapsto C \odot D$ takes any *cofiber square* in $\mathcal{C}$ to a homotopy pullback square of spaces. (A cofiber square is a commutative pushout square in which either the horizontal or the vertical arrows are cofibrations.) Bilinearity also means that $* \odot D$ is contractible.

Usually, in studying a specific Waldhausen category $\mathcal{C}$, one is compelled to introduce some other Waldhausen categories. For us it will be important to know whether and how these other Waldhausen categories can be equipped with SW products if $\mathcal{C}$ is so equipped.

For example, there is the simplicial Waldhausen category $\mathcal{S}_\bullet\mathcal{C}$ constructed from $\mathcal{C}$. See [Wald2]. Each $\mathcal{S}_m\mathcal{C}$ is a Waldhausen category whose objects are certain functors $(i, j) \mapsto C(i, j)$ from the poset of pairs (i, j) with $0 \le i, j \le m$ to $\mathcal{C}$. The functors must satisfy two conditions: $C(i, j) = *$ if $j \le i$, and

$$\begin{array}{ccc} C(i,j) & \longrightarrow & C(i,j+1) \\ \downarrow & & \downarrow \\ C(i+1,j) & \longrightarrow & C(i+1,j+1) \end{array}$$

is a pushout square where the horizontal arrows are cofibrations, for $i, j < m$. Up to isomorphism, such a functor (diagram) is determined by its top row, which can be an arbitrary string of cofibrations

$$* = C(0,0) \to C(0,1) \to C(0,2) \to C(0,3) \to \cdots \to C(0,m)\,.$$

The remaining information consists of chosen subquotients,

$$C(i,j) \cong C(0,j)/C(0,i)\,.$$

We suppose that $\mathcal{C}$ is equipped with an SW product and try to use this to define, in the simplest way possible, an SW product on the Waldhausen category $\mathcal{S}_m\mathcal{C}$, for each $m \geq 0$. A good formula is

$$C \odot D = \operatorname*{holim}_{\substack{i,j,p,q \\ i+q\geq m \\ j+p\geq m}} C(i,j) \odot D(p,q)$$

for C and D in $\mathcal{S}_m\mathcal{C}$. (The category of based spaces in the sense of 0.1 is closed under homotopy limits.) Note that the quadruples (i,j,p,q) satisfying the conditions $i+q \geq m$ and $j+p \geq m$ still form a poset, and $(i,j,p,q), \mapsto C(i,j) \odot D(p,q)$ is a covariant functor, so that the homotopy limit is at least defined. However, the formula does not seem all that simple. The justification is given in 1.3 and 1.4 below.

For now, the following may help. The main point here is that, for C and D in $\mathcal{S}_m\mathcal{C}$, an element η in $C \odot D$ determines elements in $C(i,j) \odot D(m-j, m-i)$, one for every pair i,j with $0 \leq i \leq m$ and $0 \leq j \leq m$. Under suitable conditions on $\mathcal{C}$, we will show in §6 that if η is *nondegenerate* (a notion defined in §2), then its images in the various $C(i,j) \odot D(m-j,m-i)$ are also nondegenerate. In other words, the quadruples (i,j,p,q) with $i+q=m$ and $j+p=m$ appearing in the definition of $C \odot D$ are particularly important to us. The quadruples (i,j,p,q) with $i+q>m$ and $j+p>m$ are needed mostly to give "coherence".

1.2. Lemma. *The functor $\odot$ on $\mathcal{S}_m\mathcal{C} \times \mathcal{S}_m\mathcal{C}$ defined above is an SW product.*

Proof. Given C, D in $\mathcal{S}_m\mathcal{C}$ we map $C \odot D$ isomorphically to $D \odot C$ via

$$C \odot D = \operatorname*{holim}_{\substack{i,j,p,q \\ i+q\geq m \\ j+p\geq m}} C(i,j) \odot D(p,q) \cong \operatorname*{holim}_{\substack{i,j,p,q \\ i+q\geq m \\ j+p\geq m}} D(p,q) \odot C(i,j) = D \odot C$$

using the symmetry property of $\odot$ as a functor on $\mathcal{C}\times\mathcal{C}$. For fixed i,j,p,q the functor $(C,D) \mapsto C(i,j) \odot D(p,q)$ on $\mathcal{S}_m\mathcal{C} \times \mathcal{S}_m\mathcal{C}$ has the bilinearity property required in 1.1. Therefore $\odot$ in 1.2 has the bilinearity property. $\square$

We may ask how the SW product in 1.2 varies with m. Write $[m] := \{0,1,\dots,m\}$. The category with objects $[m]$ for $m \geq 0$ and monotone maps as morphisms has an automorphism (conjugation) of order two which takes $f : [k] \to [m]$ to $\bar{f} = r_m f r_k$ where r_k and r_m are the order reversing bijections of $[k]$ and $[m]$, respectively. In other words, $\bar{f}(i) = m - f(k-i)$. Recall also that $[m] \mapsto \mathcal{S}_m\mathcal{C}$ is a *simplicial* category. Then, for C and D in $\mathcal{S}_m\mathcal{C}$, and a monotone $f : [k] \to [m]$, we have f^*C and $\bar{f}^*D$ in $\mathcal{S}_k\mathcal{C}$. For example,

$$(f^*C)(i,j) = C(f(i), f(j)), \qquad (\bar{f}^*D)(p,q) = D(\bar{f}(p), \bar{f}(q)),$$

and if $i+q \geq k$ and $j+p \geq k$, then $f(i) + \bar{f}(q) \geq m$ and $f(j) + \bar{f}(p) \geq m$. It follows that we have a map induced by f,

$$C \odot D \longrightarrow f^*C \odot \bar{f}^*D.$$

Let $u_s : [m] \to [1]$ be the unique monotone map with $u_s(s) = 1$ and $u_s(s-1) = 0$, for $0 < s \le m$. Let $v : [1] \to [m]$ be given by $v(0) = 0$ and $v(1) = m$, so that $u_s v = \mathrm{id}$ on $[1]$. Note $\bar{v} = v$. We shall use these monotone maps to compare $\odot$ on $\mathcal{S}_m\mathcal{C}$ with $\odot$ on $\mathcal{C}$, noting that $\mathcal{C} \cong \mathcal{S}_1\mathcal{C}$ and that $\odot$ on $\mathcal{C}$ is the same as $\odot$ on $\mathcal{C} \cong \mathcal{S}_1\mathcal{C}$ under this identification.

1.3. Proposition. *For C and D in $\mathcal{S}_1\mathcal{C}$ and $u_s, u_t : [m] \to [1]$ and $v : [1] \to [m]$ as above, the canonical map*

$$u_s^* C \odot u_t^* D \longrightarrow v^* u_s^* C \odot v^* u_t^* D \;=\; C \odot D$$

is a homotopy equivalence if $s + t \le m + 1$. If $s + t > m + 1$, then $u_s^* C \odot u_t^* D \simeq *$.

Proof. Let T be the poset of all quadruples (i, j, p, q) such that $i + q \ge m$ and $j + p \ge m$, and $0 \le i, j, p, q \le m$. Let F be the functor on T taking (i, j, p, q) to $u_s^* C(i, j) \odot u_t^* D(p, q)$. Let $T_0 \subset T$ consist of the (i, j, p, q) for which $i < s$ and $p < t$. Then for (i, j, p, q) not in T_0 we have $F(i, j, p, q) \simeq *$ by inspection. Also, any element in T which is $\ge$ an element in $T \smallsetminus T_0$ belongs to $T \smallsetminus T_0$. It follows that the projection

$$\operatorname{holim} F \longrightarrow \operatorname{holim} F \mid T_0$$

is a homotopy equivalence (again by inspection, going back to the definition of the holim).

Suppose now that $s+t \le m+1$. Then for $(i, j, p, q) \in T_0$ we have $q = (i+q)-i \ge m-i \ge m+1-s \ge t$ and similarly $j \ge s$, so that $F \mid T_0$ is isomorphic to a constant functor with constant value $C(0,1) \odot D(0,1)$. The nerve of T_0 is contractible because there is a maximal element. This shows that 1.3 holds when $s + t \le m + 1$.

Next suppose $s + t > m + 1$. Let $T_1 \subset T_0$ consist of the elements (i, j, p, q) for which $j < s$ or $q < t$. Note that $\omega = (s-1, m, t-1, m)$ is the maximal element in T_0. Then $T_1 \cup \omega$ is a retract of T_0 (in the category of posets, with maps preserving $\le$ as morphisms). Furthermore $F \mid T_0$ pulls back from $T_1 \cup \omega$ because $F \mid (T_0 \smallsetminus T_1)$ is isomorphic to a constant functor. This leads to maps

$$\operatorname{holim} F \mid T_0 \;\leftrightarrows\; \operatorname{holim} F \mid (T_1 \cup \omega)$$

which are easily seen to be reciprocal homotopy equivalences. From the definition of the homotopy inverse limit, there is a homotopy pullback square

$$\begin{array}{ccc} \operatorname{holim}(F \mid T_1 \cup \omega) & \longrightarrow & F(\omega) \\ \big\downarrow & & \big\downarrow \\ \operatorname{holim} F \mid T_1 & \longrightarrow & \mathbf{map}(|T_1|, F(\omega)) \end{array}$$

where the vertical arrow on the left is projection, and the one on the right is the inclusion of the constant maps. Note that T_1 has a final sub–poset consisting of the (i, j, p, q) with $i = s - 1$ and $p = t - 1$. This final sub–poset has a contractible classifying space, so that $|T_1|$ itself is contractible. Therefore (from the homotopy pullback square just above) the projection

$$\operatorname{holim} F \mid (T_1 \cup \omega) \longrightarrow \operatorname{holim} F \mid T_1$$

is a homotopy equivalence. Since $F \mid T_1$ has contractible values, we can conclude $\operatorname{holim} F \simeq \operatorname{holim} F \mid T_0 \simeq \operatorname{holim}(F \mid T_1 \cup \omega) \simeq \operatorname{holim} F \mid T_1 \simeq *$. $\square$

We see that our formula for $\odot$ on $\mathcal{S}_m\mathcal{C}$ gives good and predictable results when applied to objects of the form u_s^*C, u_t^*D with C, D in $\mathcal{S}_1\mathcal{C} \cong \mathcal{C}$. An arbitrary object E in $\mathcal{S}_m\mathcal{C}$ fits into a natural diagram

$$* = E_0 \rightarrowtail E_1 \rightarrowtail \cdots \rightarrowtail E_n = E$$

where the arrows are cofibrations and each cofiber E_s/E_{s-1} is isomorphic to an object of the form $u_s^*C_s$ for some C_s in $\mathcal{S}_1\mathcal{C}$. Because of the bilinearity, it follows that $\odot$ is "correctly defined" on all of $\mathcal{S}_m\mathcal{C}$.

1.4. Example. Let $\mathcal{C}$ be the category of chain complexes of finitely generated free abelian groups, graded over $\mathbb{Z}$ and bounded (from below and above). The morphisms are the chain maps ; a morphism is a cofibration if it is split mono in each degree, and a weak equivalence if it is a homotopy equivalence. Each C in $\mathcal{C}$ has a dual $C^* = \hom(C, \mathbb{Z})$. For C, D in $\mathcal{C}$ we have chain complexes $\hom(C, D)$ and $C \otimes D$, and an isomorphism $C \otimes D \cong \hom(C^*, D)$.

We can introduce analogous notions in $\mathcal{S}_m\mathcal{C}$. For C in $\mathcal{S}_m\mathcal{C}$ we define C^* by $C^*(i,j) := (C(m-j, m-i))^*$. For C and D in $\mathcal{S}_m\mathcal{C}$ we define $\hom(C, D)$ as a chain subcomplex of $\hom(C(0,m), D(0,m))$ whose k–chains are the homomorphisms of graded groups $C(0,m) \to D(0,m)$ raising degrees by k, and taking the image of $C(0,i)$ in $C(0,m)$ to the image of $D(0,i)$ in $D(0,m)$, for each i with $0 \le i \le m$. The restriction maps from $\hom(C, D)$ to $\hom(C(m-j, m-i), D(p,q))$, defined whenever $m - j \le p$ and $m - i \le q$, determine a chain map

$$(\bullet) \qquad \hom(C, D) \longrightarrow \lim_{\substack{i,j,p,q \\ m-j \le p \\ m-i \le q}} \hom(C(m-j, m-i), D(p,q))\,.$$

(Note that $(i,j,p,q) \mapsto \hom(C(m-j, m-i), D(p,q))$ is a covariant functor.) The chain map $(\bullet)$ is always an isomorphism. For the proof, replace $\mathcal{C}$ by the category $\mathcal{D}$ of graded f.g. free abelian groups, bounded from below and above. Think of $(\bullet)$ as a natural transformation between bi–additive functors on $\mathcal{S}_m\mathcal{D} \times \mathcal{S}_m\mathcal{D}$. It is enough to verify that it is an isomorphism when $C(0,m)$ and $D(0,m)$ have rank one. This case is easy.

Therefore, if we wish to have a natural isomorphism $C \otimes D \cong \hom(C^*, D)$ for objects C, D in $\mathcal{S}_m\mathcal{C}$, we are forced to define

$$C \otimes D := \lim_{\substack{i,j,p,q \\ j+p \ge m \\ i+q \ge m}} C(i,j) \otimes D(p,q))\,.$$

Of course, there are other ways to say the same thing. For example, we know that $C \otimes D$ injects in the ordinary tensor product of chain complexes $C(0,m) \otimes D(0,m)$, and the image of this injection is easy to identify ; this leads to

$$C \otimes D \cong \operatorname*{colim}_{\substack{j,q \\ j+q \le m+1}} C(0,j) \otimes D(0,q)\,.$$

In order to make SW products from tensor products, we use the Kan–Dold functor. The Kan–Dold functor associates to a chain complex E the simplicial abelian group whose n–simplices are the chain maps from the cellular chain complex of the CW–space Δ^n to E. We will write $E \mapsto E^\sharp$ for the composition of the Kan–Dold functor with geometric realization. The functor $E \mapsto E^\sharp$ respects finite limits.

There are now two slightly different ways of introducing an SW product in $\mathcal{S}_m\mathcal{C}$. Given C and D in $\mathcal{S}_m\mathcal{C}$, we might define

$$C \odot D := (C \otimes D)^\sharp \cong \lim_{i,j,p,q} (C(i,j) \odot D(p,q))^\sharp$$

where the limit is taken over all (i,j,p,q) with $i+q \geq m$ and $j+p \geq m$. Alternatively, we might define

$$C \odot D := \operatorname*{holim}_{i,j,p,q} (C(i,j) \odot D(p,q))^\sharp$$

(same conditions on i,j,p,q) which means that we use the formula just before 1.2, and an SW product on $\mathcal{C}$ given by $A \odot B := (A \otimes B)^\sharp$. These two slightly different definitions are not isomorphic, but they are related by a natural transformation (the usual map from a limit to the corresponding homotopy limit) respecting the symmetry τ. By 1.3, the natural transformation is a homotopy equivalence in certain cases. The filtration argument given after the proof of 1.3 shows that it is always a homotopy equivalence.

Another Waldhausen category which frequently arises in the study of a Waldhausen category $\mathcal{C}$ is $\mathcal{C}^*(X)$, the parametrization of $\mathcal{C}$ by a finite simplicial complex X. This is defined as follows (compare [RaWe], [Ra3]). Let $\mathrm{sub}(X)$ be the poset of simplicial subcomplexes of X, viewed as a category. An object of $\mathcal{C}^*(X)$ is a functor F from $\mathrm{sub}(X)$ to $\mathcal{C}$ which takes all morphism to cofibrations, takes $\emptyset$ to $*$, and takes unions to pushouts. (Such a functor is determined up to isomorphism by its restriction to the full sub–poset of $\mathrm{sub}(X)$ consisting of all the nonempty faces of X.) Any natural transformation $F_1 \to F_2$ between such functors qualifies as a *morphism* in $\mathcal{C}^*(X)$. It is a *cofibration* if $F_1(Z) \to F_2(Z)$ is a cofibration for all $Z \in \mathrm{sub}(X)$, and for each pair of subcomplexes $Z_1 \subset Z_2$ of X, the evident morphism

$$\operatorname{colim}\,(F_2(Z_1) \leftarrow F_1(Z_1) \rightarrow F_1(Z_2)) \;\longrightarrow\; F_2(Z_2)$$

is a cofibration. (This ensures that the functor $Z \mapsto F_2(Z)/F_1(Z)$ is again in $\mathcal{C}^*(X)$; it is of course the cofiber of the cofibration.) The morphism $F_1 \to F_2$ is a *weak equivalence* if $F_1(Z) \to F_2(Z)$ is a weak equivalence for all $Z \in \mathrm{sub}(X)$.

1.5. Definition. An SW product $\odot$ on $\mathcal{C}$ gives rise to one on $\mathcal{C}^*(X)$ by the formula $F_1 \odot F_2 := \mathrm{holim}_s\, F_1(s) \odot F_2(s)$ where the homotopy inverse limit is taken over the poset of all nonempty faces $s \subset X$.

2. Axioms

For motivation, we return to the setting of 1.4. Suppose that C and D are objects in $\mathcal{C}$. A component of $C \odot D = (\hom(C^*, D))^\sharp$ will be regarded as *nondegenerate* if the corresponding homotopy class of chain maps $C^* \to D$ is the class of a homotopy equivalence. It is not a trivial matter to generalize this notion to the abstract setting of 1.1. In this section we list a number of axioms, about a Waldhausen category $\mathcal{C}$ with SW product, which will make it possible.

To begin, we suppose again that $\mathcal{C}$ is an abstract Waldhausen category. We suppose also that $\mathcal{C}$ is small (the class of objects is a set), although in our examples $\mathcal{C}$ is usually only equivalent to a small category. The first few axioms do not involve an SW product. The notions *cylinder functor*, *cylinder axiom*, *saturation axiom* from [Wald2] appear in 2.1 ; we explain them in the comment after 2.1.

2.1. Axioms. $\mathcal{C}$ *is equipped with a cylinder functor satisfying the cylinder axiom* [Wald2, §1.4]. *The weak equivalences in* $\mathcal{C}$ *satisfy the saturation axiom* [Wald, §1.2].

Comment. The *saturation axiom* is the following condition: if a, b are composable morphisms in $\mathcal{C}$ and two of a, b, ab are weak equivalences, then so is the third.

An example of a Waldhausen category with cylinder functor is the category of compact based CW–spaces, where the morphisms are the based cellular maps, the weak equivalences are the homotopy equivalences, and the cofibrations are the CW–embeddings. In this case, the cylinder functor is the rule associating to a morphism $f : X \to Y$ its reduced mapping cylinder $T(f)$, and the canonical maps $X \to T(f)$ and $Y \to T(f)$ (front inclusion and back inclusion) and $T(f) \to Y$ (cylinder projection).

For the abstract version, we need two other Waldhausen categories $\mathrm{Ar}\mathcal{C}$ and $\mathcal{F}_1\mathcal{C}$ associated with $\mathcal{C}$. The objects of $\mathrm{Ar}\mathcal{C}$ are the arrows of $\mathcal{C}$, the morphisms from $f : A \to B$ to $g : C \to D$ are the commutative squares

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow{g} & D \end{array}$$

in $\mathcal{C}$, and we call such a morphism a cofibration {weak equivalence} if the two vertical arrows are cofibrations {weak equivalences}. The objects of $\mathcal{F}_1\mathcal{C}$ are the morphisms of $\mathrm{cof}\,\mathcal{C}$. Again, a morphism in $\mathcal{F}_1\mathcal{C}$ from $f : A \to B$ to $g : C \to D$ is a commutative square of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow{g} & D \end{array}$$

in $\mathcal{C}$. If, in this square, both $A \to C$ and $C \amalg_A B \to D$ are in $\mathrm{cof}\,\mathcal{C}$, then the morphism given by the square is a cofibration. If both $A \to C$ and $B \to D$ are weak equivalences, then the morphism given by the square is a weak equivalence.

A *cylinder functor* on $\mathcal{C}$ is a functor T on $\mathrm{Ar}\mathcal{C}$ taking every object $f : C \to D$ to a diagram of the form

$$\begin{array}{ccccc} C & \xrightarrow{j_1} & T(f) & \xleftarrow{j_2} & D \\ & & \big\downarrow{\scriptstyle p} & & \\ & & D & & \end{array}$$

with $pj_1 = f$ and $pj_2 = \mathrm{id}_D$, and taking morphisms in $\mathrm{Ar}\mathcal{C}$ to natural transformations between such diagrams. There are two additional conditions. The first of these (Cyl 1) requires that j_1 and j_2 assemble to an exact functor from $\mathrm{Ar}\mathcal{C}$ to $\mathcal{F}_1\mathcal{C}$,

$$(C \xrightarrow{f} D) \quad \mapsto \quad (C \vee D \to T(C \xrightarrow{f} D))\,.$$

The second (Cyl 2) requires that $j_2 : D \to T(* \to D)$ be an identity morphism for each D in $\mathcal{C}$. The cylinder functor is said to satisfy the *cylinder axiom* if the cylinder projection $p : T(f) \to D$ is a weak equivalence for any $f : C \to D$.

Suppose that $\mathcal{C}$ comes with a cylinder functor T. The *cone* of a morphism $f : C \to D$, denoted $\mathrm{cone}(f)$, is the cofiber of $j_1 : C \to T(f)$. We often write $\mathrm{cone}(C)$ instead of $\mathrm{cone}(\mathrm{id}_C)$. The *suspension* ΣC of an object C in $\mathcal{C}$ is the cone of $C \to *$. *(End of comment.)*

A *quasi–morphism* from an object C in $\mathcal{C}$ to an object D in $\mathcal{C}$ is a diagram of the form $C \to D' \leftarrow D$ where the arrow $\leftarrow$ is a weak equivalence *and* a cofibration. Diagrams of this type, with fixed C and D, are the objects of a category $\mathcal{M}(C, D)$ whose morphisms are commutative diagrams of the form

$$\begin{array}{ccccc} C & \longrightarrow & D' & \longleftarrow & D \\ \big\downarrow{\scriptstyle =} & & \big\downarrow & & \big\downarrow{\scriptstyle =} \\ C & \longrightarrow & D'' & \longleftarrow & D\,. \end{array}$$

Write $[C, D]$ for the set of components of $\mathcal{M}(C, D)$, and write $[f] \in [C, D]$ for the class of a morphism $f : C \to D$ (regarded as a quasi–morphism $C \to D = D$). The sets $[C, D]$ are the morphism sets in a new category $\mathcal{H}\mathcal{C}$, with the same objects as $\mathcal{C}$. Explicitly, the composition law $[D, E] \times [C, D] \to [C, E]$ is defined on representatives $D \to E' \leftarrow E$ and $C \to D' \leftarrow D$ as follows. Let E'' be the pushout of $D' \leftarrow D \to E'$ and form $C \to E'' \leftarrow E$ (arrow $\to$ via D', arrow $\leftarrow$ via E'). This represents an element in $[C, E]$, as required.— Clearly $C \mapsto C$, $f \mapsto [f]$ is a functor from $\mathcal{C}$ to $\mathcal{H}\mathcal{C}$. It is an exercise to show that the functor takes weak equivalences to isomorphisms, provided $\mathcal{C}$ satisfies 2.1. This is actually carried out in [Wei].

2.2. Axiom (Simplicity)**.** *Let f be a morphism in $\mathcal{C}$ such that $[f]$ is an isomorphism. Then f is a weak equivalence.*

This axiom ensures that there is a reasonably close relationship between $\mathcal{C}$ and $\mathcal{HC}$. The remaining axioms are about $\mathcal{HC}$, and they have a very familiar form reminiscent of Spanier–Whitehead duality.

2.3. Axiom (Stability). *The suspension functor $\Sigma : \mathcal{HC} \to \mathcal{HC}$ is an equivalence of categories.*

For the next two axioms, suppose that $\mathcal{C}$ is equipped with an SW product $\odot$. The rule $(C, D) \mapsto \pi_0(C \odot D)$ can be regarded as a functor on $\mathcal{HC} \times \mathcal{HC}$.

2.4. Axiom (Co–representability). *For every B in $\mathcal{HC}$, the functor on $\mathcal{HC}$ taking C to $\pi_0(B \odot C)$ is co–representable, say by an object $T(B)$ in $\mathcal{HC}$.*

Let's spell this out. We must have an isomorphism $[T(B), C] \cong \pi_0(B \odot C)$ naturally in C. The object $T(B)$ is characterized by this property up to unique isomorphism in $\mathcal{HC}$.

2.5. Axiom (Involutivity). *For every B in $\mathcal{HC}$, the canonical morphism from $T^2(B)$ to B in $\mathcal{HC}$ is an isomorphism.*

The canonical morphism is the image of id $\in [T(B), T(B)]$ under the chain of bijections $[T(B), T(B)] \cong \pi_0(B \odot T(B)) \cong \pi_0(T(B) \odot B) \cong [T^2(B), B]$. One can see that 2.5 is not a consequence of 2.4 by looking at the most trivial possible SW product, $B \odot C := *$. Then 2.4 is satisfied, but 2.5 is not unless $\mathcal{HC}$ is equivalent to a category with one object and one morphism.

Remark. When 2.4 holds, we can regard T as a functor from $\mathcal{HC}^{\mathrm{op}}$ to $\mathcal{HC}$. It is *self–adjoint* in the sense that $[TC, D] \cong [TD, C]$ in $\mathcal{HC}$, naturally. This follows from the symmetry of $\odot$. The canonical natural transformation $T^2(B) \to B$ is just the *counit* of the adjunction [MaL, IV.1]. It is therefore an isomorphism for all B in $\mathcal{HC}$ if and only if T is an equivalence $\mathcal{HC}^{\mathrm{op}} \to \mathcal{HC}$.

3. First consequences

Throughout this section, we assume that $\mathcal{C}$ satisfies the axioms of §2. The goal here is to translate statements about $\mathcal{HC}$ into statements about a topological category whose objects are those of $\mathcal{C}$, and where the space of morphisms from C to D is the classifying space $|\mathcal{M}(C, D)|$ of the category $\mathcal{M}(C, D)$. This topological category is essentially a *hammock localization* in the sense of Dwyer and Kan, [DwyKa1], [DwyKa2] ; see [Wei, 1.2] for more details.

3.1. Quotation [Wei]**.** There is a composition law in the shape of a functor

$$\mathcal{M}(D,E)\times\mathcal{M}(C,D)\to\mathcal{M}(C,E)$$

which on objects is defined as follows: the pair $(D\to E'\leftarrow E\,,\,C\to D'\leftarrow D)$ is mapped to $C\to E''\leftarrow E$, where E'' is the pushout of $D'\leftarrow D\to E'$. The morphisms $C\to E''$ and $E''\leftarrow E$ are via D' and E', respectively. Here we are assuming that pushouts of diagrams $F\leftarrow G\to H$ in $\mathcal{C}$, where one of the arrows is a cofibration, are canonically defined in $\mathcal{C}$. (The definition of a Waldhausen category requires that they exist, and the universal property makes them unique up to unique isomorphism.) The composition law is associative up to a natural isomorphism of functors. Also, the object

$$C\xrightarrow{=}C\xleftarrow{=}C$$

in $\mathcal{M}(C,C)$ acts as a two–sided identity, again up to a natural isomorphism.

It is always possible to replace $\mathcal{C}$ by an equivalent category so that the pushouts needed above are canonically defined and *associative, with units.* What this means is that, when $\mathcal{C}$ has been so improved, the composition law just described is associative and has two–sided identities. See [Isb]. We will use this below, in the proof of 3.5.

3.2. Corollary. *Let $C\xrightarrow{f}D'\xleftarrow{e}D$ be a quasi–morphism. The corresponding morphism $C\to D$ in $\mathcal{HC}$ is an isomorphism if and only if f is a weak equivalence. In that case the maps*

$$|\mathcal{M}(B,C)|\longrightarrow|\mathcal{M}(B,D)|\,,\quad|\mathcal{M}(D,E)|\longrightarrow|\mathcal{M}(C,E)|$$

given by composition with the quasi–morphism $C\xrightarrow{f}D'\xleftarrow{e}D$ are homotopy equivalences, for arbitrary B and E in $\mathcal{C}$.

Proof. Suppose that the quasi–morphism in question represents an isomorphism in $\mathcal{HC}$. Then it follows immediately from 3.1 that composition with it gives homotopy equivalences. Next, it is an exercise to show that the composition of our quasi–morphism with

$$(D\xrightarrow{e}D'\xleftarrow{=}D')$$

(representing the class $[e]$) is in the class $[f]$. But we saw in §2 that $[e]$ is an isomorphism in $\mathcal{HC}$, so that $C\to D'\leftarrow D$ represents an isomorphism in $\mathcal{HC}$ if and only if $[f]$ is an isomorphism in $\mathcal{HC}$. Now apply axiom 2.2. □

3.3. Theorem. *For fixed D in $\mathcal{C}$, the contravariant functor $C\mapsto|\mathcal{M}(C,D)|$ takes cofiber squares to homotopy pullback squares.*

This is the main result of [Wei]. It uses axiom 2.1 only. A *cofiber square* in $\mathcal{C}$ is a commutative pushout square in which either the vertical arrows or the horizontal arrows are cofibrations. With a view to 3.4, we note that $\mathcal{M}(C,D)|$ has a canonical base point, corresponding to the object

$$C \xrightarrow{\text{zero}} D \xleftarrow{=} D$$

in $\mathcal{M}(C,D)$. The base point is sufficiently natural, so that $C \mapsto |\mathcal{M}(C,D)|$ becomes a contravariant functor from $\mathcal{C}$ to based spaces.

3.4. Corollary. *For arbitrary C, D in $\mathcal{C}$, the suspension functor from $\mathcal{M}(C,D)$ to $\mathcal{M}(\Sigma C, \Sigma D)$ induces a homotopy equivalence $|\mathcal{M}(C,D)| \to |\mathcal{M}(\Sigma C, \Sigma D)|$.*

Remark. The proof uses axioms 2.1, 2.2, 2.3 only.

Proof. We begin by proving that suspension $|\mathcal{M}(C,D)| \to |\mathcal{M}(\Sigma C, \Sigma D)|$ induces an isomorphism on π_k for all k. Suppose inductively that this has been established for $k \le n$ and all C and D ; the induction start is of course axiom 2.3. From 3.3 we have the following commutative diagram in the homotopy category of based spaces:

$$(\bullet) \qquad \begin{array}{ccc} |\mathcal{M}(\Sigma C, D)| & \longrightarrow & |\mathcal{M}(\Sigma(\Sigma C), \Sigma D)| \\ \downarrow \simeq & & \downarrow \simeq \\ \Omega|\mathcal{M}(C,D)| & \longrightarrow & \Omega|\mathcal{M}(\Sigma C, \Sigma D)|\,. \end{array}$$

In more detail: the vertical arrows are extracted from the homotopy pullback squares obtained by applying $|\mathcal{M}(?,D)|$ and $|\mathcal{M}(?,\Sigma D)|$ to the cofiber squares

$$\begin{array}{ccc} C & \longrightarrow & \mathrm{cone}(C) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma C \end{array} \quad , \quad \begin{array}{ccc} \Sigma C & \longrightarrow & \Sigma\,\mathrm{cone}(C) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma(\Sigma C) \end{array}$$

respectively. This should make the horizontal arrows in $(\bullet)$ obvious. (Note that we have used $\Sigma\,\mathrm{cone}(C)$, not $\mathrm{cone}(\Sigma C)$.) Applying the inductive assumption to the upper row in $(\bullet)$, one finds that the map in the lower row induces isomorphisms in π_k for all $k \le n$, which completes the induction step. Therefore the suspension–induced map $|\mathcal{M}(C,D)| \to |\mathcal{M}(\Sigma C, \Sigma D)|$ is bijective on π_k for all $k \ge 0$ and consequently

$$\Omega|\mathcal{M}(C,D)| \to \Omega|\mathcal{M}(\Sigma C, \Sigma D)|$$

is a homotopy equivalence. Using $(\bullet)$ once more, we deduce that 3.4 is correct provided C is of the form ΣB for some B. But axiom 2.3 implies that we can indeed assume $C = \Sigma B$ without loss of generality. $\square$

Our main goal in the remainder of this section is to prove corollary 3.8 below, which is dual to 3.3 since it claims linearity of the expression $|\mathcal{M}(C,D)|$ in the *second* variable.

We begin by comparing $|\mathcal{M}(C,D)|$ and $|\mathcal{M}(C,\Sigma D)|$. There are two seemingly different but equally reasonable ways to define a partial stabilization map from $|\mathcal{M}(C,D)|$ to $\Omega|\mathcal{M}(C,\Sigma D)|$. For the first, start with the cofiber square

$$\begin{array}{ccc} D & \longrightarrow & \mathrm{cone}(D) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma D \end{array}$$

and apply $\mathcal{M}(C,?)$ to obtain a commutative square with contractible upper right–hand and lower left–hand terms,

$$\begin{array}{ccc} |\mathcal{M}(C,D)| & \longrightarrow & |\mathcal{M}(C,\mathrm{cone}(D))| \\ \downarrow & & \downarrow \\ |\mathcal{M}(C,*)| & \longrightarrow & |\mathcal{M}(C,\Sigma D)|\,. \end{array}$$

This gives $|\mathcal{M}(C,D)| \to \Omega|\mathcal{M}(C,\Sigma D)|$ as claimed. For the other definition, start with the cofiber square

$$\begin{array}{ccc} C & \longrightarrow & \mathrm{cone}(C) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma C \end{array}$$

and apply $\mathcal{M}(?,\Sigma D)$ to obtain a commutative square with contractible upper right–hand and lower left–hand terms,

$$\begin{array}{ccc} \mathcal{M}(\Sigma C,\Sigma D) & \longrightarrow & \mathcal{M}(\mathrm{cone}(C),\Sigma D) \\ \downarrow & & \downarrow \\ \mathcal{M}(*,\Sigma D) & \longrightarrow & \mathcal{M}(C,\Sigma D)\,. \end{array}$$

This results in $|\mathcal{M}(\Sigma C,\Sigma D)| \to \Omega|\mathcal{M}(C,\Sigma D)|$, a homotopy equivalence by 3.3 ; we pre–compose with $|\mathcal{M}(C,D)| \to |\mathcal{M}(\Sigma C,\Sigma D)|$ of 3.4, another homotopy equivalence.

3.5. Lemma. *The two partial stabilization maps $|\mathcal{M}(C,D)| \to \Omega|\mathcal{M}(C,\Sigma D)|$ are homotopic. (Therefore both are homotopy equivalences.)*

Proof. Let $s = s_{C,D}$ be the partial stabilization map defined first, and let $\sigma = \sigma_{C,D}$ be the other. Fix D. An easy Yoneda type argument which we leave to the reader shows that both maps are completely determined, for all C, by what they do to $\mathrm{id}_D \in |\mathcal{M}(D,D)|$. It is therefore quite enough to verify that $s(\mathrm{id}_D)$ and $\sigma(\mathrm{id}_D)$ belong to the same component of $\Omega|\mathcal{M}(D,\Sigma D)|$. Here we have to be explicit. By inspection, $s(\mathrm{id}_D)$ has the following description: Choose a path in $|\mathcal{M}(D,\mathrm{cone}(D))|$ from the vertex

$$D \xrightarrow{\subset} \mathrm{cone}(D) \xleftarrow{=} \mathrm{cone}(D)$$

to the base point (which is the zero morphism from D to $\mathrm{cone}(D)$, viewed as a quasi–morphism). The image of that path under the map from $|\mathcal{M}(D, \mathrm{cone}(D))|$ to $|\mathcal{M}(D, \Sigma D)|$ induced by $\mathrm{cone}(D) \to \Sigma D$ is a loop, equal to $s(\mathrm{id}_D)$. (The description is up to contractible choice, but the definition of s is also up to contractible choice only.) Again by inspection, $\sigma(\mathrm{id}_D)$ has the following description: Choose a path in $|\mathcal{M}(\mathrm{cone}(D), \Sigma D)|$ from the vertex

$$\mathrm{cone}(D) \longrightarrow \Sigma D \overset{=}{\longleftarrow} \Sigma D$$

to the base point. Its image under the map $|\mathcal{M}(\mathrm{cone}(D), \Sigma D)| \to |\mathcal{M}(D, \Sigma(D)|$ induced by the boundary inclusion $D \to \mathrm{cone}(D)$ is a loop, equal to $\sigma(\mathrm{id}_D)$. To see that the loops are homotopic, we introduce a third. Choose a path in $|\mathcal{M}(\mathrm{cone}(D), \mathrm{cone}(D))|$ from the identity vertex

$$\mathrm{cone}(D) \overset{=}{\longrightarrow} \mathrm{cone}(D) \overset{=}{\longleftarrow} \mathrm{cone}(D)$$

to the base point. Its image under the map $|\mathcal{M}(\mathrm{cone}(D), \mathrm{cone}\, D)| \to |\mathcal{M}(D, \Sigma(D)|$ induced by projection $\mathrm{cone}(D) \to \Sigma D$ and boundary inclusion $D \to \mathrm{cone}(D)$ is a loop, clearly homotopic (even equal with appropriate choices) to both $s(\mathrm{id}_D)$ and $\sigma(\mathrm{id}_D)$. □

For B, C, D in $\mathcal{HC}$ there is a *slant product* $[C, D] \times \pi_0(B \odot C) \longrightarrow \pi_0(B \odot D)$ defined by $(g, [\eta]) \mapsto (\mathrm{id} \odot g)[\eta]$. Not surprisingly, this is induced by a map of spaces

$$|\mathcal{M}(C, D)| \times B \odot C \longrightarrow B \odot D\,.$$

This map can be described as follows. Let X_0, X_1, X_2 be the (unreduced) homotopy colimits of the functors from $\mathcal{M}(C, D)$ to based spaces which take an object

$$C \overset{f}{\longrightarrow} D' \overset{e}{\longleftarrow} D$$

to $B \odot C$, $B \odot D'$ and $B \odot D$ respectively. The obvious natural transformations induce maps

$$X_0 \longrightarrow X_1 \overset{\simeq}{\longleftarrow} X_2$$

giving a based map $X_0 \to X_2$ well defined up to contractible choice. (The contractible space by which we parametrize the choices is the space of retractions r from the reduced mapping cylinder of $X_2 \to X_1$ to X_2. Each of these retractions r can be composed with $X_0 \to X_1$ and the front inclusion of the cylinder, giving a map $X_0 \to X_2$.) We obtain the map we have been looking for by projecting from X_2 to $B \odot D$ and observing that X_0 is homeomorphic to $|\mathcal{M}(C, D)| \times B \odot C$.

3.6. Definition. A class in $\pi_0(B \odot C)$ is *nondegenerate* if the corresponding element in $[T(B), C]$, is an isomorphism in $\mathcal{HC}$. Equivalently, $[\eta] \in \pi_0(B \odot C)$ is nondegenerate if slant product with $[\eta]$ is a bijection $[C, D] \to \pi_0(B \odot D)$ for all D.

Remark. Axiom 2.5 is equivalent to the statement that the symmetry map from $\pi_0(B \odot C)$ to $\pi_0(C \odot B)$ takes nondegenerate components to nondegenerate components. *Proof:* Without loss of generality, the nondegenerate component lives in $\pi_0(B \odot T(B))$ and corresponds to $\mathrm{id} \in [T(B), T(B)]$. Apply symmetry to obtain a class in $\pi_0(T(B) \odot B)$ corresponding to some element of $[T^2(B), B]$. This element is exactly the canonical morphism $T^2(B) \to B$ in $\mathcal{HC}$.

3.7. Proposition. *If η belongs to a nondegenerate component of $B \odot C$, then the slant product with η is a homotopy equivalence $\backslash\eta : |\mathcal{M}(C, D)| \to B \odot D$ for any D.*

Remark. The proof uses only axioms 2.1, 2.2, 2.3, and of course the SW product $\odot$ and the nondegeneracy assumption on η, which should be read as in the second sentence of 3.6.

Proof. The proof is by a bootstrap procedure similar to that used in the proof of 3.4. First we show that $\backslash\eta : |\mathcal{M}(C, D)| \to B \odot D$ induces isomorphisms on π_k, for all $k \geq 0$. Suppose inductively that this has already been established for $k \leq n$ and all D ; the induction start ($k = 0$) comes from the nondegeneracy assumption on η. By applying $|\mathcal{M}(C, ?)|$ and $B\odot ?$ to the cofiber square

$$\begin{array}{ccc} D & \xrightarrow{\subset} & \mathrm{cone}(D) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma D \end{array}$$

we obtain two homotopy pullback squares (by the second sentence in 3.5 and bilinearity of $\odot$) with contractible upper right–hand and lower left–hand vertices, and therefore *compatible* homotopy equivalences

$$|\mathcal{M}(C, D)| \longrightarrow \Omega|\mathcal{M}(C, \Sigma D)|\,, \qquad B \odot D \longrightarrow \Omega B \odot \Sigma D\,.$$

Compatibility means that the square

$$\begin{array}{ccc} |\mathcal{M}(C, D)| & \xrightarrow{\backslash\eta} & B \odot D \\ \downarrow\simeq & & \downarrow\simeq \\ \Omega|\mathcal{M}(C, \Sigma D)| & \xrightarrow{\Omega(\backslash\eta)} & \Omega B \odot \Sigma D \end{array}$$

commutes. Applying the inductive assumption to the upper row in the square, we see that $\backslash\eta : |\mathcal{M}(C, \Sigma D)| \to B \odot \Sigma D$ induces isomorphisms on π_k for $k \leq n + 1$. Since every object in $\mathcal{C}$ is related to an object of the form ΣD by a chain of weak equivalences (2.3 and 3.2), this completes the induction step. As in the proof of 3.4, we may conclude that

$$\Omega(\backslash\eta) : \Omega|\mathcal{M}(C, D)| \to \Omega B \odot D$$

is a homotopy equivalence for all D. In particular, the lower row in the square just above is a homotopy equivalence. But then the upper row must be a homotopy equivalence. □

3.8. Corollary. *For every C in $\mathcal{C}$, the covariant functor $D \mapsto |\mathcal{M}(C, D)|$ takes cofiber squares to homotopy pullback squares.*

Proof. Fix C and choose B and $\eta \in B \odot C$ in a nondegenerate component. This is possible by 2.4 and 2.5. Slant product with η is a map from $|\mathcal{M}(C, D)|$ to $B \odot D$, natural in D. By 3.7 it is a homotopy equivalence. It is therefore enough to know that $D \mapsto B \odot D$ takes cofiber squares to homotopy pullback squares ; but this is part of the definition, 1.1. □

4. The involution on $|w\mathcal{C}|$

Assume throughout this section that $\mathcal{C}$ with the SW product $\odot$ satisfies the axioms of §2. At the end of this section we will know that $|w\mathcal{C}|$ has a canonical involution. (Strictly speaking, we must replace $|w\mathcal{C}|$ by a larger space $|xw\mathcal{C}|$ mapping to $|w\mathcal{C}|$ by a homotopy equivalence, and it is $|xw\mathcal{C}|$ which has the involution. Compare [Vo].) In later sections we will see that $\mathcal{S}_m\mathcal{C}$ and $\mathcal{C}^*(X)$ also satisfy the axioms (see 1.5). When combined, these results lead to an involution on the spectrum $\boldsymbol{K}(\mathcal{C})$ and many other good things.

Write $w\mathcal{M}(C,D)$ for the union of the *invertible* components of $\mathcal{M}(C,D)$. Then $w\mathcal{M}(C,D)$ is a full subcategory of $\mathcal{M}(C,D)$, and by 3.2 its objects are those diagrams

$$C \xrightarrow{f} D' \xleftarrow{e} D$$

for which f is a weak equivalence (and e is a weak equivalence and a cofibration, as always). The following key lemma relates the results of §3 to the homotopy type of $|w\mathcal{C}|$.

4.1. Lemma. *For C in $\mathcal{C}$, let $\mathcal{F}_C$ be the functor on $w\mathcal{C}$ given by $\mathcal{F}_C(D) = |w\mathcal{M}(C,D)|$. Then* $\operatorname{hocolim} \mathcal{F}_C$ *is contractible.*

Proof. By Thomason's homotopy colimit theorem [Tho1], $\operatorname{hocolim} \mathcal{F}_C$ is homotopy equivalent to the nerve of a single category $\mathcal{K}$ with the following description. Objects are diagrams

$$C \xrightarrow{f} D' \xleftarrow{e} D$$

where f is a weak equivalence, e is a weak equivalence and a cofibration, C is fixed but D and D' are not. Morphisms are commutative diagrams

$$\begin{array}{ccccc} C & \xrightarrow{f_1} & D_1' & \xleftarrow{e_1} & D_1 \\ \downarrow{=} & & \downarrow & & \downarrow \\ C & \xrightarrow{f_2} & D_2' & \xleftarrow{e_2} & D_2 \,. \end{array}$$

The objects of the form $C \xrightarrow{f} D \xleftarrow{=} D$ are the objects of a full subcategory $\mathcal{K}_0 \subset \mathcal{K}$. The inclusion $\mathcal{K}_0 \to \mathcal{K}$ has a left adjoint $(C \to D' \leftarrow D) \mapsto (C \to D' \xleftarrow{=} D')$, so that $|\mathcal{K}_0| \simeq |\mathcal{K}|$. But $\mathcal{K}_0$ has an initial object. □

Remark. Let $w_C\mathcal{C}$ be the component of $w\mathcal{C}$ which contains the object C. It is a consequence of 4.1 that $\Omega|w_C\mathcal{C}| \simeq |w\mathcal{M}(C,C)|$. To see this note first that $\mathcal{F}_C(D)$ is empty when D is not in the component of C. Therefore $\operatorname{hocolim} \mathcal{F}_C$ is in effect a homotopy colimit taken over $w_C\mathcal{C}$. On the other hand, $\mathcal{F}_C$ takes any morphism in $w_C\mathcal{C}$ to a homotopy equivalence, so that the projection $\operatorname{hocolim} \mathcal{F}_C \to |w_C\mathcal{C}|$ is a quasifibration. Its total space is contractible according to 4.1, and its fiber over the base point is $|w\mathcal{M}(C,C)|$.

4.2. Proposition. *For B in $\mathcal{C}$ let $\mathcal{G}_B$ be the functor on $w\mathcal{C}$ given by $\mathcal{G}_B(D) = B \odot_w D$, where $B \odot_w D \subset B \odot D$ consists of the nondegenerate components. Then* $\operatorname{hocolim} \mathcal{G}_B$ *is contractible.*

Proof. Essentially this is a consequence of 4.1 together with 3.7. Choose some C and some point $\eta \in B \odot_w C$. We wish to compare the two functors $\mathcal{F} = \mathcal{F}_C$ and $\mathcal{G} = \mathcal{G}_B$ on $w\mathcal{C}$. To make the comparison easier we introduce a third functor $\mathcal{F}'$ defined as follows. For D in $\mathcal{C}$ let $\mathcal{F}'(D)$ be the homotopy colimit of the functor on $w\mathcal{M}(C,D)$ whose value on an object

$$C \xrightarrow{f} D' \xleftarrow{e} D$$

is the (contractible) homotopy fiber of $e_* : B \odot_w D \to B \odot_w D'$ over the point $f_*(\eta)$. Projection from the hocolim to $|w\mathcal{M}(C,D)|$ is a homotopy equivalence, and it is also a natural transformation $\mathcal{F}' \to \mathcal{F}$ of functors in the variable D. But there is also a forgetful transformation $\mathcal{F}' \to \mathcal{G}$, and again we know (from 3.7) that $\mathcal{F}'(D) \to \mathcal{G}(D)$ is a homotopy equivalence for every D. (It is simply the slant product with η restricted to certain components, provided we make the identification $\mathcal{F}'(D) \simeq |w\mathcal{M}(C,D)|$.) Therefore

$$\operatorname{hocolim} \mathcal{G} \simeq \operatorname{hocolim} \mathcal{F}' \simeq \operatorname{hocolim} \mathcal{F} \simeq * . \quad \square$$

Remark. The true meaning of 4.2 is that every object B in $\mathcal{C}$ has a (Spanier–Whitehead) dual which is unique up to contractible choice. Indeed, $\operatorname{hocolim} \mathcal{G}_B$ is the space of choices of duals.

4.3. Corollary. *Let $\mathcal{E}$ be the functor on $w\mathcal{C} \times w\mathcal{C}$ defined by $\mathcal{E}(B,D) = B \odot_w D$. The composition of the projection map* $\operatorname{hocolim} \mathcal{E} \longrightarrow |w\mathcal{C} \times w\mathcal{C}|$ *with projection to the first coordinate $|w\mathcal{C} \times w\mathcal{C}| \to |w\mathcal{C}|$ is a homotopy equivalence* $\operatorname{hocolim} \mathcal{E} \to |w\mathcal{C}|$.

Proof. There is a Fubini principle for homotopy colimits of functors on product categories which, applied to the present case, means that

$$\operatorname{hocolim} \mathcal{E} \cong \operatorname*{hocolim}_{B \text{ in } w\mathcal{C}} (\operatorname{hocolim} \mathcal{G}_B)$$

where $\mathcal{G}_B$ is the functor defined in 4.2. The homeomorphism is over $|w\mathcal{C}|$. $\square$

4.4. Remark. Homotopy colimits can always be interpreted as classifying spaces of topological (or simplicial) categories. This is how they appeared in the literature for the first time, in [Seg]. Specifically, let $xw\mathcal{C}$ be the topological category whose objects are triples (B, D, z) where B, D are objects in $w\mathcal{C}$ and z is a point in $B \odot_w D$. A morphism from (B_1, D_1, z_1) to (B_2, D_2, z_2) is a pair of weak equivalences $B_1 \to B_2$, $D_1 \to D_2$ so that the induced map $B_1 \odot D_1 \to B_2 \odot D_2$ takes z_1 to z_2. Then $|xw\mathcal{C}|$ is homeomorphic to $\operatorname{hocolim} \mathcal{E}$. From this point of view, the true meaning of 4.3 is that the forgetful (simplicial, continuous) functor

$$xw\mathcal{C} \longrightarrow w\mathcal{C} \qquad ; \qquad (B, D, z) \mapsto B$$

induces a homotopy equivalence of the classifying spaces. (Here $w\mathcal{C}$ is a discrete category as always.) Note that $(B, D, z) \mapsto (D, B, \tau(z))$ defines an involution on $xw\mathcal{C}$, where τ is the symmetry operator from 1.1.

5. Generating classes

Let $\mathcal{C}$ be a Waldhausen category satisfying axioms 2.1 and 2.2. Here we develop techniques for checking axiom 2.3 and, if there is an SW product $\odot$, also axioms 2.4 and 2.5.

Let $\mathcal{G}$ be a class of objects of $\mathcal{C}$. Let $\langle \mathcal{G} \rangle$ be the smallest full subcategory of $\mathcal{C}$ which

- contains $\mathcal{G}$ and the zero object
- is closed under formation of mapping cones
- is closed under isomorphism, that is, $B \cong C$ in $\mathcal{C}$ and B in $\langle \mathcal{G} \rangle$ implies C in $\langle \mathcal{G} \rangle$.

We say that $\mathcal{G}$ is a *generating class* if every object of $\mathcal{C}$ is isomorphic in $\mathcal{HC}$ (see 3.2) to an object in $\langle \mathcal{G} \rangle$.

A covariant or contravariant functor from $\mathcal{C}$ to based spaces is *w–invariant* if it takes weak equivalences to homotopy equivalences, and *linear* if, in addition, it takes cofiber squares to homotopy pullback squares and takes $*$ to a contractible space. For example, any object D in $\mathcal{C}$ determines a contravariant functor $C \mapsto |\mathcal{M}(C, D)|$ which is linear by 3.3. If $\mathcal{C}$ is equipped with an SW product $\odot$, then $C \mapsto C \odot D$ is a covariant linear functor. Lemmas 5.1 and 5.2 below can facilitate the analysis of such functors.

5.1. Lemma. *Let $\mathcal{F}_1 \to \mathcal{F}_2$ be a natural transformation between covariant linear functors from $\mathcal{C}$ to based spaces. Let $\mathcal{G}$ be a generating class for $\mathcal{C}$, closed under suspension Σ. If $\mathcal{F}_1(C) \to \mathcal{F}_2(C)$ is a homotopy equivalence for every C in $\mathcal{G}$, then $\mathcal{F}_1(C) \to \mathcal{F}_2(C)$ is a homotopy equivalence for every C in $\mathcal{C}$.*

Proof. Choose C in $\mathcal{C}$ and $m \geq 0$ and a diagram of cofibrations

$$* = C_0 \rightarrowtail C_1 \rightarrowtail \dots \rightarrowtail C_{m-1} \rightarrowtail C_m$$

such that C_i/C_{i-1} is isomorphic to some object in $\mathcal{G}$, for $0 < i \leq m$, and C_m is isomorphic to C in $\mathcal{HC}$. By induction, $\mathcal{F}_1(C_{m-1}) \to \mathcal{F}_2(C_{m-1})$ is a homotopy equivalence, and by hypothesis $\mathcal{F}_1(C/C_{m-1}) \to \mathcal{F}_2(C/C_{m-1})$ is a homotopy equivalence. Using the commutative diagram

$$\begin{array}{ccccc} \mathcal{F}_1(C_{m-1}) & \longrightarrow & \mathcal{F}_1(C_m) & \longrightarrow & \mathcal{F}_1(C_m/C_{m-1}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}_2(C_{m-1}) & \longrightarrow & \mathcal{F}_2(C_m) & \longrightarrow & \mathcal{F}_2(C_m/C_{m-1}) \end{array}$$

where the rows are fibration sequences up to homotopy, we conclude that the middle arrow becomes a homotopy equivalence after Ω has been applied (see remark below). The argument works equally well for ΣC_m, so that

$$\Omega\mathcal{F}_1(\Sigma C_m) \to \Omega\mathcal{F}_2(\Sigma C_m)$$

is a homotopy equivalence. Linearity of $\mathcal{F}_1$ and $\mathcal{F}_2$ now shows that $\mathcal{F}_1(C) \to \mathcal{F}_2(C)$ is also a homotopy equivalence. □

5.2. Lemma. *Let $\mathcal{F}_1 \to \mathcal{F}_2$ be a natural transformation between contravariant linear functors from $\mathcal{C}$ to based spaces. Let $\mathcal{G}$ be a generating class for $\mathcal{C}$ such that $\mathcal{F}_1(C) \to \mathcal{F}_2(C)$ is a homotopy equivalence for every C in $\mathcal{G}$. Then $\mathcal{F}_1(C) \to \mathcal{F}_2(C)$ is a homotopy equivalence for every C in $\mathcal{C}$.*

Proof. Choose C in $\mathcal{C}$. Arguing inductively, we can assume that C is the mapping cone of a morphism $f : C_1 \to C_2$ in $\mathcal{C}$ such that the left–hand and middle vertical arrows in the commutative diagram

$$\begin{array}{ccccc} \mathcal{F}_1(C_1) & \longleftarrow & \mathcal{F}_1(C_2) & \longleftarrow & \mathcal{F}_1(C) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}_2(C_1) & \longleftarrow & \mathcal{F}_2(C_2) & \longleftarrow & \mathcal{F}_2(C) \end{array}$$

are homotopy equivalences. Since the rows are fibration sequences up to homotopy, it follows that the right–hand vertical arrow is also a homotopy equivalence. □

We will often use 5.1 and 5.2 to establish axiom 2.3 in special cases. For the rest of the section, assume that $\mathcal{C}$ comes with an SW product $\odot$. The following lemmas and corollaries are meant to facilitate the verification of 2.4 and 2.5.

Before stating 5.3, we note that any morphism in $\mathcal{HC}$ has a mapping cone, well defined up to non–unique isomorphism. Namely, if the morphism is from C to D, represent it by a quasi–morphism $C \to D' \leftarrow D$ (second arrow a weak equivalence and a cofibration) ; its mapping cone is defined as the mapping cone of $C \to D'$.

5.3. Lemma. *Suppose that $\mathcal{C}$ satisfies 2.1, 2.2, 2.3. Let $B_3 \to B_2 \xrightarrow{g} B_1$ be a cofibration sequence in $\mathcal{C}$. If $D \mapsto \pi_0(B_i \odot D)$ is a co–representable functor on $\mathcal{HC}$ for $i = 1, 2$, then it is co–representable for $i = 3$. A co–representing object is the mapping cone of $T(g) : T(B_1) \to T(B_2)$.*

Proof. Choose C_i and η_i in $B_i \odot C_i$ in nondegenerate components, for $i = 1, 2$. Then

$$|\mathcal{M}(C_1, C_2)| \simeq B_1 \odot C_2 \simeq C_2 \odot B_1 \simeq |\mathcal{M}(B_2, B_1)|$$

by 3.7 and the remark following it. Choose a quasi–morphism $C_1 \to C_2$ in the component corresponding to that of $g : B_2 \to B_1$ (from the cofibration sequence). Replacing C_2 by something weakly equivalent if necessary, we may assume that the quasi–morphism is a genuine morphism $f : C_1 \to C_2$. It turns out that the square

$$(\bullet) \qquad \begin{array}{ccc} |\mathcal{M}(C_2, D)| & \longrightarrow & B_2 \odot D \\ \big\downarrow{\scriptstyle f^*} & & \big\downarrow{\scriptstyle g_*} \\ |\mathcal{M}(C_1, D)| & \longrightarrow & B_1 \odot D \end{array}$$

commutes up to a homotopy which is *natural* in D. In fact the Yoneda line of reasoning reduces this to the assertion that

$$\begin{array}{ccc} \{\mathrm{id}\} & \longrightarrow & B_2 \odot C_2 \\ \big\downarrow{\scriptstyle f^*} & & \big\downarrow{\scriptstyle g_*} \\ |\mathcal{M}(C_1, C_2)| & \longrightarrow & B_1 \odot C_2 \end{array}$$

commutes up to homotopy, where $\mathrm{id} \in |\mathcal{M}(C_2, C_2)|$ is the identity vertex. An equivalent assertion is that $f_*(\eta_1)$ and $g_*(\eta_2)$ are in the same component of $B_1 \odot C_2$. But that is clear from the construction of f.

Let C_3 be the mapping cone of f. Bilinearity of $\odot$ shows that there is a fibration sequence up to homotopy

$$B_3 \odot D \longrightarrow B_2 \odot D \longrightarrow B_1 \odot D$$

natural in D, and 3.3 gives a fibration sequence up to homotopy

$$|\mathcal{M}(C_3, D)| \longrightarrow |\mathcal{M}(C_2, D)| \longrightarrow |\mathcal{M}(C_1, D)|$$

natural in D. Using the homotopy commutativity of $(\bullet)$ now, we conclude that there is a homotopy equivalence $e : |\mathcal{M}(C_3, D)| \longrightarrow B_3 \odot D$, well defined up to homotopy and as such natural in D. Again the Yoneda reasoning shows that e is none other than slant product with $\eta_3 \in B_3 \odot C_3$, the image under e of the identity vertex in $|\mathcal{M}(C_3, C_3)|$. $\square$

5.4. Corollary. *Suppose that $\mathcal{C}$ satisfies 2.1, 2.2, 2.3 and that $D \mapsto \pi_0(B \odot D)$ is a co–representable functor on $\mathcal{HC}$ for every B in a generating class $\mathcal{G}$. Then $\mathcal{C}$ satisfies 2.4.*

Proof. For every object B in $\langle \mathcal{G} \rangle$, the functor $D \mapsto \pi_0(B \odot D)$ is a co–representable functor on $\mathcal{HC}$, by 5.3. □

5.5. Lemma. *Suppose that $\mathcal{C}$ satisfies 2.1, 2.2, 2.3, 2.4. Suppose also that the functor $C \mapsto |\mathcal{M}(B,C)|$ is linear, for fixed B in $\mathcal{C}$. Let $\mathcal{G}$ be a generating class for $\mathcal{C}$, closed under Σ. Suppose that whenever B is in $\mathcal{G}$ and C is in $\mathcal{C}$ and $\eta \in B \odot C$ is in a nondegenerate component, then the slant product with $\tau(\eta) \in C \odot B$ is a bijection $[B,D] \to \pi_0(C \odot D)$ for all D in $\mathcal{G}$. Then $\mathcal{C}$ satisfies 2.5.*

Proof. Fix B in $\mathcal{G}$ and C in $\mathcal{C}$ and $\eta \in B \odot C$ in a nondegenerate component. We note first that slant product with $\tau(\eta) \in C \odot B$, as a map, is a homotopy equivalence $|\mathcal{M}(B,D)| \to C \odot D$ for every D in $\mathcal{G}$. This is proved exactly like 3.7. Lifting the restriction on D, we can still say that slant product with $\tau(\eta)$ is a natural transformation

$$|\mathcal{M}(B,D)| \to C \odot D$$

of functors in the variable D. (Actually, to make it strictly natural we would have to enlarge $C \odot D$ without changing the homotopy type ; see the paragraph preceding 3.6.) The two functors are linear by assumption, and for D in $\mathcal{G}$ the natural transformation specializes to a homotopy equivalence. Therefore by 5.1 it specializes to a homotopy equivalence for arbitrary D in $\mathcal{C}$. In particular, the slant product map $[B,D] \to \pi_0(C \odot D)$ is a bijection for arbitrary D, or equivalently, the canonical map $T^2(B) \to B$ is an isomorphism in $\mathcal{HC}$. This holds for B in $\mathcal{G}$.

Let $\mathcal{B}$ be the class of all objects B in $\mathcal{C}$ for which $T^2(B) \to B$ is an isomorphism in $\mathcal{HC}$. We have seen that $\mathcal{G} \subset \mathcal{B}$. Therefore it is sufficient to show that if $f : C \to D$ is a morphism in $\mathcal{C}$, with C and D in $\mathcal{B}$, then the mapping cone of f is in $\mathcal{B}$.

We can regard T as a contravariant functor from $\mathcal{HC}$ to $\mathcal{HC}$. Let us call a diagram $B \to C \to D$ in $\mathcal{HC}$ *exact* if, up to an isomorphism of diagrams, it can be obtained from a cofibration sequence in $\mathcal{C}$ by passage to $\mathcal{HC}$. If $B \to C \to D$ is exact, then $[B,A] \leftarrow [C,A] \leftarrow [D,A]$ is an exact sequence of abelian groups for every A in $\mathcal{HC}$. (Exactness in the based set sense follows from 3.3 ; the abelian group structures are due to the fact that $[B,A] \cong [\Sigma^n B, \Sigma^n A] \cong \pi_n |\mathcal{M}(B, \Sigma^n A)|$ for any $n > 0$, by 2.4 and 3.3.) An extra careful reading of the proof of 5.3 (exercise) shows that the contravariant functor T takes exact diagrams $C \to D \to E$ to exact diagrams $T(C) \leftarrow T(D) \leftarrow T(E)$. Now fix an exact diagram $C \to D \to E$ in $\mathcal{HC}$ and form the ladder

$$\begin{array}{ccccccccc} T^2(C) & \longrightarrow & T^2(D) & \longrightarrow & T^2(E) & \longrightarrow & T^2(\Sigma C) & \longrightarrow & T^2(\Sigma D) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C & \longrightarrow & D & \longrightarrow & E & \longrightarrow & \Sigma C & \longrightarrow & \Sigma D \end{array}$$

where the vertical arrows are the canonical morphisms. Suppose that C and D are in $\mathcal{B}$. Applying $[?,A]$ for some A in $\mathcal{C}$ transforms the diagram into a diagram of abelian groups with exact rows. Four of the vertical arrows are then isomorphisms of abelian groups, and the fifth is an isomorphism by the five lemma. Since this holds for arbitrary A in $\mathcal{HC}$, the canonical morphism $T^2(E) \to E$ must be an isomorphism in $\mathcal{HC}$, so that E is in $\mathcal{B}$. $\square$

5.6. Lemma. *Suppose that $\mathcal{C}$ satisfies the axioms of §2. Let $\mathcal{G}$ be a generating class for $\mathcal{C}$. For objects B, C in $\mathcal{C}$ and a class $[\eta] \in \pi_0(B \odot C)$, the following are equivalent:*

(1) *$[\eta]$ is nondegenerate ;*
(2) *slant product with $[\eta]$ is a bijection $[C, D] \to \pi_0(B \odot D)$ for every D in $\mathcal{G}$.*

Proof. Assuming (2), let $\mathcal{B}$ be the class of all objects D in $\mathcal{C}$ for which slant product with $[\eta]$ is a bijection $[C, D] \to \pi_0(B \odot D)$. We must show that mapping cones of morphisms between objects in $\mathcal{B}$ belong to $\mathcal{B}$. This is done by a five lemma argument, as in the proof of 5.5. $\square$

6. The axioms and the $\mathcal{S}_\bullet$ construction

Assume throughout this section that $\mathcal{C}$ with the SW product $\odot$ satisfies the axioms of §2. We shall prove that $\mathcal{S}_m\mathcal{C}$ satisfies the axioms.

6.1. Theorem. *Let C, D be objects in $\mathcal{S}_m\mathcal{C}$. The canonical map*

$$|\mathcal{M}(C, D)| \longrightarrow \operatorname*{holim}_{\substack{i,j,p,q\\ i+q\geq m\\ j+p\geq m}} |\mathcal{M}(C(m-j, m-i), D(p, q))|$$

is a homotopy equivalence.

Explanation and proof. The quadruples (i, j, p, q) satisfying the stated conditions form a sub–poset of $[m]^4$, where $[m] = \{0, \ldots, m\}$ as usual. The rule taking a quadruple (i, j, p, q) to $|\mathcal{M}(C(m-j, m-i), D(p, q))|$ is a covariant functor. Note that $C(m-j, m-i)$ and $D(p, q)$ are objects of $\mathcal{C}$, and that the conditions on i, j, p, q are equivalent to $m - j \leq p$ and $m - i \leq q$.

Fix D for the moment. We are then dealing with a natural transformation between contravariant linear functors in the variable C. Using axiom 2.3 for $\mathcal{C}$, one finds that the objects in $\mathcal{S}_m\mathcal{C}$ which are iterated degeneracies of objects in $\mathcal{S}_1\mathcal{C}$ form a generating class $\mathcal{G}$. Therefore by 5.2 it suffices to establish 6.1 in the special case when C is in $\mathcal{G}$. Therefore fix $C = u_s^* A$, where A lives in $\mathcal{S}_1\mathcal{C}$ and where $u_s : [m] \to [1]$ is monotone, onto, with $u_s(s) = 1$ and $u_s(s-1) = 0$. The specialization functor $\mathcal{M}(C, D) \to \mathcal{M}(C(0, s), D(0, s))$ has a left adjoint, so that the corresponding specialization map

$$|\mathcal{M}(C, D)| \longrightarrow |\mathcal{M}(C(0, s), D(0, s)|$$

is a homotopy equivalence. All we have to prove now is that the other specialization map

$$(\bullet)\qquad \underset{\substack{i,j,p,q\\ i+q\geq m\\ j+p\geq m}}{\operatorname{holim}} |\mathcal{M}(C(m-j,m-i),D(p,q))| \longrightarrow |\mathcal{M}(C(0,s),D(0,s)|$$

is also a homotopy equivalence. Here we still assume $C = u_s^*B$, but now we also find it convenient to "unfix" D. In fact domain and codomain of $(\bullet)$ are linear functors in the variable D, by 3.8. Because of 5.1, we may assume without loss of generality that $D = u_t^*B$ for some B in $\mathcal{S}_1\mathcal{C}$ and some t. Summarizing, we may assume $C = u_s^*A$ and $D = u_t^*B$, and must show that $(\bullet)$ is a homotopy equivalence. This is done as in the proof of 1.3. □

6.2. Corollary. *For any $m \geq 0$ and objects C, D in $\mathcal{S}_m\mathcal{C}$, the suspension map from $[C,D]$ to $[\Sigma C, \Sigma D]$ is a bijection.*

Proof. We have $[C,D] = \pi_0|\mathcal{M}(C,D)|$ and $[\Sigma C,\Sigma D] = \pi_0|\mathcal{M}(\Sigma C,\Sigma D)|$. The expressions which 6.1 gives for $|\mathcal{M}(C,D)|$ and $|\mathcal{M}(\Sigma C,\Sigma D)|$ are homotopy equivalent (via the suspension) by 3.4. □

6.3. Lemma. *$\mathcal{S}_m\mathcal{C}$ satisfies 2.3.*

Proof. Let $\mathcal{Y}_m$ be the class of all objects in $\mathcal{S}_m\mathcal{C}$ which can be desuspended infinitely often in $\mathcal{H}\mathcal{S}_m\mathcal{C}$. Because of 6.2, all we have to show is that $\mathcal{Y}_m$ contains all objects of $\mathcal{H}\mathcal{S}_m\mathcal{C}$. But $\mathcal{Y}_m$ contains the generating class $\mathcal{G}$ from the proof of 6.1, and is closed under formation of mapping cones. □

6.4. Lemma. *$\mathcal{S}_m\mathcal{C}$ satisfies 2.2.*

Proof. Suppose that $f : C \to D$ is a morphism in $\mathcal{S}_m\mathcal{C}$ representing an isomorphism in $\mathcal{H}\mathcal{S}_m\mathcal{C}$. Then, for any monotone injective $v : [1] \to [m]$, the morphism v^*f from v^*C to v^*D represents an isomorphism in $\mathcal{H}\mathcal{S}_1\mathcal{C}$. Since 2.2 holds for $\mathcal{C} \cong \mathcal{S}_1\mathcal{C}$, it follows that each v^*f is a weak equivalence, and this in turn implies that f is a weak equivalence in $\mathcal{S}_m\mathcal{C}$, by the very definition of $\mathcal{S}_m\mathcal{C}$.

6.5. Lemma. *$\mathcal{S}_m\mathcal{C}$ satisfies 2.4 and 2.5.*

Proof. Assume $m \geq 2$. Let $u_s : [m] \to [1]$ be the usual monotone surjection with $u_s(s) = 1$, $u_s(s-1) = 0$. By 5.4, to check 2.4 it suffices to check that for B in $\mathcal{S}_1\mathcal{C}$ and every s with $0 < s \leq m$, the functor $D \mapsto \pi_0(u_s^*B \odot D)$ on $\mathcal{H}\mathcal{S}_m\mathcal{C}$ is co–representable. Choose C in $\mathcal{S}_1\mathcal{C}$ and η in a nondegenerate component of $B \odot C$. Let $t = m + 1 - s$. From 1.3 we have a forgetful homotopy equivalence

$u_s^*B \odot u_t^*C \to B \odot C$. Choose $\eta^! \in u_s^*B \odot u_t^*C$ in the component corresponding to that of $\eta \in B \odot C$. We establish 2.4 for $\mathcal{S}_m\mathcal{C}$ by showing that $\eta^!$ is in a nondegenerate component. To do this we use the homotopy commutative diagram

$$(\bullet)\qquad \begin{array}{ccc} |\mathcal{M}(u_t^*C, D)| & \longrightarrow & u_s^*B \odot D \\ \big\downarrow & & \big\downarrow \\ |\mathcal{M}(C, D(0,t))| & \longrightarrow & B \odot D(0,t)) \end{array}$$

where the horizontal arrows are slant products (with $\eta^!$ and η, respectively) and the vertical arrows are specialization maps. (In the lower row, make the identifications $B \cong (u_s^*B)(s-1,m)$ and $C \cong (u_t^*C)(0,t)$.) We saw in the proof of 6.1 that the left–hand vertical arrow in $(\bullet)$ is a homotopy equivalence. The other vertical arrow is a homotopy equivalence when D is an iterated degeneracy of an object in $\mathcal{S}_1\mathcal{C}$, by 1.3. It is therefore a homotopy equivalence in general, by 5.1. Therefore $\eta^!$ is nondegenerate.

Finally we note that the arguments given for nondegeneracy of $\eta^!$ apply equally well to $\tau(\eta^!)$. It follows that the canonical morphism $T^2(u_s^*B) \to u_s^*B$ is an isomorphism in $\mathcal{H}\mathcal{S}_m\mathcal{C}$. Now 5.5 shows that 2.5 holds for $\mathcal{S}_m\mathcal{C}$. Note that the linearity hypothesis needed in 5.5 is satisfied, thanks to 6.1 and 3.8. □

6.6. Proposition. *For B, C in $\mathcal{S}_m\mathcal{C}$, and a class $[\eta]$ in $\pi_0(B \odot C)$, the following conditions are equivalent:*

1. *$[\eta]$ is nondegenerate ;*
2. *the image of $[\eta]$ in $\pi_0(B(0,t) \odot C(m-t,m))$ is nondegenerate, $\forall t$;*
3. *the image of $[\eta]$ in $\pi_0(B(m-t,m) \odot C(0,t))$ is nondegenerate, $\forall t$;*
4. *the image of $[\eta]$ in $\pi_0(B(s,t) \odot C(m-t,m-s))$ is nondegenerate $\forall s,t$.*

Proof. (1)⇔ (2) and (1)⇔ (3) : It is enough to prove (1)⇔ (2). Choose $t > 0$ and choose some D in the image of $u_{m-t+1}^* : \mathcal{S}_1\mathcal{C} \to \mathcal{S}_m\mathcal{C}$. Then D belongs to $\mathcal{G}$, the generating class used in the proof of 6.1. We have a homotopy commutative diagram

$$\begin{array}{ccc} |\mathcal{M}(C, D)| & \longrightarrow & B \odot D \\ \big\downarrow & & \big\downarrow \\ |\mathcal{M}(C(m-t,m), D(m-t,m)| & \longrightarrow & B(0,t) \odot D(m-t,m) \end{array}$$

where the vertical arrows are specialization maps, and the horizontal ones are slant product with η and the image of η in $B(0,t) \odot C(m-t,m)$, respectively. It is enough to show that the two vertical arrows are homotopy equivalences. (For the implication (1)⇒(2), note that $D(m-t,m)$ is arbitrary ; for (2)⇒(1), note that D is arbitrary in $\mathcal{G}$ and use §5.) We will in fact show that each of the two vertical arrows taken by itself is a homotopy equivalence, without assuming any relationship between the objects B and C in $\mathcal{S}_m\mathcal{C}$. Using §5, we may assume without loss of generality that B, C are also in $\mathcal{G}$. Then 1.3 takes care of the right–hand vertical arrow, and an argument completely analogous to 1.3 (but starting from 6.1 rather than the definition of $\odot$ on $\mathcal{S}_m\mathcal{C}$) takes care of the left–hand vertical arrow.

(1) $\Rightarrow$ (4) : Given B, C and nondegenerate $[\eta]$ as in (1) and integers s, t as in (3) with $0 < s < t \leq m$, let $w : [2] \to [m]$ be given by $w(0) = 0$, $w(1) = s$, $w(2) = t$. Then $\bar{w}$ is given by $\bar{w}(0) = m - t$, $\bar{w}(1) = m - s$, $\bar{w}(2) = m$. Let μ be the image of η in $\pi_0(w^*B \odot \bar{w}^*C)$. Then (2) holds for $[\mu]$ because it holds for $[\eta]$, because (1) holds for $[\eta]$. Therefore (1) and (3) hold for $[\mu]$. In particular, the image of $[\mu]$ in π_0 of

$$w * B(1,2) \odot \bar{w}^*C(0,1) \;=\; B(s,t) \odot C(m-t, m-s)$$

is nondegenerate ; but of course it equals the image of $[\eta]$ there. $\square$

6.7. Corollary. *Let $w : [m] \to [n]$ be monotone. Suppose that $[\eta] \in \pi_0(B \odot C)$ is nondegenerate, where B and C are objects in $\mathcal{S}_n\mathcal{C}$. Then the image of $[\eta]$ in $\pi_0(w^*B \odot \bar{w}^*C)$ is nondegenerate.*

Proof. Use the equivalence (1)$\Leftrightarrow$ (4) in 6.6. $\square$

7. The involution on $\boldsymbol{K}(\mathcal{C})$

7.1. Definitions. Waldhausen's definition of the K–theory space $K(\mathcal{C})$ for a Waldhausen category $\mathcal{C}$ is

$$K(\mathcal{C}) := \Omega|w\mathcal{S}_\bullet\mathcal{C}|\,.$$

Here $w\mathcal{S}_\bullet\mathcal{C} := \{w\mathcal{S}_m\mathcal{C} \mid m \geq 0\}$ is regarded as a simplicial category. Suppose now that $\mathcal{C}$ comes with an SW product $\odot$ and satisfies the axioms of §2. Using the notation from the remark following 4.3, we modify Waldhausen's definition as follows:

$$xK(\mathcal{C}) := \Omega|xw\mathcal{S}_\bullet\mathcal{C}|\,.$$

Explanations: By §6, for each $m \geq 0$, the Waldhausen category $\mathcal{S}_m\mathcal{C}$ comes with an SW product and satisfies the axioms of §2. Therefore the topological category $xw\mathcal{S}_m\mathcal{C}$ is defined ; see 4.4. By 6.7, each monotone $[m] \to [n]$ determines an operator from $xw\mathcal{S}_n\mathcal{C}$ to $xw\mathcal{S}_m\mathcal{C}$, so that $xw\mathcal{S}_\bullet\mathcal{C}$ is a *simplicial* topological category. By 4.3 and 4.4, the forgetful functors

$$xw\mathcal{S}_m\mathcal{C} \to w\mathcal{S}_m\mathcal{C} \qquad ; \qquad (C, D, z) \mapsto C$$

for $m \geq 0$ induce homotopy equivalences of the classifying spaces. They commute with simplicial operators, and so define a (simplicial, continuous) functor

$$xw\mathcal{S}_\bullet\mathcal{C} \to w\mathcal{S}_\bullet\mathcal{C}$$

which induces a homotopy equivalence of the classifying spaces. Applying Ω, we obtain a map $xK(\mathcal{C}) \to K(\mathcal{C})$ which is a homotopy equivalence.

Each $xw\mathcal{S}_m\mathcal{C}$ comes with an involution ι, as in 4.4. The involution ι *anticommutes* with the simplicial operators relating the $xw\mathcal{S}_m\mathcal{C}$ for $m \geq 0$. That is, for a monotone $f : [m] \to [n]$, we have

$$\bar{f}^*\iota = \iota f^* : xw\mathcal{S}_n\mathcal{C} \longrightarrow xw\mathcal{S}_m\mathcal{C}$$

(see §1 after 1.2). Nevertheless, ι determines an involution on $|xw\mathcal{S}_\bullet\mathcal{C}|$. Here is an explanation in abstract terms. First of all, for each $m \geq 0$ the order–reversing bijection $[m] \to [m]$ determines a linear homeomorphism $\Delta^m \to \Delta^m$ which we indicate by $v \mapsto \bar{v}$ for $v \in \Delta^m$. Now suppose that Y and Z are simplicial spaces, and $g = \{g_n : Y(n) \to Z(n)\}$ is a collection of maps which anticommutes with the simplicial operators, i.e.,

$$\bar{f}^* g_n = g_m f^*$$

for monotone $f : [m] \to [n]$. Define $|g| : |Y| \to |Z|$ by mapping the point with coordinates $y \in Y(n)$ and $v \in \Delta^n$ to the point with coordinates $g_n(y) \in Z(n)$ and $\bar{v} \in \Delta^n$.

The special case we are interested in here is $Y(n) = Z(n) = |xw\mathcal{S}_n\mathcal{C}|$ and $g = \iota$. In this case $|g|$ is an involution, by inspection.

Remark. One can still see the similarity with [Vo]. However, our construction is clearly more general, better adapted to [Wald2], and it does produce an actual involution as opposed to a self–map whose square is *homotopic to* the identity.

Waldhausen shows in [Wald2, §1.3] that $K(\mathcal{C})$ is the zero term of an Ω–spectrum whose n–th term is the space

$$\Omega|w\underbrace{\mathcal{S}_\bullet \dots \mathcal{S}_\bullet}_{\leftarrow n \rightarrow}\mathcal{C}|\,.$$

The same argument shows that $xK(\mathcal{C})$ is the zero term of an Ω–spectrum *with involution* whose n–th term is the space with involution

$$\Omega|xw\underbrace{\mathcal{S}_\bullet \dots \mathcal{S}_\bullet}_{\leftarrow n \rightarrow}\mathcal{C}|\,.$$

Here $|xw\mathcal{S}_\bullet \dots \mathcal{S}_\bullet\mathcal{C}|$ is the geometric realization of the n–simplicial space

$$(k_1, \dots, k_n) \mapsto xw\mathcal{S}_{k_1} \dots \mathcal{S}_{k_n}\mathcal{C}$$

and the (multi)–simplicial operators anticommute with the involution, as before.

7.2. Summary. *$xK(\mathcal{C})$ is an infinite loop space with involution and the forgetful map $xK(\mathcal{C}) \to K(\mathcal{C})$ of infinite loop spaces is a homotopy equivalence.*

Suppose now that $\mathcal{A}$ and $\mathcal{B}$ are Waldhausen subcategories of $\mathcal{C}$. What this means for $\mathcal{A}$, say, is that $\mathcal{A}$ is a Waldhausen category in its own right, with notions of cofibration and weak equivalence restricted from $\mathcal{C}$, and the inclusion functor $\mathcal{A} \to \mathcal{C}$ is *exact.* See [Wald2, 1.1, 1.2]. Suppose moreover that $\mathcal{A}$ and $\mathcal{B}$ are full subcategories of $\mathcal{C}$, closed under weak equivalence in the following sense: if $C \to D$ is a weak equivalence in $\mathcal{C}$, and one of C, D is in $\mathcal{A}$ (in $\mathcal{B}$), then the other is in $\mathcal{A}$ (in $\mathcal{B}$). Then $\mathcal{A}$ and $\mathcal{B}$ also inherit the cylinder functor from $\mathcal{C}$. Suppose finally that whenever C and D are objects in $\mathcal{C}$ such that a nondegenerate class exists in $\pi_0(C \odot D)$, then C belongs to $\mathcal{A}$ if and only if D belongs to $\mathcal{B}$. Informally, $\mathcal{A}$ and $\mathcal{B}$ are dual subcategories of $\mathcal{C}$.

Let $xw\mathcal{S}_m(\mathcal{C} : \mathcal{A}, \mathcal{B}) \subset xw\mathcal{S}_m(\mathcal{C})$ be the full topological subcategory consisting of the objects (C, D, z) with C in $\mathcal{A}$ and D in $\mathcal{B}$. See 4.4. Let

$$xK(\mathcal{C} : \mathcal{A}, \mathcal{B}) := \Omega|xw\mathcal{S}_\bullet(\mathcal{C} : \mathcal{A}, \mathcal{B})|$$

so that $xK(\mathcal{C} : \mathcal{A}, \mathcal{B})$ is an infinite loop subspace of $xK(\mathcal{C})$. By 4.3 and 4.4, the forgetful maps

$$\begin{array}{lcl} xw\mathcal{S}_m(\mathcal{C} : \mathcal{A}, \mathcal{B}) \to w\mathcal{S}_m(\mathcal{A}) & ; & (C, D, z) \mapsto C \\ xw\mathcal{S}_m(\mathcal{C} : \mathcal{A}, \mathcal{B}) \to w\mathcal{S}_m(\mathcal{B}) & ; & (C, D, z) \mapsto D \end{array}$$

induce homotopy equivalences of the classifying spaces. This leads to the following generalization of 7.2, which we shall need in §9.

7.3. Proposition. *The forgetful maps $K(\mathcal{A}) \leftarrow xK(\mathcal{C} : \mathcal{A}, \mathcal{B}) \to K(\mathcal{B})$ of infinite loop spaces are homotopy equivalences. There is an involutory homeomorphism*

$$xK(\mathcal{C} : \mathcal{A}, \mathcal{B}) \cong xK(\mathcal{C} : \mathcal{B}, \mathcal{A})\,.$$

8. The axioms and parametrization

The goal is to check that the category $\mathcal{C}^*(X)$ defined just before 1.5 satisfies the axioms of §2 provided $\mathcal{C}$ does. The overall strategy is similar to the one used in §6. In particular, we rely on 5.1 and 5.2. Assume throughout this section that $\mathcal{C}$ satisfies the axioms of §2.

8.1. Proposition (compare 6.1)**.** *Let C, D be objects in $\mathcal{C}^*(X)$. The canonical map*

$$|\mathcal{M}(C, D)| \longrightarrow \operatorname*{holim}_{\substack{s,t \\ s\subset t}} |\mathcal{M}(C(s), D(t))|$$

(where s and t are faces of X) is a homotopy equivalence.

Explanation and proof. The pairs (s, t) with $s \subset t$ form a poset where $(s, t) \le (s', t')$ if $t \subset t'$ and $s' \subset s$. The rule $(s, t) \mapsto |\mathcal{M}(C(s), D(t)|$ is a covariant functor from the poset to spaces.

Let s_0 be a face of X. We say that an object C of $\mathcal{C}^*(X)$ is *concentrated on s_0* if $C(s) = *$ for any face s not containing s_0, and $C(s_0) \to C(s)$ is an isomorphism if s does contain s_0. The objects in $\mathcal{HC}^*(X)$ which are isomorphic to objects concentrated on some face of X form a generating class. Therefore, by 5.2, it is enough to establish 8.1 in the special case where C is concentrated on s_0.

Note that the specialization functor $D \mapsto D(s_0)$ from $\mathcal{C}^*(X)$ to $\mathcal{C}$ has a left adjoint λ which embeds $\mathcal{C}$ in $\mathcal{C}^*(X)$ as the full subcategory of the objects concentrated on s_0. It follows that the specialization functor $\mathcal{M}(C,D) \to \mathcal{M}(C(s_0),D(s_0))$ (where C is concentrated on s_0 but D is arbitrary) also has a left adjoint, given on objects by

$$(C(s_0) \to D' \xleftarrow{e} D(s_0)) \mapsto (C \to D'' \leftarrow D)$$

where D'' is the pushout of $\lambda(D') \leftarrow \lambda(D(s_0)) \to D$ and we have identified C with $\lambda(C(s_0))$. Consequently the specialization map

$$|\mathcal{M}(C,D)| \to |\mathcal{M}(C(s_0),D(s_0))|$$

is a homotopy equivalence. All we have to prove now is that the other specialization map

$$\operatorname*{holim}_{\substack{s,t\\ s\subset t}} |\mathcal{M}(C(s),D(t))| \longrightarrow |\mathcal{M}(C(s_0),D(s_0)|$$

is also a homotopy equivalence. This is true because

$$\begin{aligned} &\operatorname*{holim}_{\substack{s,t\\ s\subset t}} |\mathcal{M}(C(s),D(t))| \\ \simeq\ &\operatorname*{holim}_{\substack{s,t\\ s_0\subset s\subset t}} |\mathcal{M}(C(s),D(t))| \\ \cong\ &\operatorname*{holim}_{\substack{s,t\\ s_0\subset s\subset t}} |\mathcal{M}(C(s_0),D(t))| \\ \cong\ &\operatorname*{holim}_{\substack{t\\ t\supset s_0}} \operatorname*{holim}_{\substack{s\\ s_0\subset s\subset t}} |\mathcal{M}(C(s_0),D(t))| \\ \simeq\ &\operatorname*{holim}_{\substack{t\\ t\supset s_0}} |\mathcal{M}(C(s_0),D(t))| \\ \simeq\ &|\mathcal{M}(C(s_0),D(s_0))|\,. \quad \square \end{aligned}$$

8.2. Corollary (compare 6.2)**.** *For objects C, D in $\mathcal{C}^*(X)$, the suspension map from $[C,D]$ to $[\Sigma C, \Sigma D]$ is a bijection.* $\square$

8.3. Lemma (compare 6.3)**.** $\mathcal{C}^*(X)$ *satisfies 2.3.*

Proof. Let $\mathcal{Y}$ be the class of all objects in $\mathcal{HC}^*(X)$ which can be desuspended infinitely often. Because of 8.2, all we have to show is that $\mathcal{Y}$ contains all objects of $\mathcal{HC}^*(X)$. We note that

(1) the mapping cone of a morphism in $\mathcal{HC}^*(X)$ with domain and codomain in $\mathcal{Y}$ belongs to $\mathcal{Y}$;
(2) if ΣC belongs to $\mathcal{Y}$, then C belongs to $\mathcal{Y}$.

Finally we note that if C is an object of $\mathcal{HC}^*(X)$ concentrated on a face s of X, then C belongs to $\mathcal{Y}$. $\square$

8.4. Lemma (compare 6.4)**.** *$\mathcal{C}^*(X)$ satisfies 2.2.* □

For a face $r \subset X$, let $\mathcal{G}_r$ be the class of all objects B in $\mathcal{C}^*(X)$ such that $B(s)$ is weakly contractible for $s \neq r$. Then $\bigcup_r \mathcal{G}_r$ is a generating class for $\mathcal{HC}^*(X)$, and we want to use it to check axioms 2.4 and 2.5 following the method of 5.4 and 5.5. Note that $B \odot C \simeq \Omega^{|r|}(B(r) \odot C(r))$ for B in $\mathcal{G}_r$ and arbitrary C in $\mathcal{C}^*(X)$, from the definition of $\odot$ in $\mathcal{C}^*(X)$. It is this fact which makes $\bigcup_r \mathcal{G}_r$ such a convenient generating class.

Notation: For a face r of X, we denote by ∂r the union of the proper faces of r, a simplicial subcomplex of X of dimension $|r|-1$. For C in $\mathcal{C}^*(X)$, write $C(r/\partial r)$ for the cofiber $C(r)/C(\partial r)$. More generally, if r and s are faces of X with $s \subset r$, write $C(r/s)$ for the cofiber $C(r)/C(s)$.

8.5. Lemma. *For C in $\mathcal{C}^*(X)$ and D in $\mathcal{G}_r$, the evaluation functor from $\mathcal{M}(C,D)$ to $\mathcal{M}(C(r/\partial r), D(r/\partial r))$ induces a homotopy equivalence of the classifying spaces.*

Proof. By 5.2 it suffices to check this when C is concentrated on a face s of X. We have seen before (proof of 8.1) that in this case the specialization functor

$$\mathcal{M}(C,D) \longrightarrow \mathcal{M}(C(s), D(s))$$

induces a homotopy equivalence of the classifying spaces. It follows that $|\mathcal{M}(C,D)|$ is contractible for $r \neq s$, in agreement with what we are trying to prove, since then $C(r/\partial r) = *$. In the case $r = s$ we note that $C(\partial r) = *$ and obtain the factorization

$$\mathcal{M}(C,D) \longrightarrow \mathcal{M}(C(r), D(r)) \longrightarrow \mathcal{M}(C(r/\partial r), D(r/\partial r))\,.$$

Here the first functor induces a homotopy equivalence of the classifying spaces, and so does the second because the projection from $D(r)$ to $D(r/\partial r)$ is a weak equivalence in $\mathcal{C}$. □

8.6. Proposition. *$\mathcal{C}^*(X)$ satisfies 2.4.*

Proof. Given a face $r \subset X$ and B in $\mathcal{G}_r$, choose C in $\mathcal{C}^*(X)$ so that C is concentrated on r and such that there exists a nondegenerate class $[\eta]$ in $\pi_0(B(r) \odot C(r))$. Then

$$B \odot \Sigma^{|r|} C \;\simeq\; \Omega^{|r|}(B(r) \odot \Sigma^{|r|} C(r)) \;\simeq\; B(r) \odot C(r)\,.$$

Therefore $[\eta]$ determines $[\eta^!] \in \pi_0(B \odot \Sigma^{|r|} C)$, and it turns out that $[\eta^!]$ is again nondegenerate. Namely, for arbitrary D in $\mathcal{C}^*(X)$ we have a commutative square of sets

$$\begin{array}{ccc} [\Sigma^{|r|} C, D] & \longrightarrow & \pi_0(B \odot D) \\ \downarrow & & \downarrow \\ [\Sigma^{|r|} C(r), D(r)] & \longrightarrow & \pi_{|r|}(B(r) \odot D(r)) \end{array}$$

in which the horizontal arrows are slant products (with $[\eta^!]$ and $[\eta]$, respectively). The vertical arrows are bijections and the lower horizontal arrow is also a bijection by 3.7, here applied to $\mathcal{C}$. We have made the identification $[\Sigma^{|r|}C(r), D(r)] \simeq \pi_{|r|}|\mathcal{M}(C(r), D(r))|$.

Summarizing, for each B in $\bigcup_r \mathcal{G}_r$ the functor $D \mapsto \pi_0(B \odot D)$ on $\mathcal{HC}^*(X)$ is co–representable. We complete the proof by applying 5.4. □

For C, D in $\mathcal{C}^*(X)$ and a face r of X, we have the specialization map from $C \odot D$ to $C(r) \odot D(r)$. In the proof of the next proposition, we need an enhanced specalization map of the form

$$C \odot D \to \Omega^{|r|}\left(C(r) \odot D(r/\partial r)\right).$$

This is well defined up to homotopy, as the composition of an obvious map

$$C \odot D \;=\; \operatorname*{holim}_{s} C(s) \odot D(s) \;\longrightarrow\; \operatorname*{holim}_{s \subset r} C(s) \odot D(s/s \cap \partial r)$$

(where r is fixed) with a homotopy inverse of the inclusion

$$\Omega^{|r|}\left(C(r) \odot D(r/\partial r)\right) \;\cong\; \operatorname*{holim}_{s \subset r} F(s) \quad\longrightarrow\quad \operatorname*{holim}_{\substack{s\\ s \subset r}} C(s) \odot D(s/s \cap \partial r)$$

where $F(s) = *$ if $s \neq r$ and $F(r) = C(r) \odot D(r/\partial r)$. Strictly speaking, $\operatorname{holim}_s F(s)$ is canonically homeomorphic to the space of pointed maps $r/\partial r \to C(r) \odot D(r/\partial r)$, so one needs to choose an identification of $r/\partial r$ with $\mathbb{S}^{|r|}$.

8.7. Proposition. $\mathcal{C}^*(X)$ *satisfies 2.5.*

Proof. Let B, C and $[\eta^!]$ be as in the proof of 8.6. By 5.5 it is sufficient to show that the slant product with $[\tau(\eta^!)] \in \pi_0(\Sigma^{|r|}C \odot B)$ is a bijection

$$[B, D] \longrightarrow \pi_0(\Sigma^{|r|}C \odot D)$$

for every face $s \subset X$ and every $D \in \mathcal{G}_s$. The slant product with $[\tau(\eta^!)]$ fits into a commutative diagram

$$\begin{array}{ccc}
[B, D] & \longrightarrow & \pi_0(\Sigma^{|r|}C \odot D) \\
\big\downarrow & & \big\downarrow \\
[B(s/\partial s), D(s/\partial s)] & \longrightarrow & \pi_{|s|}(\Sigma^{|r|}C(s) \odot D(s/\partial s))
\end{array}$$

in which the lower horizontal arrow is the slant product with $[\mu]$, the image of $[\tau(\eta^!)]$ under the specialization map

$$\pi_0(\Sigma^{|r|}C \odot B) \to \pi_{|s|}(\Sigma^{|r|}C(s) \odot B(s/\partial s)).$$

The left–hand vertical arrow in the diagram is a bijection by 8.5. (Here we use the restrictive assumption on D.) The right–hand one is a bijection by inspection.

So now we need to know that slant product with $[\mu]$ (lower horizontal arrow) is a bijection.

If s does not contain r, then both $C(s)$ and $B(s/\partial s)$ are isomorphic to $*$ in $\mathcal{HC}$, so that slant product with $[\mu]$ is a bijection for trivial reasons. Assume therefore that s contains r. Then $C(s) \cong C(r)$ and $B(s/\partial s)$ is isomorphic to $\Sigma^{|s|-|r|}B(r)$ in $\mathcal{HC}$. With these identifications we rewrite the lower row in the square as

$$\pi_{|s|-|r|}|\mathcal{M}(B(r), D(s/\partial s))| \longrightarrow \pi_{|s|-|r|}(C(r) \odot D(s/\partial s))$$

and recognize (exercise) that it is the slant product with $[\tau(\eta)] \in \pi_0(C(r) \odot B(r))$. But this is a bijection by 3.7, since $\mathcal{C}$ satisfies 2.5 and $[\eta] \in \pi_0(B(r) \odot C(r))$ was nondegenerate to begin with. (See the remark after 3.6). □

8.8. Proposition. *For B, C in $\mathcal{C}^*(X)$ and $[\eta] \in \pi_0(B \odot C)$ the following are equivalent:*

(1) *$[\eta]$ is nondegenerate ;*
(2) *for each face $s \subset X$, the image of $[\eta]$ under the specialization map*

$$\pi_0(B \odot C) \to \pi_{|s|}(B(s) \odot C(s/\partial s)) \cong \pi_0(\Sigma^{-|s|}B(s) \odot C(s/\partial s))$$

is nondegenerate.

Proof. We apply 5.6, with $\mathcal{G} = \bigcup_s \mathcal{G}_s$ as in 8.5. For D in $\mathcal{G}_s$ and $[\eta] \in \pi_0(B \odot C)$ as above we have a commutative diagram of slant products (horizontal arrows) and specialization maps (vertical arrows)

$$\begin{array}{ccc} [C, D] & \longrightarrow & \pi_0(B \odot D) \\ \downarrow & & \downarrow \\ [C(s/\partial s), D(s/\partial s)] & \longrightarrow & \pi_{|s|}(B(s) \odot D(s/\partial s)) \,. \end{array}$$

By 8.5, one of the vertical arrows is a bijection, and we have noted many times before that the other is a bijection, too. Therefore the upper horizontal arrow is bijective if and only if the lower one is ; in other words, (1) is equivalent to (2). □

We need the following application of 8.8 in §9. It will help us to verify that certain (incomplete) simplicial sets have the Kan extension property.

8.9. Application. *Suppose that $X = \Delta^m$, that $Y \subset X$ is the i–th face (of dimension $m-1$) for some i, and $Z \subset X$ is the union of all proper faces of X except Y. Suppose that B and C are in $\mathcal{C}^*(X)$, and a nondegenerate class in $\pi_0(B \odot C)$ exists. Then the following are equivalent:*

(1) *$B(Y) \rightarrowtail B(X)$ is a weak equivalence ;*
(2) *$C(Z) \rightarrowtail C(X)$ is a weak equivalence.*

Proof. For a face r of X, of any dimension, define $\mathcal{G}_r$ as in 8.5 and let $\mathcal{G}'_r$ be the class of all objects D in $\mathcal{C}^*(X)$ for which $D(t) \simeq *$ if t does not contain r, and $D(r) \rightarrowtail D(t)$ is a weak equivalence if t contains r. Up to weak equivalence, objects in $\mathcal{G}'_r$ are objects concentrated on r. By 8.8, if E_1 and E_2 are objects of $\mathcal{C}^*(X)$ and a nondegenerate class in $\pi_0(E_1 \odot E_2)$ exists, then E_1 is in $\mathcal{G}_r$ if and only if E_2 is in $\mathcal{G}'_r$. In other words, $\mathcal{G}'_r$ and $\mathcal{G}_r$ are "dual" to each other. Now assume that (1) holds. Then we can find a morphism $B_1 \to B$ in $\mathcal{C}^*(X)$ such that B_1 is in $\langle \cup_{r\subset Y} \mathcal{G}'_r \rangle$ and $B_1(Y) \to B(Y)$ as well as $B_1(X) \to B(X)$ are weak equivalences. The mapping cone of $B_1 \to B$ is therefore weakly equivalent to an object in $\langle \cup_{r\subset Z} \mathcal{G}'_r \rangle$. In short, B is weakly equivalent to an object in

$$\langle \bigcup_{r\subset Z} \mathcal{G}_r \cup \bigcup_{r\subset Y} \mathcal{G}'_r \rangle\,.$$

Then C is weakly equivalent to an object in

$$\langle \bigcup_{r\subset Y} \mathcal{G}_r \cup \bigcup_{r\subset Z} \mathcal{G}'_r \rangle$$

and this implies (2). The converse, (2) $\Rightarrow$ (1) , can be proved similarly. $\square$

8.10. Corollary/Definitions. Let $Y \subset X$ be a simplicial subcomplex. Restriction of parameters is an exact functor $\rho : \mathcal{C}^*(X) \to \mathcal{C}^*(Y)$, and from the definition of $\odot$ in $\mathcal{C}^*(X)$, the restriction functor ρ is accompanied by a binatural transformation

$$B \odot C \longrightarrow \rho(B) \odot \rho(C) \qquad\qquad (B, C \text{ in } \mathcal{C}^*(X)).$$

By 8.8, this takes nondegenerate components to nondegenerate components. Let $\mathcal{C}^*(X,Y)$ be the inverse image of $*$ under the exact functor ρ. Then $\mathcal{C}^*(X,Y)$ inherits from $\mathcal{C}^*(X)$ the structure of a Waldhausen category with SW product. It follows from 8.8 that the dual $T(B)$ in $\mathcal{HC}^*(X)$ of some B in $\mathcal{HC}^*(X,Y)$ belongs to $\mathcal{HC}^*(X,Y)$, up to isomorphism, so that $\mathcal{C}^*(X,Y)$ satisfies axioms 2.4 and 2.5. Axiom 2.1 for $\mathcal{C}^*(X,Y)$ is obvious, and axioms 2.2 and 2.3 for $\mathcal{C}^*(X,Y)$ can be established as in 8.1, 8.2, 8.3.

For any $m \geq 0$ and any face $s \subset X$ not contained in Y, we define an exact functor $p_s : \mathcal{C}^*(X,Y) \to \mathcal{C}$ by $C \mapsto C(s/\partial s)$. Note that p_s has a left inverse q_s given by $(q_s D)(t) = D$ for faces $t \subset X$ containing s, and $(q_s D)(t) = *$ for faces t not containing s. The following lemma will be useful in §9, in constructing the map Ξ (see introduction).

8.11. Lemma. *The map $\boldsymbol{K}(\mathcal{C}^*(X,Y))) \to \prod_s \boldsymbol{K}(\mathcal{C})$ induced by the p_s for all faces $s \subset X$ not contained in Y is a homotopy equivalence. Equivalently, the map from $\bigvee_s \boldsymbol{K}(\mathcal{C})$ to $\boldsymbol{K}(\mathcal{C}^*(X,Y))$ induced by the q_s for all faces s of X not contained in Y is a homotopy equivalence.*

Proof (compare [Lü, §10]). For C in $\mathcal{C}^*(X,Y)$ let C_i in $\mathcal{C}^*(X,Y)$ be defined by $C_i(Z) := C(Z^i)$, where Z^i is the i–skeleton of the subcomplex $Z \subset X$. Then each C in $\mathcal{C}^*(X,Y)$ fits into a natural diagram of cofibrations

$$* \cong C_{-1} \rightarrowtail C_1 \rightarrowtail \dots \rightarrowtail C_m = C$$

where m is the dimension of X. The additivity theorem of [Wald2] now implies that the identity map of $\boldsymbol{K}(\mathcal{C}^*(X,Y))$ is homotopic to the sum of the maps induced by the exact functors

$$F_i : \mathcal{C}^*(X,Y) \to \mathcal{C}^*(X,Y) \quad ; \quad C \mapsto C_i/C_{i-1}$$

for $-1 \le i \le m$. Each F_i factors through $\prod_{|s|=i} p_s : \mathcal{C}^*(X,Y) \to \prod_{|s|=i} \mathcal{C}$ where the product is over all i–simplices $s \subset \Delta^m$. The conclusion is that the map from $\boldsymbol{K}(\mathcal{C}^*(X,Y))$ to $\prod_s \boldsymbol{K}(\mathcal{C})$ induced by the p_s for *all* faces $s \subset X$ not contained in Y has a homotopy left inverse. But clearly it has a homotopy right inverse also, namely, the map from $\bigvee_s \boldsymbol{K}(\mathcal{C}) \simeq \prod_s \boldsymbol{K}(\mathcal{C})$ to $\boldsymbol{K}(\mathcal{C}^*(X,Y))$ induced by the functors q_s . □

8.12. Corollary. *The diagram $\boldsymbol{K}(\mathcal{C}^*(X,Y)) \to \boldsymbol{K}(\mathcal{C}^*(X)) \to \boldsymbol{K}(\mathcal{C}^*(Y))$ is a fibration sequence up to homotopy.* □

9. Symmetric L–theory and the map Ξ

Suppose throughout this section that $\mathcal{C}$ is a Waldhausen category with SW product, satisfying the axioms of §2. We abbreviate $\mathcal{C}^*(m) := \mathcal{C}^*(\Delta^m)$.

9.1. Definition/Notation. A 0–dimensional *symmetric Poincaré object* in $\mathcal{C}$ is an object C in $\mathcal{C}$, together with a point ϕ in $(C \odot C)^{h\mathbb{Z}/2}$ whose image ϕ_0 in $C \odot C$ is in a nondegenerate component. The set of 0–dimensional symmetric Poincaré objects in $\mathcal{C}$ is denoted by $\mathrm{sp}_0(\mathcal{C})$.

Further explanations. The superscript $h\mathbb{Z}/2$ means: form the space of homotopy fixed points of the action of $\mathbb{Z}/2$. Here the action is on $C \odot C$, and the generator of $\mathbb{Z}/2$ acts by τ of 1.1. The homotopy fixed point ϕ is a $\mathbb{Z}/2$–map from $E\mathbb{Z}/2$ to $C \odot C$, and ϕ_0 is the value of ϕ at the base point of $E\mathbb{Z}/2$.

In the next definition, a Δ*–set* is a simplicial set without degeneracy operators, i.e., a contravariant functor $[m] \mapsto X[m]$ from the category Δ (with objects $[m]$ for $m \ge 0$, and monotone injections as morphisms) to the category of sets.

9.2. Definition. $L^\bullet(\mathcal{C})$ is the realization of the Δ–set $[m] \mapsto \mathrm{sp}_0(\mathcal{C}^*(m))$.

(Note that this depends not only on $\mathcal{C}$, but also very much on $\odot$.) One should think of a 0–dimensional symmetric Poincaré object in $\mathcal{C}^*(1)$, for example, as a *bordism* between two 0–dimensional symmetric objects in $\mathcal{C}^*(0) \cong \mathcal{C}$. Similarly, 0–dimensional symmetric Poincaré objects in $\mathcal{C}^*(m)$ are to be thought of as m–parameter bordisms, so that $L^\bullet(\mathcal{C}, 0)$ is the *bordism theory* of symmetric Poincaré objects in $\mathcal{C}$.

Remark. $L^\bullet(\mathcal{C})$ is (the realization of) a *fibrant* Δ–set, i.e., one having the Kan extension property. *Proof:* Fix $m > 0$. Let $X = \Delta^m$ and define $Y, Z \subset X$ as in 8.9. We will sometimes regard X, Y and Z as simplicial complexes, and sometimes as Δ-sets.

Extending a Δ–map $Z \to L^\bullet(\mathcal{C})$ to a Δ–map $X \to L^\bullet(\mathcal{C})$ is the same as lifting a 0–dimensional Poincaré object (C, ϕ) in $\mathcal{C}^*(Z)$ to a 0–dimensional Poincaré object (D, ψ) in $\mathcal{C}^*(X)$, so that $\rho(D, \psi) = (C, \phi)$. Here $\rho : \mathcal{C}^*(X) \to \mathcal{C}^*(Z)$ is the restriction functor. (See 8.10.) To construct (D, ψ), we first construct D. Note that $D(W) = C(W)$ is prescribed for all subcomplexes $W \subset Z$. Let $D(X)$ be the pushout of

$$C(Y \cap Z) \quad \overset{p}{\longleftarrow} \quad \mathrm{cyl}\,[\,C(Y \cap Z) \xrightarrow{\mathrm{id}} C(Y \cap Z)\,] \quad \rightarrowtail \quad \mathrm{cyl}\,[\,C(Z) \xrightarrow{\mathrm{id}} C(Z)\,]$$

where *cyl* denotes mapping cylinders and p is a cylinder projection. It is good to think of $D(X)$ as the *relative mapping cylinder* of $\mathrm{id} : C(Z) \to C(Z)$, relative to $C(Y \cap Z)$. We define $D(Z) \rightarrowtail D(X)$ as the back inclusion and $D(Y) \rightarrowtail D(X)$ as the front inclusion of the relative cylinder. This completes the construction of D ; all we really need to know about it is summarized in the two properties

(1) the cofibration $C(Z) = D(Z) \rightarrowtail D(X)$ is a weak equivalence ;
(2) the cofibration $D(Y) \rightarrowtail D(X)$ is also a weak equivalence.

These properties ensure that $\phi \in (C \odot C)^{h\mathbb{Z}/2}$ has a lift $\psi \in (D \odot D)^{h\mathbb{Z}/2}$. It remains to show that the class $[\psi_0]$ in $\pi_0(D \odot D)$ is nondegenerate. Let $f : TD \to D$ be a morphism in $\mathcal{C}^*(X)$ representing the adjoint of $[\psi_0]$, a morphism in $\mathcal{HC}^*(X)$. By 8.9, the cofibrations $(TD)(Z) \rightarrowtail (TD)(X)$ and $(TD)(Y) \rightarrowtail (TD)(X)$ are weak equivalences. By 8.10, the specialization $f_W : (TD)(W) \to D(W)$ is a weak equivalence for any subcomplex W of Z, in particular for $W = Z$. Combining this information with (1) and (2) above, we see that $f_W : (TD)(W) \to D(W)$ is a weak equivalence for all $W \subset X$, so that f is a weak equivalence in $\mathcal{C}^*(X)$. $\square$

Each 0–dimensional symmetric Poincaré object (C, ϕ) in $\mathcal{C}$ can be viewed as a point in $|xw\mathcal{C}|^{h\mathbb{Z}/2}$ projecting to the vertex (C, C, ϕ_0) in $|xw\mathcal{C}|$. (The notation comes from §4, in particular, the prefix x indicates enlarged models designed so that $\mathbb{Z}/2$ can act.) Further, we have the inclusions $|xw\mathcal{C}| \subset xK(\mathcal{C})$ and $|xw\mathcal{C}|^{h\mathbb{Z}/2} \subset xK(\mathcal{C})^{h\mathbb{Z}/2}$. Consequently $\mathrm{sp}_0(\mathcal{C}) \subset xK(\mathcal{C})^{h\mathbb{Z}/2}$ and for the same reason $\mathrm{sp}_0(\mathcal{C}^*(m))$ is contained in $xK(\mathcal{C}^*(m))^{h\mathbb{Z}/2}$, and so we see an inclusion map

$$(\maltese) \qquad \Xi : \big|\,[m] \mapsto \mathrm{sp}_0(\mathcal{C}^*(m))\,\big| \quad \hookrightarrow \quad \big|\,[m] \mapsto xK(\mathcal{C}^*(m))^{h\mathbb{Z}/2}\,\big|.$$

(The domain of Ξ is the realization of a Δ–set and the codomain is the realization of a Δ-0-space.) The domain of Ξ is $L^\bullet(\mathcal{C})$ and the codomain will eventually (in 9.3–9.14) be identified with Ω^∞ of the *Tate* spectrum $\boldsymbol{K}(\mathcal{C})^{th\mathbb{Z}/2}$ (prefix x suppressed). This is the mapping cone of a certain norm map from a homotopy orbit spectrum to a homotopy fixed point spectrum,

$$\boldsymbol{K}(\mathcal{C})_{h\mathbb{Z}/2} \longrightarrow \boldsymbol{K}(\mathcal{C})^{h\mathbb{Z}/2}\,.$$

More details about Tate spectra are given later in this section. The identification of the codomain of Ξ with $\Omega^\infty(\boldsymbol{K}(\mathcal{C})^{th\mathbb{Z}/2})$ goes back to [WW2], although not in this generality. We will not use results from [WW2].

9.3. Lemma. *The geometric realization of the Δ–spectrum $[m] \mapsto \boldsymbol{K}(\mathcal{C}^*(m))$ is contractible.*

Proof. By 8.11, the geometric realization in question is homotopy equivalent to the geometric realization of

$$[m] \mapsto \boldsymbol{K}(\mathcal{C})^{\mathcal{R}(m)}$$

where $\mathcal{R}(m)$ denotes the set of nonempty subsets of $\{0, \dots, m\}$. Here the face operator f^* induced by a monotone injection $f : [m] \to [n]$ is given by $(f^*(x))(s) = x(f(s))$ for a nonempty $s \subset [m]$.

The geometric realization of any Δ–spectrum $[m] \to \boldsymbol{Y}[m]$ has a canonical filtration (by skeletons) which leads to a spectral sequence converging to the homotopy groups. Its E^1 term including differential is isomorphic to the chain complex of *graded* abelian groups

$$\cdots \longrightarrow \pi_*\boldsymbol{Y}[m+1] \longrightarrow \pi_*\boldsymbol{Y}[m] \longrightarrow \cdots$$

where the differentials are alternating sums of homomorphisms induced by face operators d_i. In the case of the Δ–spectrum $[m] \mapsto \boldsymbol{K}(\mathcal{C})^{\mathcal{R}(m)}$, the E^1 term is therefore isomorphic to the tensor product of the graded abelian group $\pi_*\boldsymbol{K}(\mathcal{C})$ with a chain compex D of ungraded abelian groups,

$$\cdots \longrightarrow \mathbb{Z}^{\mathcal{R}(m+1)} \longrightarrow \mathbb{Z}^{\mathcal{R}(m)} \longrightarrow \mathbb{Z}^{\mathcal{R}(m-1)} \longrightarrow \cdots\,.$$

The differential in D is given by $(\partial x)(s) = \sum_i (-1)^i x(e_i(s))$ where e_i is the monotone injection from $[m]$ to $[m+1]$ with $i \notin \operatorname{im}(e_i)$. Now all we have to show is that D is acyclic, because then the E^2 term of our spectral sequence will vanish.

Let $\mathcal{R}(m)_+ \subset \mathcal{R}(m)$ consist of the $s \subset \{0, \dots, m\}$ which contain 0, and let $\mathcal{R}(m)_-$ be the complement of $\mathcal{R}(m)_+$. The splittings $\mathcal{R}(m) = \mathcal{R}(m)_+ \cup \mathcal{R}(m)_-$ determine a splitting of D as a graded group: $D = D_- \oplus D_+$. The differential maps D_- isomorphically to D/D_-. Therefore D is acyclic. ☐

Call a CW–spectrum $\boldsymbol{V}$ with an action of a discrete group G *induced* if there exists a CW–spectrum $\boldsymbol{U}$ and a G–map $\boldsymbol{U} \wedge G_+ \to \boldsymbol{V}$ which is an ordinary homotopy equivalence.

9.4. Lemma. *Let Φ be the homotopy fiber of the face map $\boldsymbol{K}(\mathcal{C}^*(m)) \to \boldsymbol{K}(\mathcal{C}^*(0))$ induced by the functor "restriction to the 0–th vertex". Then Φ is induced as a spectrum with $\mathbb{Z}/2$–action.*

Proof. Remember that $\boldsymbol{K}(\mathcal{C}^*(m))$ is short for $x\boldsymbol{K}(\mathcal{C}^*(m))$, a spectrum with $\mathbb{Z}/2$–action. The inclusion of $\boldsymbol{K}(\mathcal{C}^*(\Delta^m, \Delta^0))$ in Φ is a homotopy equivalence by 8.12. (Here we embed Δ^0 in Δ^m as the 0–th vertex.) So it is enough to show that $\boldsymbol{K}(\mathcal{C}^*(\Delta^m, \Delta^0))$ is induced as a spectrum with $\mathbb{Z}/2$–action.

Next we reduce to the case $m = 1$, as follows. Let $Z \subset \Delta^m$ be the 0–th face, opposite the 0–th vertex. There is an isomorphism of Waldhausen categories from $\mathcal{C}^*(\Delta^m, \Delta^0)$ to $\mathcal{D}^*(\Delta^1, \Delta^0)$ where $\mathcal{D} = \mathcal{C}^*(Z)$. The isomorphism takes C in $\mathcal{C}^*(\Delta^m, \Delta^0)$ to the cofibration in $\mathcal{D}$, alias object in $\mathcal{D}^*(\Delta^1, \Delta^0)$, given by

$$(s \mapsto C(s)) \quad \rightarrowtail \quad (s \mapsto C(s \cup 0))$$

where s runs through the faces of Z. The inverse isomorphism $\imath$ from $\mathcal{D}^*(\Delta^1, \Delta^0)$ to $\mathcal{C}^*(\Delta^m, \Delta^0)$ comes with a binatural inclusion

$$D \odot E \to \imath(D) \odot \imath(E)$$

which is a homotopy equivalence, respects the symmetry operators τ, and takes nondegenerate components to nondegenerate components. This is clear from 1.5 and 8.8. Therefore the inclusion of $\boldsymbol{K}(\mathcal{D}^*(\Delta^1, \Delta^0))$ in $\boldsymbol{K}(\mathcal{C}^*(\Delta^m, \Delta^0))$ is a homotopy equivalence, and respects the actions of $\mathbb{Z}/2$.

Having reduced to the case $m = 1$, we finally show that $\boldsymbol{K}(\mathcal{C}^*(\Delta^1, \Delta^0))$ is induced as a $\mathbb{Z}/2$–spectrum. Let $s \subset \Delta^1$ be the 0–th face (opposite the 0–th vertex) and let $t \subset \Delta^1$ be the unique face of dimension 1. Let $\mathcal{A} \subset \mathcal{C}^*(\Delta^1, \Delta^0)$ be the full subcategory consisting of the objects C for which $C(s) \rightarrowtail C(t)$ is a weak equivalence in $\mathcal{C}$, and let $\mathcal{B} \subset \mathcal{C}^*(\Delta^1, \Delta^0)$ be the full subcategory consisting of the objects D for which $D(t)$ is weakly equivalent to the zero object of $\mathcal{C}$. By 8.8, if C and D are objects in $\mathcal{C}^*(\Delta^1, \Delta^0)$ and a nondegenerate class in $C \odot D$ exists, then C belongs to $\mathcal{A}$ if and only if D belongs to $\mathcal{B}$.

We now use the functors $p_s : \mathcal{C}(\Delta^1, \Delta^0) \to \mathcal{C}$ and $p_t : \mathcal{C}(\Delta^1, \Delta^0) \to \mathcal{C}$ defined in 8.11. Of the functors $p_s \mid \mathcal{A}$ and $p_t \mid \mathcal{A}$, the first induces a homotopy equivalence $e : \boldsymbol{K}(\mathcal{A}) \to \boldsymbol{K}(\mathcal{C})$ by the approximation theorem [Wald2, 1.6] and the second induces a nullhomotopic map. Of the functors $p_s \mid \mathcal{B}$ and $p_t \mid \mathcal{B}$, the first induces a homotopy equivalence $f : \boldsymbol{K}(\mathcal{B}) \to \boldsymbol{K}(\mathcal{C})$ by the approximation theorem, and the second induces a map $g : \boldsymbol{K}(\mathcal{B}) \to \boldsymbol{K}(\mathcal{C})$ with the property $f + g \simeq 0$, by the additivity theorem. (Namely, $f + g$ is homotopic to the map induced by the exact functor $C \mapsto C(t)$ from $\mathcal{B}$ to $\mathcal{C}$; but $C(t) \simeq *$ for all C in $\mathcal{B}$.) It follows that g is also a homotopy equivalence. We conclude that the composition

$$\boldsymbol{K}(\mathcal{A}) \vee \boldsymbol{K}(\mathcal{B}) \to \boldsymbol{K}(\mathcal{C}^*(\Delta^1, \Delta^0)) \to \boldsymbol{K}(\mathcal{C}) \times \boldsymbol{K}(\mathcal{C})$$

(first map induced by the inclusions, second map induced by p_s and p_t) is a homotopy equivalence. Therefore, by 8.11,

$$\boldsymbol{K}(\mathcal{A}) \vee \boldsymbol{K}(\mathcal{B}) \to \boldsymbol{K}(\mathcal{C}^*(\Delta^1, \Delta^0))$$

is a homotopy equivalence. By 7.3, the duality involution on $\boldsymbol{K}(\mathcal{C})$, as a homotopy class of maps, takes the wedge summand $\boldsymbol{K}(\mathcal{A})$ to the wedge summand $\boldsymbol{K}(\mathcal{B})$, and takes the wedge summand $\boldsymbol{K}(\mathcal{B})$ to the wedge summand $\boldsymbol{K}(\mathcal{A})$. $\square$

9.5. Definition. Let $\mathcal{C}^*[m]$ be the following minor variation on $\mathcal{C}^*(m)$: objects of $\mathcal{C}^*[m]$ are covariant functors from the poset of nonempty faces of Δ^m to $\mathcal{C}$, and morphisms are natural transformations between such functors. A morphism $f : C \to D$ is a cofibration if $f_s : C(s) \to D(s)$ is a cofibration for all faces s, and a weak equivalence if $f_s : C(s) \to D(s)$ is a weak equivalence for all faces s.

Our reason for introducing $\mathcal{C}^*[m]$ is this. In some situations it is annoying that the rule $[m] \mapsto \boldsymbol{K}(\mathcal{C}^*(m))$ is not a simplicial spectrum (it is only a Δ–spectrum, since the degeneracy operators are missing). Fortunately the inclusions $\boldsymbol{K}(\mathcal{C}^*(m)) \to \boldsymbol{K}(\mathcal{C}^*[m])$ are homotopy equivalences (see 9.7 below), and the rule $[m] \mapsto \boldsymbol{K}(\mathcal{C}^*[m])$ is a simplicial spectrum because $[m] \mapsto \mathcal{C}^*[m]$ is a simplicial Waldhausen category.

9.6. Lemma. *The inclusion* $\mathcal{C}^*(m) \hookrightarrow \mathcal{C}^*[m]$ *has the approximation property* [Wald2, 1.6].

Proof. The first part, *App1*, holds by definition. For the second part, *App2*, suppose given C in $\mathcal{C}^*(m)$ and E in $\mathcal{C}^*[m]$ and a morphism $x : C \to E$ in $\mathcal{C}^*[m]$. We must find a factorization

$$C \xrightarrow{f} D \xrightarrow{g} E$$

of x such that D is in $\mathcal{C}^*(m)$ and f is a cofibration in $\mathcal{C}^*(m)$ and g is a weak equivalence in $\mathcal{C}^*[m]$. Suppose that $D(s)$, f_s, g_s have already been constructed for all faces $s \subset \Delta^m$ of dimension i. Let $t \subset \Delta^m$ be a face of dimension $i+1$. Let $D_1(t)$ be the colimit of

$$C(t) \leftarrow C(\partial t) \xrightarrow{f} D(\partial t)\,.$$

This may also be described as a colimit of the objects $C(t)$ and $C(s)$, $D(s)$ for faces $s \subset \partial t$. It follows that there is a unique map $u : D_1(t) \to E(t)$ extending $x_t : C(t) \to E(t)$ and the compositions

$$D(s) \xrightarrow{g_s} E(s) \to E(t)$$

for faces $s \subset \partial t$. Let $D(t)$ be the mapping cylinder of u. The cylinder projection is a weak equivalence $g_t : D(t) \to E(t)$ and the inclusion $C(t) \to D_1(t) \to D(t)$ is a cofibration $f_t : C(t) \to D(t)$. Proceed in the same way for all other faces of dimension $i+1$. $\square$

9.7. Corollary. *The inclusion* $\boldsymbol{K}(\mathcal{C}^*(m)) \to \boldsymbol{K}(\mathcal{C}^*[m])$ *is a homotopy equivalence.*

9.8. Corollary. *The inclusion* $\mathcal{HC}^*(m) \to \mathcal{HC}^*[m]$ *is an equivalence of categories.*

We introduce an SW product in $\mathcal{C}^*[m]$ using the formula of 1.5, that is, $C \odot D := \operatorname{holim}_s C(s) \odot D(s)$ where the homotopy limit is taken over the faces of Δ^m.

9.9. Corollary. $\mathcal{C}^*[m]$ *with the SW product defined just above satisfies the axioms of §2.*

Apart from a description of $\boldsymbol{K}(\mathcal{C})^{th\mathbb{Z}/2}$, lemmas 9.3, 9.4 and 9.6 (with corollaries 9.7–9.9) are all we need in order to identify the codomain of Ξ in (✠) with $\Omega^\infty(\boldsymbol{K}(\mathcal{C})^{th\mathbb{Z}/2})$. We turn to a description of the Tate functor $\boldsymbol{X} \mapsto \boldsymbol{X}^{thG}$. For us, $\boldsymbol{X}$ will be a CW–spectrum with an action of the discrete group G by cellular automorphisms, and $\boldsymbol{X}^{thG}$ will be a CW–spectrum. (See the remark after 9.11.) CW–spectra and maps between CW–spectra are defined in [Ad, Part III, §2].

9.10. Definitions.

- For CW–spectra $\boldsymbol{X}$ and $\boldsymbol{Y}$, let $\mathbf{map}(\boldsymbol{X},\boldsymbol{Y})$ be the geometric realization of the simplicial set whose k–simplices are the maps from $\Delta^k_+ \wedge \boldsymbol{X}$ to $\boldsymbol{Y}$. If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are both equipped with an action of the discrete group G by cellular maps, then we have $\mathbf{map}_G(\boldsymbol{X},\boldsymbol{Y}) \subset \mathbf{map}(\boldsymbol{X},\boldsymbol{Y})$, the subspace of G–maps.
- Suppose that $\boldsymbol{X}$ is a CW–spectrum with an action of the discrete group G by cellular maps. Let BG be the standard classifying CW–space for G, with universal cover EG. The spaces

$$\mathbf{map}_G(EG_+ \wedge \boldsymbol{S}^0, \mathbb{S}^n \wedge \boldsymbol{X})$$

for $n \geq 0$ form a CW–Ω–spectrum which we call $\boldsymbol{X}^{hG}$. Note that $\boldsymbol{X}^{hG}$ is always an Ω–spectrum by definition, even though $\boldsymbol{X}$ might not be an Ω–spectrum. We also define $\boldsymbol{X}_{hG} := EG \wedge_G \boldsymbol{X}$.
- Suppose that $\boldsymbol{X}$ is a CW–spectrum with an action of the discrete group G by cellular maps. For each $i \geq 0$ there is a cofibration sequence $EG^i_+ \longrightarrow \mathbb{S}^0 \longrightarrow EG^i * \mathbb{S}^0$ of pointed G–spaces, where $*$ denotes the join and EG^i is the i–skeleton of EG. This leads to a fibration sequence up to homotopy of spectra,

$$\cup_i\,(EG^i_+ \wedge \boldsymbol{X})^{hG} \longrightarrow \boldsymbol{X}^{hG} \longrightarrow \cup_i\,((EG^i * \mathbb{S}^0) \wedge \boldsymbol{X})^{hG}\,.$$

The right–hand term in this sequence is the Tate spectrum $\boldsymbol{X}^{thG}$.

9.11. Properties.

- A map $\boldsymbol{X} \to \boldsymbol{Y}$ of CW–spectra with cellular G–actions which is an ordinary homotopy equivalence induces a homotopy equivalence $\boldsymbol{X}^{thG} \to \boldsymbol{Y}^{thG}$.
- The functor $\boldsymbol{X} \mapsto \boldsymbol{X}^{thG}$ takes homotopy pushout squares of CW–spectra with cellular G–action to homotopy pushout squares of spectra. It takes $*$ to $*$.
- If G is finite and $\boldsymbol{X}$ is an induced G–spectrum, then $\boldsymbol{X}^{thG}$ is contractible.
- For finite G, there is a chain of homotopy equivalences

$$\cup_i\,(EG^i_+ \wedge \boldsymbol{X})^{hG} \;\simeq\; \ldots \;\simeq\; \boldsymbol{X}_{hG}\,.$$

natural in the variable $\boldsymbol{X}$. In other words, the homotopy fiber of $\boldsymbol{X}^{hG} \hookrightarrow \boldsymbol{X}^{thG}$ is related by a chain of natural homotopy equivalences to $\boldsymbol{X}_{hG}$.

Proof. The first two properties listed are obvious. For the last two, we assume that G is finite and use the following fact: there *exist* a functor V from CW–spectra $\boldsymbol{X}$ with cellular G–action to CW–spectra and a natural transformation $\boldsymbol{n} : V(\boldsymbol{X}) \to \boldsymbol{X}^{hG}$, the *norm map*, such that

a) $V(\boldsymbol{X})$ is related through a chain of natural homotopy equivalences to $\boldsymbol{X}_{hG}$;
b) $\boldsymbol{n} : V(\boldsymbol{X}) \to \boldsymbol{X}^{hG}$ is a homotopy equivalence if $\boldsymbol{X}$ is induced as a G–spectrum.

Such a pair $(V, \boldsymbol{n})$ is provided by [GM, 5.10], where $\boldsymbol{n}$ is denoted $\bar{\tau}$ and $V(\boldsymbol{X})$ is informally identified with $\boldsymbol{X}_{hG}$. We use it to set up a commutative square

$$\begin{array}{ccc} \mathrm{hocolim}_i \; V(EG^i_+ \wedge \boldsymbol{X}) & \longrightarrow & V(\boldsymbol{X}) \\ \downarrow \boldsymbol{n} & & \downarrow \boldsymbol{n} \\ \cup_i \, (EG^i_+ \wedge \boldsymbol{X})^{hG} & \longrightarrow & \boldsymbol{X}^{hG} \end{array}$$

where the horizontal arrows are induced by the projections from $EG^i_+ \wedge \boldsymbol{X}$ to $\boldsymbol{X}$. Since $EG^i_+ \wedge \boldsymbol{X}$ has a finite G–invariant filtration by CW–subspectra for which the filtration quotients are induced, the norm map for $EG^i_+ \wedge \boldsymbol{X}$ is a homotopy equivalence by property b) of V and $\boldsymbol{n}$. Therefore the left–hand vertical arrow in the square is a homotopy equivalence. Also, the upper horizontal arrow is a homotopy equivalence by property a) of V and $\boldsymbol{n}$. As a result we can identify the lower horizontal arrow with the right–hand vertical arrow (the norm map), at least in the homotopy category. The last two properties listed in 9.11 are now restatements of properties a) and b). $\square$

Remark. An early version of the Tate construction appears in [WW2]. More conceptual versions appear in [AdCoDw] and [GM]. The norm map plays a central role in [AdCoDw], but unfortunately property b) which we used in the proof of 9.11 is not explicitly stated. In the terminology of [GM], the Tate construction is a functor from suitable G–spectra to suitable G–spectra, and our Tate spectrum is the fixed point spectrum (in the equivariant sense) of theirs.

In [AdCoDw] and [GM], the group G is a compact Lie group, so that our account above is less general in one respect and more general in another respect (since we allow *infinite* discrete groups). Allowing infinite groups in this context is an idea going back to Pierre Vogel. However, we only need the case $G = \mathbb{Z}/2$.

We return to our business: understanding the codomain of Ξ in (✠).

9.12. Theorem. *The following inclusions of spectra are homotopy equivalences:*

$$\big|\, [m] \mapsto \boldsymbol{K}(\mathcal{C}^*(m))^{h\mathbb{Z}/2} \,\big| \quad \hookrightarrow \quad \big|\, [m] \mapsto \boldsymbol{K}(\mathcal{C}^*(m))^{th\mathbb{Z}/2} \,\big| \quad \hookleftarrow \quad \boldsymbol{K}(\mathcal{C})^{th\mathbb{Z}/2}\,.$$

Proof. Abbreviate $\boldsymbol{\kappa}(m) = \boldsymbol{K}(\mathcal{C}^*(m))$. By 9.4, the homotopy fiber of the 0–th vertex operator from $\boldsymbol{\kappa}(m)$ to $\boldsymbol{\kappa}(0)$ is induced as a spectrum with $\mathbb{Z}/2$–action. Since the Tate functor $\boldsymbol{X} \mapsto \boldsymbol{X}^{th\mathbb{Z}/2}$ respects fibration sequences up to homotopy and annihilates induced spectra, the 0–th vertex operator $\boldsymbol{\kappa}(m)^{th\mathbb{Z}/2} \to \boldsymbol{\kappa}(0)^{th\mathbb{Z}/2}$ is a homotopy equivalence for every m. Therefore all face operators in the Δ–spectrum $[m] \mapsto \boldsymbol{\kappa}(m)^{th\mathbb{Z}/2}$ are homotopy equivalences, so that the inclusion of $\boldsymbol{\kappa}(0)^{th\mathbb{Z}/2}$ in $|\,[m] \mapsto \boldsymbol{\kappa}(m)^{th\mathbb{Z}/2}\,|$ is a homotopy equivalence. Next we look at the cofibration sequence

$$\big|\,[m] \mapsto \boldsymbol{\kappa}(m)^{h\mathbb{Z}/2}\,\big| \;\to\; \big|\,[m] \mapsto \boldsymbol{\kappa}(m)^{th\mathbb{Z}/2}\,\big| \;\to\; \big|\,[m] \mapsto \boldsymbol{\kappa}(m)^{th\mathbb{Z}/2}/\boldsymbol{\kappa}(m)^{h\mathbb{Z}/2}\,\big|\,.$$

By the last property in 9.11, its last term is homotopy equivalent to the geometric realization of $[m] \mapsto \mathbb{S}^1 \wedge \boldsymbol{\kappa}(m)_{h\mathbb{Z}/2}$. This simplifies to

$$\mathbb{S}^1 \wedge \big|\,[m] \mapsto \boldsymbol{\kappa}(m)\,\big|_{h\mathbb{Z}/2}$$

which is contractible by 9.3. $\square$

Obviously what we really want is an infinite loop space version of 9.12, and so what we need is a lemma stating that under certain conditions, Ω^∞ commutes with geometric realization. The following is enough.

9.13. Lemma [Wald1, Lemma 5.2], [May, Thm. 12.7]**.** *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a diagram of based simplicial CW–spaces. Suppose that*

- *$g_m f_m : X_m \to Z_m$ is the zero map for each $m \geq 0$;*
- *$X_m \to Y_m \to Z_m$ is a fibration sequence up to homotopy for each $m \geq 0$;*
- *Z_m is connected for each $m \geq 0$.*

Then $|X| \xrightarrow{|f|} |Y| \xrightarrow{|g|} |Z|$ is a fibration sequence up to homotopy.

Remark. Let $[m] \mapsto \boldsymbol{X}[m]$ be a simplicial CW–spectrum such that $\boldsymbol{X}[m]$ is 0–connected for all $m \geq 0$. Let $\boldsymbol{X}'[m]$ be a functorial Ω–spectrification of $\boldsymbol{X}[m]$. Lemma 9.13 shows that the geometric realization of $[m] \mapsto \boldsymbol{X}'[m]$ is again a CW–Ω–spectrum. In particular:

$$\Omega^\infty|\boldsymbol{X}| \simeq \Omega^\infty|\boldsymbol{X}'| \simeq |\Omega^\infty \boldsymbol{X}'| \simeq |\Omega^\infty \boldsymbol{X}|\,.$$

9.14. Theorem/Summary. *All arrows in the following diagram of spaces are homotopy equivalences:*

$$\begin{array}{c}
\big|\,[m] \mapsto K(\mathcal{C}^*(m))^{h\mathbb{Z}/2}\,\big| \\
\big\downarrow{=} \\
\big|\,[m] \mapsto \Omega^\infty(\boldsymbol{K}(\mathcal{C}^*(m))^{h\mathbb{Z}/2})\,\big| \\
\big\downarrow{\subset} \\
\big|\,[m] \mapsto \Omega^\infty(\boldsymbol{K}(\mathcal{C}^*(m))^{th\mathbb{Z}/2})\,\big| \\
\big\uparrow{\subset} \\
\Omega^\infty(\boldsymbol{K}(\mathcal{C})^{th\mathbb{Z}/2})\,.
\end{array}$$

Proof. Abbreviate $\boldsymbol{\kappa}(m) = \boldsymbol{K}(\mathcal{C}^*(m))$ and $\boldsymbol{\kappa}[m] = \boldsymbol{\kappa}(\mathcal{C}^*[m])$. The last arrow in the diagram is a homotopy equivalence since all face operators in the Δ–space $m \mapsto \Omega^\infty(\boldsymbol{K}(\mathcal{C}^*(m))^{th\mathbb{Z}/2})$ are homotopy equivalences (see proof of 9.12). Using 9.6–9.9, we can replace the middle arrow by the inclusion of geometric realizations of *simplicial* spaces

$$\big|\,[m] \mapsto \Omega^\infty(\boldsymbol{\kappa}[m]^{h\mathbb{Z}/2})\,\big| \quad \hookrightarrow \quad \big|\,\Omega^\infty(\boldsymbol{\kappa}[m]^{th\mathbb{Z}/2})\,\big|\,.$$

By 9.13, the diagram

$$\begin{array}{c} \big|\,[m] \mapsto \Omega^\infty(\boldsymbol{\kappa}[m]^{h\mathbb{Z}/2})\,\big| \\ \big\downarrow \subset \\ \big|\,[m] \mapsto \Omega^\infty(\boldsymbol{\kappa}[m]^{th\mathbb{Z}/2})\,\big| \\ \big\downarrow \\ \big|\,[m] \mapsto \Omega^\infty(\boldsymbol{\kappa}[m]^{th\mathbb{Z}/2}/\boldsymbol{\kappa}[m]^{h\mathbb{Z}/2})\,\big| \end{array}$$

is a fibration sequence up to homotopy, and its last term is homotopy equivalent to Ω^∞ of the realization of $[m] \mapsto \boldsymbol{\kappa}[m]^{th\mathbb{Z}/2}/\boldsymbol{\kappa}[m]^{h\mathbb{Z}/2}$, which is contractible by 9.12. $\square$

Summarizing, 9.14 is our license to write

$$\Xi : L^\bullet(\mathcal{C}) \longrightarrow \Omega^\infty(\boldsymbol{K}(\mathcal{C})^{th\mathbb{Z}/2})\,.$$

10. Stable SW products

10.1. Definition (compare 1.1)**.** By a *stable SW product* on a Waldhausen category $\mathcal{C}$ we shall mean a functor

$$(C, D) \mapsto C \odot_\bullet D$$

from $\mathcal{C} \times \mathcal{C}$ to the category of Ω–spectra (details below) which takes pairs of weak equivalences to homotopy equivalences, and which is *symmetric* and *bilinear* in the following sense.

- *Symmetry* means that the functor comes with an isomorphism $\tau : C \odot_\bullet D \cong D \odot_\bullet C$, natural in both variables, whose square is the identity on $C \odot_\bullet D$.
- *Bilinearity* means (in the presence of symmetry) that, for fixed but arbitrary D, the functor $C \mapsto C \odot_\bullet D$ takes any cofiber square in $\mathcal{C}$ to a homotopy pullback square of Ω–spectra. Bilinearity also means that $* \odot D$ is contractible.

In 10.1, an Ω–spectrum is a collection of based spaces E_m for $m \in \mathbb{Z}$, together with based maps $\varepsilon_m : E_m \to \Omega E_{m+1}$, for $m \in \mathbb{Z}$. (To be even more precise, Σ means smash product on the left with $[-1,+1]/\{-1,+1\}$, and Ω is right adjoint to Σ.) A morphism from an Ω–spectrum $\{E_m\}$ to another, say $\{E'_m\}$, is a sequence of maps $f_m : E_m \to E'_m$ such that $\Omega f_{m+1}\varepsilon_m = \varepsilon'_m f_m$ for all $m \in \mathbb{Z}$. The morphism is a *homotopy equivalence* if each f_m is a based homotopy equivalence.

The m–th term of the Ω–spectrum $C \odot_\bullet D$ in 10.1 will be denoted by $C \odot_m D$.

10.2. Observation. *Let $\odot_\bullet$ be a stable SW product on $\mathcal{C}$. Then for each $m \in \mathbb{Z}$, the functor $(C, D) \mapsto C \odot_m D$ is an (unstable) SW product on $\mathcal{C}$.* □

10.3. Proposition. *Suppose that the SW product $\odot_m$ satisfies the axioms of §2 for $m = 0$. Then $\odot_m$ satisfies the axioms of §2 for all $m \in \mathbb{Z}$.*

Proof. We have $\pi_0(C \odot_m D) \cong \pi_1(\Sigma C \odot_m D) \cong \pi_0(\Sigma C \odot_{m-1} D)$, showing that $D \mapsto \pi_0(C \odot_m D)$ is a co–representable on $\mathcal{HC}$ if and only if $D \mapsto \pi_0(\Sigma C \odot_{m-1} D)$ is co–representable. Therefore 2.4 holds for $\odot_m$ if and only if it holds for $\odot_{m-1}$.

Given C in $\mathcal{HC}$ and $m \in \mathbb{Z}$, write $T_m C$ for the object (unique up to unique isomorphism) which co–represents $D \mapsto \pi_0(C \odot_m D)$. Then

$$[T_m C, D] \cong \pi_0(C \odot_m D) \cong \pi_0(\Sigma C \odot_{m-1} D) \cong [T_{m-1}\Sigma C, D]$$

which shows that $T_m C \cong T_{m-1}\Sigma C$. Therefore $T_m : \mathcal{HC}^{\mathrm{op}} \to \mathcal{HC}$ is an equivalence of categories if and only if T_{m-1} is an equivalence. In other words (see remark after 2.5), axiom 2.5 holds for $\odot_m$ if and only if it holds for $\odot_{m-1}$. □

10.4. Proposition. *Suppose that the Waldhausen category $\mathcal{C}$ has a cylinder functor and an SW product $\odot$ as in 1.1. There exists another SW product $\odot_1$ on $\mathcal{C}$ and a natural transformation $C \odot D \longrightarrow \Omega(C \odot_1 D)$ which is a based homotopy equivalence and respects symmetry. Consequently $\odot$ is the zero–th term of a stable SW product $\odot_\bullet$ on $\mathcal{C}$.*

Proof. Given C and D in $\mathcal{C}$ let $X(C, D)$ be the reduced homotopy colimit of the commutative diagram of based spaces

$$(\bullet) \qquad \begin{array}{ccccc} * \odot * & \longrightarrow & \Sigma C \odot * & \longleftarrow & \mathrm{cone}(C) \odot * \\ \downarrow & & \downarrow & & \downarrow \\ * \odot \Sigma D & \longrightarrow & \Sigma C \odot \Sigma D & \longleftarrow & \mathrm{cone}(C) \odot \Sigma D \\ \uparrow & & \uparrow & & \uparrow \\ * \odot \mathrm{cone}(D) & \longrightarrow & \Sigma C \odot \mathrm{cone}(D) & \longleftarrow & \mathrm{cone}(C) \odot \mathrm{cone}(D) \end{array}$$

where all maps are induced by $* \to \Sigma C$, $\mathrm{cone}(C) \to \Sigma C$, $* \to \Sigma D$, $\mathrm{cone}(D) \to \Sigma D$. Let $Y(C,D) \subset X(C,D)$ be the reduced homotopy colimit of the smaller diagram obtained from the above by deleting the term in the center, $\Sigma C \odot \Sigma D$, and all arrows touching it. Since $Y(C,D)$ is a *reduced* homotopy colimit of contractible spaces, it is homotopy equivalent to the reduced homotopy colimit of the corresponding diagram with a point at each vertex. Therefore $Y(C,D)$ is contractible and the inclusion $\Sigma C \odot \Sigma D \to X(C,D)$ is a homotopy equivalence, so that

$$X(C,D)/Y(C,D) \simeq \Sigma C \odot \Sigma D\,.$$

The symmetry τ of $\odot$ leads to an involutory natural isomorphism

$$X(C,D)/Y(C,D) \xrightarrow{u} X(D,C)/Y(D,C)$$

which involves a diagram flip. We define $C \odot_1 D := \Omega(X(C,D)/Y(C,D))$. The symmetry $\tau : C \odot_1 D \to D \odot_1 C$ is given by $f \mapsto ufv$, where f denotes a based map from $[-1,+1]/\{-1,+1\}$ to $X(C,D)/Y(C,D)$ and $v : [-1,+1] \to [-1,+1]$ is the reflection at 0.

Let $V(C,D)$ be the reduced homotopy colimit of the diagram of based spaces

$$(\bullet\bullet)\qquad \begin{array}{ccccc} C\odot D & \xrightarrow{=} & C\odot D & \xleftarrow{=} & C\odot D \\ \downarrow{=} & & \downarrow{=} & & \downarrow{=} \\ C\odot D & \xrightarrow{=} & C\odot D & \xleftarrow{=} & C\odot D \\ \uparrow{=} & & \uparrow{=} & & \uparrow{=} \\ C\odot D & \xrightarrow{=} & C\odot D & \xleftarrow{=} & C\odot D \end{array}$$

and let $W(C,D) \subset V(C,D)$ be the reduced homotopy colimit of the subdiagram obtained by deleting the term in the middle and all arrows touching it. Then clearly

$$V(C,D)/W(C,D) \cong (\Psi/\partial\Psi) \wedge (C \odot D)$$

where Ψ is the classifying space of the indexing category (with 9 objects) we are using, and $\partial\Psi$ is the classifying space of the subcategory obtained by deleting the terminal object.

We want to produce a map $\jmath : V(C,D)/W(C,D) \to X(C,D)/Y(C,D)$ with good symmetry properties. To do so we use the cofibrations $C \rightarrowtail \mathrm{cone}(C)$, $D \rightarrowtail \mathrm{cone}(D)$ and the zero maps $C \to *$, $C \to \Sigma C$, $D \to *$, $D \to \Sigma D$ to get a map from diagram $(\bullet\bullet)$ to diagram $(\bullet)$. Passage to reduced homotopy colimits then gives the map $\jmath$ we want. By inspection, the following diagram commutes:

$$\begin{array}{ccc} X(C,D)/Y(C,D) & \xrightarrow{\cong} & X(D,C)/Y(D,C) \\ \uparrow{\jmath} & & \uparrow{\jmath} \\ V(C,D)/W(C,D) & & V(D,C)/W(D,C) \\ \uparrow{\cong} & & \uparrow{\cong} \\ (\Psi/\partial\Psi)\wedge(C\odot D) & \xrightarrow{\phi\wedge\tau} & (\Psi/\partial\Psi)\wedge(D\odot C) \end{array}$$

where $\phi : \Psi/\partial\Psi \to \Psi/\partial\Psi$ is the involution induced by the diagram flip mentioned earlier. Now Ψ as a space with $\mathbb{Z}/2$–action is homeomorphic to $[-1,+1]\times[-1,+1]$, with $\mathbb{Z}/2$ acting by permutation of the factors, and also to $[-1,+1]\times[-1,+1]$ with $\mathbb{Z}/2$ acting by reflection v on the first factor and trivially on the second factor. Using this second homeomorphism, we can write $\jmath$ in the form

$$\Sigma\Sigma(C\odot D)\to X(C,D)/Y(C,D)$$

with adjoint $C\odot D\to\Omega\Omega(X(C,D)/Y(C,D)) = \Omega(C\odot_1 D)$. So we have our natural map from $C\odot D$ to $\Omega(C\odot_1 D)$. It is a homotopy equivalence and has all the symmetry properties we need. $\square$

11. Quadratic L–theory, and Ξ revisited

Suppose again that $\mathcal{C}$ is a Waldhausen category equipped with an SW product $\odot$, and that $\mathcal{C}$ and $\odot$ satisfy the axioms of §2. We will use the stabilization $\odot_\bullet$ of $\odot=\odot_0$ given by 10.4 to improve on §9 in two respects. First, the space $L^\bullet(\mathcal{C})$ (defined using $\odot_0$) turns out to be an infinite loop space, and the map Ξ from it to $\Omega^\infty(\boldsymbol{K}(\mathcal{C})^{th\mathbb{Z}/2})$ is a map of infinite loop spaces. Second, we can define a *quadratic* L–theory space $L_\bullet(\mathcal{C})$, whose realization is again an infinite loop space, and a symmetrization map $L_\bullet(\mathcal{C})\to L^\bullet(\mathcal{C})$, which is a map of infinite loop spaces. Usually $L_\bullet(\mathcal{C})$ has greater geometric significance than $L_\bullet(\mathcal{C},0)$, and one tends to be interested in the composition

$$L_\bullet(\mathcal{C})\longrightarrow L^\bullet(\mathcal{C})\xrightarrow{\Xi}\Omega^\infty(\boldsymbol{K}(\mathcal{C})^{th\mathbb{Z}/2})$$

which has enormous geometric significance.

It will be necessary to handle several SW products at the same time, so we will for example write $L^\bullet(\mathcal{C},\odot_i)$ to mean $L^\bullet(\mathcal{C})$, constructed using the SW product $\odot_i$.

Notation, preliminaries. For a based Δ–space X let $\bar\Sigma X$ be the Δ–space given by $n\mapsto X(n-1)$ for $n>0$, and $0\mapsto *$. For $i\ge 0$ the $(i+1)$–th face operator $\bar\Sigma X(n+1)\to\bar\Sigma X(n)$ agrees with the i–th face operator $X(n)\to X(n-1)$, and the 0–th face operator is zero. Compare [Cu]. Note that every simplex in $\bar\Sigma X$ has its 0–th vertex and 0–th face (opposite the 0–th vertex) at $*$. There is a continuous bijection $|\bar\Sigma X|\cong\Sigma|X|$ provided geometric realizations are taken in the based sense ; for example, $|X| := (\coprod_n X(n)\wedge\Delta^n_+)/\sim$. If each $X(n)$ is a based CW–space and the face operators $X(n)\to X(m)$ are cellular, then $|\bar\Sigma X|\cong\Sigma|X|$.

In a similar spirit, we introduce the full subcategory of $\mathcal{C}^*(m+1)$ consisting of all objects B for which $B(s)=*$ if s is the 0–th vertex or the 0–th face. It is isomorphic to $\mathcal{C}^*(m)$, and so we have the inclusion functor $\sigma:\mathcal{C}^*(m)\to\mathcal{C}^*(m+1)$. Explicitly: σ takes B in $\mathcal{C}^*(m)$ to $\sigma B\in\mathcal{C}^*(m+1)$ defined by

$$(\sigma B)(t)=\begin{cases} * & \text{if } t \text{ is the 0–th vertex}\\ * & \text{if } t \text{ is contained in the 0–th face}\\ B(t^\flat) & \text{otherwise}\end{cases}$$

where $t^\flat \subset \Delta^m$ has vertices $x_1 - 1, \dots, x_r - 1$ if $t \subset \Delta^{m+1}$ has vertices $0, x_1, \dots, x_r$. The functor $\sigma : \mathcal{C}^*(m) \to \mathcal{C}^*(m+1)$ intertwines the SW products on $\odot_{i-1}$ on $\mathcal{C}^*(m)$ and $\odot_i$ on $\mathcal{C}^*(m+1)$ (see 1.5) in the following way. There is a forgetful fibration

$$\operatorname*{holim}_t \ (\sigma C)(t) \odot_i (\sigma D)(t) \longrightarrow \operatorname*{holim}_s \ (\sigma C)(s) \odot_i (\sigma D)(s)$$

where t runs through all faces but s only runs through those contained in the 0–th face or equal to the 0–th vertex. The codomain of this fibration is contractible since it is a homotopy limit of contractible spaces. The fiber over the base point is homeomorphic to $\Omega(C \odot_i D)$ by inspection. The inclusion of the fiber is therefore a homotopy equivalence $\Omega(C \odot_i D) \to (\sigma C) \odot_i (\sigma D)$. We compose

$$C \odot_{i-1} D \to \Omega(C \odot_i D) \xrightarrow{\subset} (\sigma C) \odot_i (\sigma D)$$

to get our intertwining map. (Occasionally we denote it by σ also.) It follows from 8.8 that the intertwining map takes nondegenerate components to nondegenerate components ; moreover it is clearly compatible with the symmetries τ of 1.1.

11.1. Proposition. *The spaces $L^\bullet(\mathcal{C}, \odot_i)$ for $i \in \mathbb{Z}$ form an Ω–spectrum.*

Proof. For the purpose of this proof, think of $L^\bullet(\mathcal{C}, \odot_i)$ as an (unrealized) Δ–set. Map $\bar{\Sigma} L^\bullet(\mathcal{C}, \odot_{i-1})$ to $L^\bullet(\mathcal{C}, \odot_i)$ by sending the $(m+1)$–simplex (C, ϕ) to the $(m+1)$–simplex $(\sigma C, \sigma\phi)$. Here (C, ϕ) is a 0–dimensional symmetric Poincaré object in $\mathcal{C}^*(m)$ with respect to $\odot_{i-1}$. By inspection, the induced homomorphism of homotopy groups from $\pi_m L^\bullet(\mathcal{C}, \odot_{i-1})$ to $\pi_{m+1} L^\bullet(\mathcal{C}, \odot_i)$ is an isomorphism for $m \geq 0$. $\square$

Next, we repeat the exercise with $\Omega^\infty(\boldsymbol{K}(\mathcal{C})^{th\mathbb{Z}/2})$ instead of $L^\bullet(\mathcal{C})$. Here again it will be important to declare the SW product, so we write $\boldsymbol{K}(\mathcal{C}, \odot_i)$ instead of just $\boldsymbol{K}(\mathcal{C})$, to indicate that $\mathbb{Z}/2$ acts by an action constructed using the SW product $\odot_i$. Our model for $\Omega^\infty(\boldsymbol{K}(\mathcal{C}, \odot_i)^{th\mathbb{Z}/2})$ is the geometric realization of the Δ–space $[m] \mapsto K(\mathcal{C}^*(m), \odot_i)^{h\mathbb{Z}/2}$, as explained in 9.14.

11.2. Proposition. *The spaces $|\,[m] \mapsto K(\mathcal{C}^*(m), \odot_i)^{h\mathbb{Z}/2}\,|$ for $i \in \mathbb{Z}$ form an Ω–spectrum.*

Proof. Much as in the proof of 11.1, the functors σ induce a map of Δ–spaces from $\bar{\Sigma}$ of $[m] \mapsto K(\mathcal{C}^*(m), \odot_{i-1})^{h\mathbb{Z}/2}$ to $[m] \mapsto K(\mathcal{C}^*(m), \odot_i)^{h\mathbb{Z}/2}$. This gives the spectrum structure, but it does not show that the spectrum so obtained is an Ω–spectrum. Using 9.14, we reduce to the statement that the spaces

$$|\,[m] \mapsto \Omega^\infty(\boldsymbol{K}(\mathcal{C}^*(m), \odot_i)^{th\mathbb{Z}/2})\,|$$

for $i \in \mathbb{Z}$ form an Ω–spectrum (with structure maps induced by the functors σ as before). Since all face operators in the Δ–spectrum $[m] \mapsto \boldsymbol{K}(\mathcal{C}^*(m), \odot_i)^{th\mathbb{Z}/2}$ are homotopy equivalences, it becomes irrelevant whether we apply Ω^∞ before or after realization. So all we have to show is that for fixed $i \in \mathbb{Z}$, the map of Δ–spectra

$$\begin{array}{c} \bar{\Sigma}\big(\,[m] \mapsto \boldsymbol{K}(\mathcal{C}^*(m), \odot_{i-1})^{th\mathbb{Z}/2}\,\big) \\ \Big\downarrow \\ [m] \mapsto \boldsymbol{K}(\mathcal{C}^*(m), \odot_i)^{th\mathbb{Z}/2} \end{array}$$

induced by the functors σ turns into a homotopy equivalence. For this purpose we abbreviate $\boldsymbol{X}(m) := \boldsymbol{K}(\mathcal{C}^*(m), \odot_{i-1})^{th\mathbb{Z}/2}$ and $\boldsymbol{Y}(m) := \boldsymbol{K}(\mathcal{C}^*(m), \odot_i)^{th\mathbb{Z}/2}$. We have to show that a certain map $|\bar{\Sigma}\boldsymbol{X}| \to |\boldsymbol{Y}|$ is a homotopy equivalence. Since all face operators in $\boldsymbol{Y}$ are homotopy equivalences, the canonical map

$$\mathrm{hocolim}\,[\,\boldsymbol{Y}(0) \xleftarrow{d_0} \boldsymbol{Y}(1) \xrightarrow{d_1} \boldsymbol{Y}(0)\,] \quad \longrightarrow \quad |\boldsymbol{Y}|$$

is a homotopy equivalence. There is an analogous map for $\bar{\Sigma}X$; it has the form

$$\mathrm{hocolim}\,[\,* \leftarrow \boldsymbol{X}(0) \rightarrow *\,] \quad \longrightarrow \quad |\bar{\Sigma}\boldsymbol{X}|$$

and it is also a homotopy equivalence because all face operators in $\boldsymbol{X}$ are homotopy equivalences. Now it only remains to show that the map between homotopy colimits of the rows in the commutative diagram

$$\begin{array}{ccccc} * & \longleftarrow & \boldsymbol{X}(0) & \longrightarrow & * \\ \Big\downarrow & & \Big\downarrow{\scriptstyle\sigma} & & \Big\downarrow \\ \boldsymbol{Y}(0) & \xleftarrow{\;d_0\;} & \boldsymbol{Y}(1) & \xrightarrow{\;d_1\;} & \boldsymbol{Y}(0) \end{array}$$

is a homotopy equivalence, or equivalently, that

$$\boldsymbol{X}(0) \xrightarrow{\sigma} \boldsymbol{Y}(1) \xrightarrow{(d_0,d_1)} \boldsymbol{Y}(0) \times \boldsymbol{Y}(1)$$

is a fibration sequence up to homotopy. Since the Tate functor respects fibration sequences up to homotopy of spectra, it is more than enough to show that the K–theory functor turns

$$\mathcal{C}^*(0) \;\xrightarrow{\sigma}\; \mathcal{C}^*(1) \;\xrightarrow{(d_0,d_1)}\; \mathcal{C}^*(0) \times \mathcal{C}^*(0)$$

into a fibration sequence (up to homotopy) of spectra. But this is clear from 8.11. $\square$

Remark. The proof of 11.2 also shows that the spectrum described in 11.2 is homotopy equivalent to $\boldsymbol{K}(\mathcal{C})^{th\mathbb{Z}/2}$.

For each $i \in \mathbb{Z}$, the constructions of §9 give us a map Ξ from $L^\bullet(\mathcal{C}, \odot_i)$ of 11.1 to the space $|\,[m] \mapsto K(\mathcal{C}^*(m), \odot_i)^{h\mathbb{Z}/2}|$ of 11.2. As i varies, these maps constitute a map of spectra which we write in the form

$$\Xi : \boldsymbol{L}^\bullet(\mathcal{C}) \to \boldsymbol{K}(\mathcal{C})^{th\mathbb{Z}/2}\,.$$

We turn to the construction of $L_\bullet(\mathcal{C})$, the quadratic L–theory space. It will be necessary to expand the Ω–spectrum $C \odot_\bullet D$ into an Ω–*bispectrum* $C \odot_\bullet^\bullet D$, with terms $C \odot_n^m D$ so that $C \odot_n^0 D = C \odot_n D$. One way to achieve this, naturally in C and D, is to define $C \odot_n^m D$ for $m > 0$ inductively as the n–th term in a functorial Ω–spectrification of $\mathbb{S}^1 \wedge (C \odot_\bullet^{m-1} D)$.

Given C in $\mathcal{C}$, a convenient model or replacement for $\Omega^\infty((C \odot_0^\bullet C)_{h\mathbb{Z}/2})$ is the union $\cup_i$ hofiber$[X \to X_i]$ where X is the space of $\mathbb{Z}/2$–maps $E\mathbb{Z}/2 \to C \odot_0^0 C$ and X_i is the space of $\mathbb{Z}/2$–maps of spectra

$$E\mathbb{Z}/2_+ \wedge \boldsymbol{S}^0 \to ((E\mathbb{Z}/2)^i * \mathbb{S}^0) \wedge (C \odot_0^\bullet C)\,.$$

Both mapping spaces are to be constructed as geometric realizations of simplicial sets. Correctness of this model or replacement follows from 9.10 and 9.11 ; that is, it has the right homotopy type. It is convenient because it comes with a forgetful map to $(C \odot_0^0 C)^{h\mathbb{Z}/2}$. This is of course the norm map, as a map of infinite loop spaces. — These conventions are understood in the next definition.

11.3. Definition/Notation. A 0–dimensional *quadratic Poincaré object* in $\mathcal{C}$ is an object C in $\mathcal{C}$, together with a vertex ψ in $\Omega^\infty((C \odot_0^\bullet C)_{h\mathbb{Z}/2})$ whose image under the composition

$$\Omega^\infty((C \odot_0^\bullet C)_{h\mathbb{Z}/2}) \xrightarrow{\text{norm}} (C \odot_0 C)^{h\mathbb{Z}/2} \to C \odot_0 C$$

is in a nondegenerate component. The set of 0–dimensional quadratic Poincaré objects in $\mathcal{C}$ is denoted by $\mathrm{qp}_0(\mathcal{C})$.

11.4. Definition. $L_\bullet(\mathcal{C}) = L_\bullet(\mathcal{C}, \odot_0^\bullet)$ is the geometric realization of the Δ–set given by $[m] \mapsto \mathrm{qp}_0(\mathcal{C}^*(m))$.

Remark. $L_\bullet(\mathcal{C})$ is a *fibrant* Δ–set.

Implicit in definitions 11.3 and 11.4 is a Δ–map $L_\bullet(\mathcal{C}) \to L^\bullet(\mathcal{C})$, given by converting all quadratic Poincaré objects in sight into symmetric ones via the norm map. This is the symmetrization map, and for the usual reasons it is an infinite loop map. For example, a delooping of $L_\bullet(\mathcal{C}) = L_\bullet(\mathcal{C}, \odot_0^\bullet)$ is $L_\bullet(\mathcal{C}, \odot_1^\bullet)$.

12. Naturality

Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor between Waldhausen categories equipped with SW products $\odot_\mathcal{C}$ and $\odot_\mathcal{D}$, respectively. Suppose that the axioms of §2 are satisfied for both $\mathcal{C}$ and $\mathcal{D}$, and that in addition to F we are given a natural transformation

$$\phi : A \odot_\mathcal{C} B \longrightarrow F(A) \odot_\mathcal{D} F(B)$$

commuting with the symmetry operators τ (see 1.1) and taking nondegenerate components to nondegenerate components. Then the map $\boldsymbol{K}(\mathcal{C}) \to \boldsymbol{K}(\mathcal{D})$ induced by F becomes a $\mathbb{Z}/2$–map between spectra with $\mathbb{Z}/2$–action provided the big models of $\boldsymbol{K}(\mathcal{C})$ and $\boldsymbol{K}(\mathcal{D})$ of §7 are used. Also, there are maps of spectra $\boldsymbol{L}^\bullet(\mathcal{C}) \to \boldsymbol{L}^\bullet(\mathcal{D})$, $\boldsymbol{L}_\bullet(\mathcal{C}) \to \boldsymbol{L}_\bullet(\mathcal{D})$ induced by F and ϕ, and the diagram

$$\begin{array}{ccc}
\boldsymbol{L}_\bullet(\mathcal{C}) & \xrightarrow{(F,\phi)_*} & \boldsymbol{L}_\bullet(\mathcal{D}) \\
{\scriptstyle \text{symmetrization}}\downarrow & & \downarrow{\scriptstyle \text{symmetrization}} \\
\boldsymbol{L}^\bullet(\mathcal{C}) & \xrightarrow{(F,\phi)_*} & \boldsymbol{L}^\bullet(\mathcal{D}) \\
{\scriptstyle \Xi}\downarrow & & \downarrow{\scriptstyle \Xi} \\
\boldsymbol{K}(\mathcal{C})^{th\mathbb{Z}/2} & \xrightarrow{(F,\phi)_*} & \boldsymbol{K}(\mathcal{D})^{th\mathbb{Z}/2}
\end{array}$$

commutes. All of this is straightforward, but often the construction of a pair (F, ϕ) is laborious. We shall discuss two examples in 12.A.

1.A. Examples

1.A.1. Example. We already have the example 1.4, where $\mathcal{C}$ is the category of bounded chain complexes of finitely generated abelian groups, and $C \odot D := (C \otimes D)^\sharp$.

More generally, suppose that $\mathcal{A}$ is an additive category, and let $\mathcal{C}$ be the category of bounded chain complexes in $\mathcal{A}$. A contravariant additive functor $T : \mathcal{A} \to \mathcal{C}$ has a canonical extension $T : \mathcal{C} \to \mathcal{C}$. Following [Ra3], a *chain duality* on $\mathcal{A}$ is a contravariant additive functor $T : \mathcal{A} \to \mathcal{C}$ together with a natural transformation e from $T^2 : \mathcal{A} \to \mathcal{C}$ to the inclusion $\mathcal{A} \to \mathcal{C}$ such that for each object M in $\mathcal{A}$

i) $e_{T(M)} \cdot T(e_M) = \text{id} : T(M) \to T^3(M) \to T(M)$,
ii) $e_M : T^2(M) \longrightarrow M$ is a homotopy equivalence.

Assume now that $\mathcal{A}$ is equipped with a chain duality (T, e). Given objects C, D in $\mathcal{C}$ define $C \otimes_{\mathcal{A}} D := \hom_{\mathcal{A}}(T(C), D)$ (a chain complex of abelian groups). Ranicki shows that there is a canonical involutory isomorphism $\tau : C \otimes_{\mathcal{A}} D \to D \otimes_{\mathcal{A}} C$. It follows that an SW product on $\mathcal{C}$ can be defined by

$$C \odot D := (C \otimes_{\mathcal{A}} D)^\sharp \,.$$

1.A.2. Example. Let $\mathcal{C}$ be the category of based compact CW–spaces (base points are understood to be 0–cells). Morphisms are all cellular maps, the cofibrations are those morphisms which up to CW–isomorphism are inclusions of subcomplexes, and the weak equivalences are the morphisms which are homotopy equivalences. For objects X, Y in $\mathcal{C}$ we let $X \odot Y := \Omega^\infty\Sigma^\infty(X \wedge Y)$.

In more detail: Let $\Omega^n\Sigma^n(X \wedge Y)$ be the geometric realization of the simplicial set whose k–simplices are the based maps

$$(\mathbb{R}^n)^\bullet \wedge \Delta^k_+ \longrightarrow (\mathbb{R}^n)^\bullet \wedge (X \wedge Y)$$

where $(\mathbb{R}^n)^\bullet$ is the one–point compactification of $\mathbb{R}^n$. Let $X \odot Y$ be the colimit of the based spaces $\Omega^n\Sigma^n(X \wedge Y)$. The conditions in 1.1 are easily verified.

1.A.3. Example. Fix a space B. Let $\mathcal{C}$ be the category of retractive relative CW–spaces over B, with finitely many cells relative to B. In other words, an object of $\mathcal{C}$ is a retractive space

$$X \underset{i}{\overset{r}{\rightleftarrows}} B$$

($ri = \mathrm{id}$) where X has the structure of a CW–space relative to B, with finitely many cells. A morphism in $\mathcal{C}$, from $X \rightleftarrows B$ to $Y \rightleftarrows B$, is a map $X \to Y$ rel B which is cellular relative to B, and respects the retractions.

A morphism is a *cofibration* if, up to isomorphism, it is an inclusion of a relative CW–subspace. A morphism is a *weak equivalence* if it becomes an isomorphism in the relative homotopy category of spaces containing B. These definitions are taken from [Wald2].

For $X = (X \rightleftarrows B)$ in $\mathcal{C}$ let $X^? = X \smallsetminus B$. For $X = (X \rightleftarrows B)$ and $Y = (Y \rightleftarrows B)$ in $\mathcal{C}$ let $X \curlywedge Y := Z \cup \{\infty\}$, where $Z = \mathrm{holim}(X^? \to B \leftarrow Y^?)$. The union $Z \cup \{\infty\}$ is to be topologized in such a way that

- Z with the obvious topology is an open subspace
- a set $W \subset X \curlywedge Y$ containing ∞ is a neighborhood of ∞ if and only if the image of its complement under $Z \to X \times Y$ has closure disjoint from $B \times B$.

Points in $X \curlywedge Y$, other than ∞, are triples (x, ω, y) where $x \in X^?$ and $y \in Y^?$ and ω is a path $[-1, 1] \to B$ such that $\omega(-1) = r(x)$ and $\omega(1) = r(y)$. We do *not* claim that $X \curlywedge Y$ is compactly generated Hausdorff. We do *not* claim that the base point in $X \curlywedge Y$ is nondegenerate. Let $X \odot Y := \Omega^\infty\Sigma^\infty(X \curlywedge Y)$, using the conventions of 1.A.2 to make $\Omega^\infty\Sigma^\infty$ precise. Note that $X \odot Y$ is a CW–space.

Remark. The space $X \curlywedge Y$ has a canonical filtration whose q–th stage is the union of $X^i \curlywedge Y^j$ for all i, j with $i + j = q$. Here X^i is the relative i–skeleton of X, still a retractive space over B. Sometimes it is convenient to take this filtration into account and to re–define $\Omega^n\Sigma^n(X \curlywedge Y)$ as the geometric realization of the simplicial set whose k–simplices are *filtration preserving* based maps

$$(\mathbb{R}^n)^\bullet \wedge \Delta^k \to (\mathbb{R}^n)^\bullet \wedge (X \curlywedge Y)$$

where Δ^k has the skeleton filtration. Again, $\Omega^\infty\Sigma^\infty(X \curlywedge Y)$ would be defined as the colimit of the based spaces $\Omega^n\Sigma^n(X \curlywedge Y)$.

Illustration. Suppose that $X = B \amalg \{x\}$ and $Y = B \amalg \{y\}$. That is, both X and Y are spaces obtained from B by adding a disjoint point. Then $X \odot Y$ is homotopy equivalent to $\Omega^\infty\Sigma^\infty(\wp(r(x), r(y))_+)$, where $\wp(r(x), r(y))$ is the space of paths in B from $r(x)$ to $r(y)$, with the compact–open topology. Note that with our definitions, $\Omega^\infty\Sigma^\infty(\wp(r(x), r(y))_+)$ is a CW–space even though B and hence $\wp(r(x), r(y))$ can be pathological.

We now check that $\odot$ in 1.A.3 satisfies the conditions in 1.1. Symmetry is obvious. For the w–invariance, suppose that $f : X \to X'$ is a weak equivalence in $\mathcal{C}$ and Y is another object of $\mathcal{C}$. Then there exists a map $g : X' \to X$ which is relative to B but perhaps not over B, and homotopies $h : gf \simeq \mathrm{id}$, $j : fg \simeq \mathrm{id}$, also relative to B but perhaps not over B. Using g and the homotopies, one shows easily that

$$X \curlywedge Y \xrightarrow{f \curlywedge \mathrm{id}} X' \curlywedge Y$$

is a based homotopy equivalence. Therefore $f \odot \mathrm{id}$ from $X \odot Y$ to $X' \odot Y$ is a homotopy equivalence.

Showing bilinearity is harder. Since we have the w–invariance, it is enough to consider a cofiber square in $\mathcal{C}$ of the form

$$(*) \qquad \begin{array}{ccc} U & \xrightarrow{g} & V \\ \downarrow{\subset} & & \downarrow{\subset} \\ \mathrm{cone}(U) & \longrightarrow & \mathrm{cone}(g)\,. \end{array}$$

where $\mathrm{cone}(U)$ and $\mathrm{cone}(g)$ are defined using the cylinder functor in $\mathcal{C}$. We must show that for any Z in $\mathcal{C}$ the resulting square

$$(**) \qquad \begin{array}{ccc} U \odot Z & \xrightarrow{g \odot \mathrm{id}} & V \odot Z \\ \downarrow & & \downarrow \\ \mathrm{cone}(U) \odot Z & \longrightarrow & \mathrm{cone}(g) \odot Z \end{array}$$

is a homotopy pullback square. Now (**) is obtained from

$$(***) \qquad \begin{array}{ccc} U \curlywedge Z & \longrightarrow & V \curlywedge Z \\ \downarrow & & \downarrow \\ \mathrm{cone}(U) \curlywedge Z & \longrightarrow & \mathrm{cone}(g) \curlywedge Z \end{array}$$

by applying $\Omega^\infty\Sigma^\infty$, and inspection shows that (***) is a pushout square of based spaces, with $\mathrm{cone}(U) \curlywedge Z$ homeomorphic to the reduced cone on $U \curlywedge Z$. So it is enough to show that the functor $\Omega^\infty\Sigma^\infty$ takes squares of based spaces of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ \mathrm{cone}(X) & \longrightarrow & Y \cup_f \mathrm{cone}(X) \end{array}$$

to homotopy pullback squares. An equivalent but more informal way to say the same thing: the stable homotopy groups of the based spaces X, Y and $Y \cup_f \mathrm{cone}(X)$ fit into a long exact sequence. This may seem like a familiar statement. But at this level of generality it is probably not so familiar, since we are working with *reduced* cones and mapping cones, taken in the category of "all" based spaces (including based spaces with degenerate base points). Adams proves it in [Ad, Part III, 3.10].

(Here is some guidance. We need III.3.10 of [Ad] with X and Y equal to suspension spectra of possibly pathological based spaces, and W equal to the sphere spectrum. So we need 3.10 in the case where the domain is a CW–spectrum but the codomains are just *spectra* as in [Ad, Part III, §2]. Since his 3.10 relies on his 3.8, 3.7 and 3.6, these must be interpreted in the same way—domains are CW, codomains need not be.)

For the next examples we introduce control. As a preliminary, let B be a set and let U, V be subsets of $B \times B$. Let

$$\begin{aligned} UV &:= \{(x,z) \in B \times B \mid \exists y \in B \text{ such that } (x,y) \in U, (y,z) \in V\} \\ U^{-1} &:= \{(x,y) \in B \times B \mid (y,x) \in U\}\,. \end{aligned}$$

1.A.4. Definitions. A *control structure* on a set B is a collection $\mathcal{U}$ of subsets of $B \times B$ with the following properties.

- $U, V \in \mathcal{U} \Rightarrow UV \in \mathcal{U}$ and $U \cup V \in \mathcal{U}$.
- $U \in \mathcal{U} \Rightarrow U^{-1} \in \mathcal{U}$.
- $U \subset V \subset B \times B, V \in \mathcal{U} \Rightarrow U \in \mathcal{U}$.
- the diagonal of $B \times B$ is in $\mathcal{U}$.

A subset W of B is *$\mathcal{U}$–bounded* if $W \times W$ belongs to $\mathcal{U}$.

Example (bounded control). Suppose that B is equipped with a metric. Let $\mathcal{U}$ consist of all subsets $U \subset B \times B$ for which $\sup\{d(x,y) \mid (x,y) \in U\} < \infty$. This defines a control structure. The $\mathcal{U}$–bounded subsets of B are exactly the subsets of finite diameter.

Example (continuous control). Suppose that B is an open dense subset of a space C. Let $\mathcal{U}$ consist of all $U \subset B \times B$ with the following property: for any (Moore–Smith) sequence of points (x_α, y_α) in U such that one of the (Moore–Smith) sequences (x_α) and (y_α) in B converges to a point in $C \smallsetminus B$, the other converges to the same point in $C \smallsetminus B$. Again, this defines a control structure on B. The $\mathcal{U}$–bounded subsets of B are the ones whose closure in B agrees with their closure in C.

(A Moore–Smith sequence is a map from a directed set ; if every point in C has a countable neighborhood base, ordinary sequences indexed by $\mathbb{N}$ will do.)

In these examples, B is a topological space and the control structure is *compatible* with the topology in the following sense: Each $U \in \mathcal{U}$ is contained in some $V \in \mathcal{U}$ which is *open* in $B \times B$.

Suppose now that B is a space equipped with a control structure $\mathcal{U}$, compatible with the topology. Let $p : X \to B$ and $q : Y \to B$ be spaces over B. A map $f : X \to Y$ is $\mathcal{U}$*–controlled* (or just *controlled* for short) if the set $\{(p(x), qf(x)) \mid x \in X\}$ belongs to $\mathcal{U}$.

A homotopy of maps from X to Y is *controlled* if it is controlled as a map from $X \times [0,1]$ to Y, where $X \times [0,1]$ is viewed as a space over B in the most obvious way.

Example. Let $\mathcal{U}$ be the control structure on B determined as above by an inclusion $B \to C$, where B is open dense in C. Let $p : X \to B$ and $q : Y \to B$ be spaces over B. Then a map $f : X \to Y$ is $\mathcal{U}$–controlled if and only if the following holds: for every $z \in C \smallsetminus B$ and every neighborhood V of z in C, there exists another neighborhood W of z in C such that $p(x) \in W$ implies $qf(x) \in V$ and $qf(x) \in W$ implies $p(x) \in V$. This is the definition of controlled map used in [ACFP] and [CaPe].

1.A.5. Definition. Let $\mathcal{U}_i$ be a control structure on B_i, for $i = 1, 2$. The *product control structure* $\mathcal{U}_1 \otimes \mathcal{U}_2$ on $B_1 \times B_2$ consists of those subsets of $(B_1 \times B_2) \times (B_1 \times B_2)$ whose images in $B_1 \times B_1$ and $B_2 \times B_2$ under the appropriate projections belong to $\mathcal{U}_1$ and $\mathcal{U}_2$, respectively.

1.A.6. Preliminaries. Suppose that the space B is equipped with a control structure $\mathcal{U}$, compatible with the topology. Let $\mathcal{C}$ be the category of retractive spaces $X \rightleftarrows B$ for which

- X comes with a structure of finite dimensional CW–space relative to B ;
- for every $\mathcal{U}$–bounded $W \subset B$, the number of cells $e \subset X \smallsetminus B$ for which $r(e) \cap W \neq \emptyset$ is finite ;
- the sizes of the cells $e \subset X \smallsetminus B$ are *controlled* in the sense that $\bigcup_e r(e) \times r(e)$ belongs to $\mathcal{U}$, where $r : X \to B$ is the retraction.

The morphisms in $\mathcal{C}$ are the retractive cellular maps. A morphism is a *cofibration* if, up to isomorphism, it is the inclusion of a relative CW–subspace. A morphism

$$(X \rightleftarrows B) \xrightarrow{f} (Y \rightleftarrows B)$$

is a *weak equivalence* if there exists a map $g : Y \to X$ (relative to B but not necessarily over B) and controlled homotopies $h : gf \simeq \mathrm{id}_X$, $k : fg \simeq \mathrm{id}_Y$ relative to B.

It is not completely obvious that $\mathcal{C}$ satisfies the axioms of a Waldhausen category. Axioms *Cof1, Cof2, Cof3* and *Weq1* are obviously satisfied, but *Weq2* is not quite so easy. However, *Weq2* for $\mathcal{C}$ is a direct consequence of the following *controlled homotopy extension property* (CHEP) which can be established by induction over skeletons:

> Let $Y \rightarrowtail Z$ be a cofibration in $\mathcal{C}$ (we assume $Y \subset Z$). Let W be any other retractive space over B. Let $f : Z \to W$ be a retractive map and let $\{h_t : Y \to W \mid 0 \leq t \leq 1\}$ be a controlled homotopy of maps relative to B, with $h_0 = f$ on Y. Then there exists a controlled homotopy $\{H_t : Z \to W\}$ of maps relative to B, extending $\{h_t\}$, with $H_0 = f$.

Note that if B is equipped with the trivial control structure, so that $\mathcal{U}$ consists of all subsets of $B \times B$, then $\mathcal{C}$ in 1.A.6 is identical with $\mathcal{C}$ in 1.A.3. In this connection, note also:

- The control structure determined by a metric on a space B is trivial if and only if B has finite diameter.
- Any control structure on a compact space B which is compatible with the topology is trivial (exercise).

1.A.7. Example. We continue to work in $\mathcal{C}$ of 1.A.6, but we assume also that B has a countable base for its topology. For X, Y in $\mathcal{C}$ and open $U \subset B \times B$, $U \in \mathcal{U}$, let

$$X \curlywedge_U Y := Z \cup \{\infty\}$$

where Z is the space of triples (x, ω, y) with $x \in X^?$ and $y \in Y^?$ and $\omega : [-1, 1] \to B$ such that $\omega(-1) = r(x)$, $\omega(1) = r(y)$ and $\mathrm{im}(\omega) \times \mathrm{im}(\omega) \subset U$. The union $Z \cup \{\infty\}$ is to be topologized in such a way that

- Z with the obvious topology is an open subspace
- a set $W \subset X \curlywedge_U Y$ containing ∞ is a neighborhood of ∞ if and only if the image of its complement under $Z \to X \times Y$ has closure disjoint from $B \times B$, and the image of its complement under $Z \to X \to B$ is $\mathcal{U}$–bounded.

We do *not* claim that $X \curlywedge_U Y$ is compactly generated Hausdorff. However, using the conventions of 1.A.2 and 1.A.3 to make sense of $\Omega^\infty\Sigma^\infty$, we find that

$$X \odot Y := \underset{U}{\mathrm{hocolim}}\ \Omega^\infty\Sigma^\infty(X \curlywedge_U Y)$$

is a CW–space and that $(X, Y) \mapsto X \odot Y$ is an SW product on $\mathcal{C}$.

Remark. Suppose that $V \subset B$ is $\mathcal{U}$–bounded and $U \in \mathcal{U}$. Then the set of all $b \in B$ for which there exists a $b_1 \in B$ with $(b, b_1) \in U$ is also $\mathcal{U}$–bounded. This shows that, for a subset $W \subset X \curlywedge_U Y$ containing ∞, the complement of W has $\mathcal{U}$–bounded image under $Z \to X \to B$ if and only if it has $\mathcal{U}$–bounded image under $Z \to Y \to B$. Consequently $X \odot Y \cong Y \odot X$.

Remark. All we care about when we define the neighborhoods of the base point ∞ in $X \curlywedge_U Y$ is this: what is a based continuous map f from $(\mathbb{R}^n)^\bullet \wedge \Delta^k_+$ to $(\mathbb{R}^n)^\bullet \wedge (X \curlywedge_U Y)$ going to be ? Such an f will of course be fully described by its restriction to $f^{-1}(\mathbb{R}^n \times Z)$, an open subset of $\mathbb{R}^n \times \Delta^k$. The restriction is a continuous map from $f^{-1}(\mathbb{R}^n \times Z)$ to $\mathbb{R}^n \times Z$, subject to a condition: $f^{-1}(C)$ is compact whenever $C \subset \mathbb{R}^n \times Z$ is the complement of an open neighborhood of the base point in $(\mathbb{R}^n)^\bullet \wedge (X \curlywedge_U Y)$.

Illustration. Suppose that $X = B \amalg S$ and $Y = B \amalg T$ where S and T are discrete (therefore countable). Then $X \odot Y$ is homotopy equivalent to

$$\underset{U}{\mathrm{hocolim}} \prod_{(s,t) \in S \times T} \Omega^\infty\Sigma^\infty(\wp_U(r(s), r(t))_+)$$

where the hocolim is taken over those $U \in \mathcal{U}$ which are open in $B \times B$, and $\wp_U(r(s), r(t))$ is the space of paths $\omega : [0,1] \to B$ from $r(s)$ to $r(t)$ such that $\mathrm{im}(\omega) \times \mathrm{im}(\omega) \subset U$.

To show that $\odot$ in 1.A.7 satisfies conditions 1.1, we proceed as in 1.A.3. In particular, the observation that the functor $X \mapsto X \curlywedge_U Y$ from $\mathcal{C}$ to based spaces respects mapping cylinders and mapping cones is crucial.

1.A.8. Variation. To make 1.A.6 and 1.A.7 more user–friendly we introduce a larger category $\mathcal{D} \supset \mathcal{C}$ (with $\mathcal{C}$ as in 1.A.6) whose objects are certain retractive spaces $Y \rightleftarrows B$ with good homotopy theoretic properties, but without a relative CW–structure. More precisely, a retractive space $Y \rightleftarrows B$ is an object of $\mathcal{D}$ if the following holds.

> There exist an object $X \rightleftarrows B$ in $\mathcal{C}$ and a retractive map $f : X \to Y$ which is a *weak equivalence* (read this as in 1.A.6).

The notion of *weak equivalence* (in $\mathcal{D}$) is defined exactly as in 1.A.6. A morphism in $\mathcal{D}$, from $(Y \rightleftarrows B)$ to $(Z \rightleftarrows B)$, is a *cofibration* if, up to isomorphism, it is the inclusion of a closed subspace $Y \subset Z$, and has the CHEP described in 1.A.6.

With these definitions, $\mathcal{D}$ is a Waldhausen category. We shall now verify that the inclusion $\mathcal{C} \to \mathcal{D}$ has the *approximation property* [Wald2, 1.6]. This consists of two parts, of which the first stipulates that an arrow in $\mathcal{C}$ be a weak equivalence in $\mathcal{C}$ if it is a weak equivalence in $\mathcal{D}$. This is true by construction. For the second part, we have to check the following. Given any object X in $\mathcal{C}$ and a morphism $f : X \to Y$ in $\mathcal{D}$, there exist a cofibration $f_1 : X \to X'$ in $\mathcal{C}$ and a weak equivalence $f_2 : X' \to Y$ in $\mathcal{D}$ such that $f = f_2 f_1$. To construct f_1 and f_2, we begin with a weak equivalence $g : Y_1 \to Y$ where Y_1 is in $\mathcal{C}$. Since g is a weak equivalence, it is then easy to construct a controlled map $u : X \to Y_1$ (relative to B, but not necessarily over B) and a controlled homotopy h (relative to B, but not necessarily over B) from f to gu. By induction on the skeletons of X, the map u is controlled homotopic (relative to B) to a cellular map ; so we may assume without loss of generality that u is cellular to begin with. Let X' be the relative mapping cylinder of u. Define $f_2 : X' \to Y$ so that it extends the identity on B, agrees with the homotopy h on $(X \smallsetminus B) \times [0, 1)$, and with g on Y'. There is a unique retraction $X' \to B$ such that f_2 becomes a morphism in $\mathcal{D}$. Let $f_1 : X \to X'$ be the front inclusion of the cylinder. Clearly f_2 is a weak equivalence and f_1 is a cofibration.

Therefore $\mathcal{C} \to \mathcal{D}$ induces a homotopy equivalence of the K–theory spectra.

Suppose that X and Y are objects in $\mathcal{D}$. We define $X \odot Y$ literally as in 1.A.7. It is not difficult to show that if X and Y are in $\mathcal{D}$, and $X_\natural \to X$, $Y_\natural \to Y$ are weak equivalences with $X_\natural$, $Y_\natural$ in $\mathcal{C} \subset \mathcal{D}$, then the induced map

$$X_\natural \odot Y_\natural \longrightarrow X \odot Y$$

is a homotopy equivalence. This in turn implies at once that $\mathcal{D}$ with the SW product $\odot$ satisfies the conditions of 1.1.

1.A.9. Example. We conclude the list of examples with a twisted version of 1.A.8. Suppose that $E \to B$ is a spherical fibration, with fibers homotopy equivalent to $\mathbb{S}^n$, and with a distinguished section which is a fiberwise cofibration [Jm,§22].

For X and Y in $\mathcal{D}$ (of 1.A.8), and open $U \subset B \times B$, $U \in \mathcal{U}$, we (re–)define $X \curlywedge_U Y$ as $Z \cup \infty$, where Z is the space of all quadruples (x, ω, y, e) for which

- $x \in X^?$, $y \in Y^?$
- $\omega : [-1, 1] \to B$ is a path such that $\omega(-1) = r(x)$, $\omega(1) = r(y)$
- e is an element of the fiber of $E \to B$ over $\omega(0)$.

The topology on $X \curlywedge_U Y$ is defined in such a way that

- Z with the obvious topology is an open subspace
- a set $W \subset X \curlywedge_U Y$ containing ∞ is a neighborhood of ∞ if and only if the image of its complement under $Z \to X \times Y$ has closure disjoint from $B \times B$, and the image of its complement under $Z \to X \to B$ is $\mathcal{U}$–bounded.

Then we let $X \odot Y := \mathrm{colim}_U\ \Omega^\infty \Sigma^\infty (X \curlywedge_U Y)$ and note that $\odot$ so defined is an SW product on $\mathcal{D}$. It specializes to $\odot$ of 1.A.8 in the case where γ is the trivial bundle $\mathbb{S}^0 \times B \to B$.

Illustration. Suppose that $X = B \amalg S$ and $Y = B \amalg T$ where S and T are discrete. Then $X \odot Y$ of 1.A.9 is homotopy equivalent to

$$\operatorname*{hocolim}_U \prod_{(s,t) \in S \times T} \Omega^\infty \Sigma^\infty (\wp_U(r(s), r(t))_+ \wedge E_{r(t)})$$

where $E_{r(t)}$ is the fiber of $E \to B$ over $r(t)$. Apart from that, notation is as in the illustration following 1.A.7.

2.A. Examples

2.A.1. Example. The axioms of §2 are satisfied for $\mathcal{C}$ and $\odot$ from example 1.A.1. This is obvious.

All other examples in §1.A fail to satisfy axiom 2.3. However, this can be repaired. For simplicity, we concentrate on $\mathcal{C}$ from 1.A.3, the category of retractive spaces over B which are CW–spaces relative to B with finitely many cells. To avoid confusion later on, we denote the suspension functor $\mathcal{C} \to \mathcal{C}$ by Σ_B instead of Σ.

2.A.2. Example. Let $\mathcal{C}_\infty$ be the *stabilization* of $\mathcal{C}$ under suspension. In detail, an object of $\mathcal{C}_\infty$ is a pair (k, X) with $k \in \mathbb{Z}$ and X in $\mathcal{C}$. We think of (k, X) as a formal (de)suspension, $\Sigma_B^{-k} X$, and accordingly define the set of morphisms from (k, X) to (ℓ, Y) as

$$\operatorname*{colim}_{i \to \infty}\ \mathrm{mor}_{\mathcal{C}}(\Sigma_B^{i-k} X, \Sigma_B^{i-\ell} Y)\,.$$

Then $\mathcal{C}_\infty$ is a Waldhausen category in its own right, with a cylinder functor.

(A morphism $f : (k,X) \to (\ell,Y)$ is a cofibration if it can be represented by a cofibration $f_i : \Sigma_B^{i-k}X \to \Sigma_B^{i-\ell}Y$ for some i. It is a weak equivalence if it can be represented by a weak equivalence $f_i : \Sigma_B^{i-k}X \to \Sigma_B^{i-\ell}Y$ for some i. To define the cylinder of $f : (k,X) \to (\ell,Y)$ find the minimal $i \geq 0$ such that f can be represented by some $f_i : \Sigma_B^{i-k}X \to \Sigma_B^{i-\ell}Y$ in $\mathcal{C}$, and form the cylinder Z of f_i in $\mathcal{C}$. Then (i,Z) is the cylinder of f in $\mathcal{C}_\infty$.)

2.A.3. Lemma. $\mathcal{C}_\infty$ *satisfies axioms 2.1, 2.2 and 2.3.*

Proof. 2.1 and 2.3 are obviously satisfied. To prove 2.2 we use [Wei, §3]. Denote objects in $\mathcal{C}_\infty$ by single letters $A, B, C, \dots$ for brevity. The hypothesis [Wei, 3.2] clearly holds for $\mathcal{C}_\infty$. That is, if in a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' \end{array}$$

in $\mathcal{C}_\infty$ the rows are cofibration sequences, and two of the vertical arrows are weak equivalences, then so is the third. (Note that this does not hold in $\mathcal{C}$.) Therefore by [Wei, 3.6], a morphism $f : C \to D$ in $\mathcal{C}_\infty$ is a weak equivalence if and only if it becomes invertible in $\mathcal{HC}_\infty$ and has zero torsion in $\mathrm{Wh}(\mathcal{C}_\infty)$. Here $\mathrm{Wh}(\mathcal{C}_\infty)$ is K_0 of the full subcategory of $\mathcal{C}_\infty$ consisting of the objects which become isomorphic to the zero object in $\mathcal{HC}_\infty$.

We now show that $\mathrm{Wh}(\mathcal{C}_\infty)$ is zero. If C is an object in $\mathcal{C}_\infty$ which becomes isomorphic to $*$ in $\mathcal{HC}_\infty$, then $C \amalg \Sigma_B C$ (the categorical coproduct) is also isomorphic to $*$ in $\mathcal{HC}_\infty$ and has zero torsion in $\mathrm{Wh}(\mathcal{C}_\infty)$. Therefore by [Wei, 3.6] again, the unique morphism $* \to C \amalg \Sigma_B C$ is a weak equivalence in $\mathcal{C}$. But then, from the definition of *weak equivalence* in $\mathcal{C}_\infty$, the unique morphism $* \to C$ in $\mathcal{C}_\infty$ is also a weak equivalence. So the torsion of C is zero. □

Having stabilized $\mathcal{C}$, we must redefine the SW product.

2.A.4. Definition/Observation. For (k,X) and (ℓ,Y) in $\mathcal{C}_\infty$ let

$$(k,X) \odot (\ell,Y) := \operatorname*{colim}_{i,j} \Omega^{i+j}(\Sigma_B^{i-k}X \curlywedge \Sigma_B^{j-\ell}Y) \;\cong\; \operatorname*{colim}_{i,j} \Omega^{i+j}\Sigma^{i+j-k-\ell}(X \curlywedge Y)$$

(where $\Omega^{i+j}(\dots)$ is defined as the geometric realization of an appropriate simplicial set). This is natural in (k,X) and (ℓ,Y) and has the properties of an SW product on $\mathcal{C}_\infty$ listed in 1.1.

In §5.A we will show that $\mathcal{C}_\infty$ with the SW product $\odot$ satisfies axioms 2.4 and 2.5. For now let us look at some specific morphism sets in $\mathcal{HC}_\infty$ and $\mathcal{HC}$. Note that

$$[\,(k,X),(\ell,Y)\,] \cong \operatorname*{colim}_{i}\,[\Sigma_B^{i-k}X, \Sigma_B^{i-\ell}Y]$$

for objects X and Y in $\mathcal{HC}$ (so that (k,X) and (ℓ,Y) are objects of $\mathcal{HC}_\infty$). Therefore it is quite enough to look at some specific morphism sets in $\mathcal{HC}$.

2.A.5. Calculation. Suppose that $X = B \amalg \{x\}$ and $Y = B \amalg \{y\}$ as in the illustration following 1.A.3. That is, both X and Y are spaces obtained from B by adding a disjoint point. Then

$$(*) \qquad [\Sigma^m_B X, \Sigma^n_B Y] \cong \pi_m(\Sigma^n(\wp(r(x), r(y))_+))$$

where, as usual, $\wp(r(x), r(y))$ is the space of paths in B from $r(x)$ to $r(y)$.

For the proof we introduce the space P of pairs (z, ω) where $z \in \Sigma^n_B Y$ and ω is a path from $r(x)$ to $r(z)$. Since $\Sigma^n_B Y \cong B \vee_y \mathbb{S}^n$, the space P is the union of closed subspaces P_0 and P_1 where P_0 consists of the pairs (z, ω) with $z \in B$, and P_1 consists of the pairs (z, ω) with $z \in \mathbb{S}^n$. We note that $P_0 \cap P_1 \hookrightarrow P_1$ is a cofibration, and that

$$P_1/(P_0 \cap P_1) \to P/P_0$$

is a homeomorphism. Also, $P_0 \to P$ is a cofibration and P_0 is contractible, so that $P \to P/P_0$ is a homotopy equivalence. Altogether, $P \simeq P_1/(P_0 \cap P_1) = \Sigma^n(\wp(r(x), r(y))_+)$. Hence we may replace $\Sigma^n(\wp(r(x), r(y))_+)$ by P in (*). In i) below we produce a map ξ from $[\Sigma^m_B X, \Sigma^n_B Y]$ to $\pi_m(P)$, in ii) we produce a map ζ from $\pi_m(P)$ to $[\Sigma^m_B X, \Sigma^n_B Y]$, and in iii) we show that ζ and ξ are inverses of one another.

i) Let $\Sigma^m_B X \xrightarrow{f} Z \xleftarrow{e} \Sigma^n_B Y$ represent $\alpha \in [\Sigma^m_B X, \Sigma^n_B Y]$. Since e is a cofibration and a weak equivalence, there exists a homotopy $\{h_t : Z \to Z \mid 0 \le t \le 1\}$ rel $\mathrm{im}(e)$ such that $h_0 = \mathrm{id}_Z$ and $\mathrm{im}(h_1) = \mathrm{im}(e)$. Then

$$u \mapsto (h_1 f(u), \{r h_t f(u) \mid 0 \le t \le 1\})$$

is a map from $\Sigma^m_B X/B \cong \mathbb{S}^m$ to P. Its homotopy class ξ depends only on α, not on the chosen representative of α, for the following reason. We lose no generality by allowing only *special* quasi–morphisms $\Sigma^m_B X \to Z \leftarrow \Sigma^n_B Y$, that is, quasi–morphisms for which the resulting map $\Sigma^m_B X \vee \Sigma^n_B Y \to Z$ is a cofibration in $\mathcal{C}$. (If the map in question is not a cofibration, replace Z by the relative mapping cylinder of $f : \Sigma^m_B X \to Z$ and replace f by the front inclusion of the cylinder. By inspection, the homotopy class ξ does not change.) Any two special quasi–morphisms representing α can be connected by a chain of morphisms (between special quasi–morphisms) of the form

$$\begin{array}{ccccc} \Sigma^m_B X & \longrightarrow & Z_0 & \longleftarrow & \Sigma^n_B Y \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma^m_B X & \longrightarrow & Z_1 & \longleftarrow & \Sigma^n_B Y \end{array}$$

where the arrow in the middle of the diagram is a cofibration. This uses another mapping cylinder construction, explained in [Wei,§2]. Again, upper row and lower row of the diagram give rise to the same class in $\pi_m(P)$. This completes the verification, so that we may write $\xi = \xi(\alpha)$.

ii) We start with a based map $g : \mathbb{S}^m \to P$ such that the composite map $\mathbb{S}^m \to P \to \Sigma^n_B Y$ is cellular. Let Z be the reduced mapping cylinder of that composition. Make Z into a retractive space over B as follows. On $\Sigma^n_B Y \subset Z$, use the given retraction. Points of Z not in $\Sigma^n_B Y \subset Z$ can be written in the form (a,t) with $a \in \mathbb{S}^n$ and $t \in [0,1]$, and we map those to $g_2(a)(t) \in B$. Here $g_2(a)$ is a path in B, the second component of $g(a) \in P$. Note that the front inclusion of the cylinder extends automatically to a map $\Sigma^m_B X \to Z$ of retractive spaces over B. Therefore front inclusion and back inclusion of the cylinder define a quasi–morphism $\Sigma^m_B X \to Z \leftarrow \Sigma^n_B Y$ and an element $\zeta(g)$ in $[\Sigma^m_B X, \Sigma^n_B Y]$. It is not hard to see that it only depends on the homotopy class of g.

iii) By construction, $\xi\zeta$ is the identity on $\pi_m(P)$. Suppose that we start with some quasi—morphism $\Sigma^m_B X \to Z_0 \leftarrow \Sigma^n_B Y$ as in i), representing a class α, then construct from it a map $\mathbb{S}^m \to P$ representing $\xi(\alpha)$, and then construct from that map as in ii) a quasi–morphism $\Sigma^m_B X \to Z_1 \leftarrow \Sigma^n_B Y$ representing $\zeta\xi(\alpha)$. Then clearly there exists a morphism in $\mathcal{M}(\Sigma^m_B X, \Sigma^n_B Y)$, given by the vertical arrows in

$$\begin{array}{ccccc} \Sigma^m_B X & \longrightarrow & Z_1 & \longleftarrow & \Sigma^n_B Y \\ \downarrow{=} & & \downarrow & & \downarrow{=} \\ \Sigma^m_B X & \longrightarrow & Z_0 & \longleftarrow & \Sigma^n_B Y\,. \end{array}$$

Therefore $\zeta\xi(\alpha) = \alpha$ and $\zeta\xi$ is the identity. □

Remark. The reader may find our use of arbitrary (compactly generated Hausdorff) spaces in the calculation 2.A.5 scary. Here is a way to avoid pathological spaces. We note that the two sides of (*) are invariant under weak homotopy equivalences in the following sense. Given a weak homotopy equivalence $v : B \to C$, and X, Y as in (*), the maps

$$[\Sigma^m_B X, \Sigma^n_B Y] \to [\Sigma^m_C v_* X, \Sigma^n_C v_* Y]$$
$$\pi_m(\Sigma^n(\wp(r(x), r(y))_+)) \to \pi_m(\Sigma^n(\wp(vr(x), vr(y))_+))$$

are bijections. It follows immediately that (*) in the special case where B is a CW–space implies the general case of (*).

2.A.6. Example. The construction of $\mathcal{C}_\infty$ from $\mathcal{C}$ and the SW product on $\mathcal{C}_\infty$ from that on $\mathcal{C}$ go through without essential changes if we assume that B comes with a control structure and define $\mathcal{C}$ as in 1.A.6., and $\odot$ on $\mathcal{C}$ as in 1.A.7. Even more generally, $\mathcal{C}$ of 1.A.6 and $\odot$ of 1.A.7 could be replaced by the larger $\mathcal{D}$ of 1.A.8 with $\odot$ as in 1.A.8 ; so $\mathcal{D}_\infty$ would be defined as the stabilization of

$$\cdots \xrightarrow{\Sigma_B} \mathcal{D} \xrightarrow{\Sigma_B} \mathcal{D} \xrightarrow{\Sigma_B} \mathcal{D} \xrightarrow{\Sigma_B} \cdots .$$

Still more generally, we can start with the SW product of 1.A.9, and use it to make an SW product on $\mathcal{D}_\infty$, as in 2.A.4.

Note in passing that the inclusions $\mathcal{HC} \to \mathcal{HD}$ and $\mathcal{HC}_\infty \to \mathcal{HD}_\infty$ are equivalences of categories.

2.A.7. Calculation. Suppose that $X = B\amalg S$ and $Y = B\amalg T$ as in the illustration following 1.A.7. Then, in the notation used there,

$$[\Sigma^m_B X, \Sigma^n_B Y] \cong \operatorname*{colim}_U \prod_s \bigoplus_t \pi_m(\Sigma^n(\wp_U(r(s), r(t))_+))\,.$$

The proof resembles that of (*) in 2.A.4. Later we will want to know that the direct sum $\bigoplus_t$ can be replaced by a product $\prod_t$, so that we can also write

$$[\Sigma^m_B X, \Sigma^n_B Y] \cong \operatorname*{colim}_U \prod_{s,t} \pi_m(\Sigma^n(\wp_U(r(s), r(t))_+))\,.$$

Namely, for each $s \in S$ and $U \in \mathcal{U}$, there exist only finitely many $t \in T$ for which $\wp_U(r(s), r(t))$ is nonempty. *Proof:* Fix $s \in S$ and $U \in \mathcal{U}$ and let $W_s \subset B$ consist of all points $b \in B$ such that $\wp_U(r(s), b)$ is nonempty. Then certainly $\{b\} \times W_s \subset U$ and therefore $\{b\} \times W_s$ and $W_s \times \{b\}$ belong to $\mathcal{U}$. But

$$W_s \times W_s = (W_s \times \{b\})(\{b\} \times W_s),$$

so $W_s \times W_s$ belongs to $\mathcal{U}$, which means that W_s is $\mathcal{U}$–bounded. Since the second condition in 1.A.6 must hold for $Y = B \cup T$, the set of all $t \in T$ for which $r(t) \in W_s$ is finite. $\square$

5.A. Examples

5.A.1. Example. Let $\mathcal{C}_\infty$ be the category of 2.A.2, the stabilization of $\mathcal{C}$ in 1.A.3. Equip this with the SW product $\odot$ of 2.A.4. Let $\mathcal{G}$ be the class of all objects which are isomorphic in $\mathcal{HC}_\infty$ to an object of the form (n, X) with $n \in \mathbb{Z}$ and $X = B\amalg\{x\}$ as in the illustration following 1.A.3. Then $\mathcal{G}$ is a generating class. We shall use this fact to establish axioms 2.4 and 2.5 for $\mathcal{C}_\infty$.

Suppose that $X = B \amalg \{x\}$ and $Y = B \amalg \{y\}$ are retractive spaces over B. Then (m, X) and (n, Y) belong to $\mathcal{G}$, and from 1.A.3 and illustration and 2.A.4 we have

$$(m, X) \odot (n, Y) \simeq \Omega^{\infty+m+n}\Sigma^\infty(\wp(r(x), r(y))_+)\,.$$

In particular, the constant path from $r(x)$ to $r(x)$ is an element η_m in

$$\wp(r(x), r(x)) \subset \wp(r(x), r(x))_+ \subset \Omega^\infty\Sigma^\infty(\wp(r(x), r(x))_+) \simeq (m, X) \odot (-m, X)\,.$$

The component of η_m is *nondegenerate.* To prove this it suffices by 5.6 to show that slant product with $[\eta_m]$ is a bijection from $[(-m, X), (n, Y)]$ to π_0 of $(m, X) \odot (n, Y)$ for arbitrary n and $Y = B\amalg\{y\}$ as above. Now 2.A.5 and the observations preceding it give

$$[\,(-m, X), (n, Y)\,] \cong \pi^s_{m+n}(\wp(r(x), r(y))_+)$$

showing that $[(-m,X),(n,Y)]$ and π_0 of $(m,X)\odot(n,Y)$ are in "abstract" bijection. An easy inspection shows that the slant product with $[\eta_m]$ is exactly this bijection. Now 5.6 applies, so $[\eta_m]$ is indeed nondegenerate.

We see that the conditions of 5.4 are met and conclude that $\mathcal{C}_\infty$ satisfies axiom 2.4. Further, the symmetry involution τ takes the component of $\eta_m \in (m,X)\odot(-m,X)$ to the component of $\eta_{-m} \in (-m,X)\odot(m,X)$, which is also nondegenerate. Therefore the conditions of 5.5 are met, and $\mathcal{C}_\infty$ satisfies 2.5. □

5.A.2. Example. Similar reasoning shows that $\mathcal{C}_\infty$ and $\mathcal{D}_\infty$ of 2.A.6 with the SW product $\odot$ defined there satisfy axioms 2.4 and 2.5. The appropriate generating class $\mathcal{G}$ consists of all objects isomorphic in $\mathcal{HC}_\infty$ or $\mathcal{HD}_\infty$ to one of type (m,X) where $X = B \amalg S$ for discrete S. One finds, using 5.6, that $(-m,X)\odot(m,X)$ has a nondegenerate component which remains nondegenerate when τ is applied. Then 5.4 can be used, and finally 5.5. □

5.A.3. Example. Continuing in the notation of 5.A.2, suppose that $\mathcal{C}_\infty$ and/or $\mathcal{D}_\infty$ are equipped with the SW product $\odot_\gamma$ of 1.A.9. Here γ is a spherical fibration on B, with distinguished section, and with fibers homotopy equivalent to $\mathbb{S}^n$, say. For $X = B \amalg S$ as in 5.A.2, one finds (as in 1.A.9, illustration) that

$$(m,X)\odot_\gamma(n-m,X) \simeq \operatorname*{hocolim}_U \prod_{(s,t)\in S\times S} \Omega^\infty\Sigma^\infty(\wp_U(r(s),r(t))_+)\,.$$

The right–hand side contains an element η_m whose (s,t)–coordinate is zero for $s \neq t$, and equal to the constant path from $r(s)$ to $r(s)$ if $s = t$. (Some open U containing the diagonal must be selected for this to make sense, but the component of η_m does not depend on the choice of U.) Think of $[\eta_m]$ as a component of $(m,X)\odot_\gamma(n-m,X)$. It is nondegenerate (use 5.6) and maps under τ to the component $[\eta_{n-m}]$ of $(n-m,X)\odot_\gamma(m,X)$, which is also nondegenerate. Now 5.4 and 5.5 can be used as before, and the conclusion is that $\mathcal{C}_\infty$ and $\mathcal{D}_\infty$ with the SW product $\odot_\gamma$ satisfy axioms 2.4 and 2.5. □

12.A. Examples

12.A.1. Example (change of control space). Suppose that B, B' are spaces equipped with control structures $\mathcal{U}$, $\mathcal{U}'$ respectively. Assume that $\mathcal{U}$ and $\mathcal{U}$ are compatible with the topologies of B and B', respectively. Let $f : B \to B$ be a (continuous) map with the properties

(1) $f_*(\mathcal{U}) \subset \mathcal{U}'$;
(2) $f^{-1}(W)$ is $\mathcal{U}$–bounded for any $\mathcal{U}'$–bounded $W \subset B_2$.

Then the pushout with f (see remark below) is an exact functor $F : \mathcal{C}_\infty \to \mathcal{C}'_\infty$, where $\mathcal{C}_\infty$ and $\mathcal{C}'_\infty$ are the Waldhausen categories made from $(B, \mathcal{U})$ and $(B', \mathcal{U}')$ as in 2.A.6. One finds that literally

$$C \odot D \subset F(C) \odot' F(D)$$

for objects C, D in $\mathcal{C}_1$, provided the SW products $\odot$ on $\mathcal{C}$ and $\odot'$ on $\mathcal{C}'$ are constructed as in 2.A.6. Call the inclusion ϕ. It commutes with the symmetry operators.

It remains to show that ϕ takes nondegenerate components to nondegenerate components. We can think of this as a statement about $\mathcal{H}F : \mathcal{H}\mathcal{C} \to \mathcal{H}\mathcal{C}'$ and $\phi_* : \pi_0(C \odot D) \to \pi_0(F(C) \odot' F(D))$, a natural transformation between functors on $\mathcal{H}\mathcal{C} \times \mathcal{H}\mathcal{C}$. It is equivalent to the statement that a certain natural transformation

$$\nu : T'F \to FT_1$$

defined just below is a natural isomorphism ; here T from $(\mathcal{H}\mathcal{C})^{\mathrm{op}}$ to $\mathcal{H}\mathcal{C}$ and T' from $(\mathcal{H}\mathcal{C}')^{\mathrm{op}}$ to $\mathcal{H}\mathcal{C}'$ are the duality functors. The natural transformation evaluated on C in $\mathcal{H}\mathcal{C}$ is the morphism $T_2F(C) \to FT_1(C)$ which is the image of $\mathrm{id} \in [T_1C, T_1C]$ under

$$[T_1C, T_1C] \cong \pi_0(C \odot T_1C) \xrightarrow{\phi} \pi_0(F(C) \odot' FT_1(C)) \cong [T_2F(C), FT_1(C)]\,.$$

Now a five lemma argument shows that ν will always be an isomorphism if it is an isomorphism for C of the form $(0, X)$, where $X = B_1 \amalg S$ and S is discrete. (Objects of this type, and isomorphic objects in $\mathcal{H}\mathcal{C}_1$, and their iterated suspensions and desuspensions, form a generating class in the sense of §5.) In the case $C = (0, X)$ and $X = B_1 \amalg S$ we know from 5.A.2 that T_1C is isomorphic to C and $T_2F(C)$ is isomorphic to $F(C)$. In short, we can proceed by inspection. □

Remark. In working with several categories of retractive spaces it is a good idea to let the underlying sets of the base spaces (here B, B') be subsets of a large set V_1, and to allow only retractive spaces X (here: over B or B') for which the underlying set of $X^?$ is contained in some other large set V_2 disjoint from V_1. Then the pushout of a retractive space $X \rightleftarrows B$ along f, for example, can be defined set–theoretically as $X^? \cup B' = (X \smallsetminus B) \cup B'$.

12.A.2. Example (linearization). Let $\mathcal{C}$ be the Waldhausen category of 1.A.3. For objects X and Y in $\mathcal{C}$ we define $X \odot Y = \Omega^\infty\Sigma^\infty(X \curlywedge Y)$ as in the remark following 1.A.3.

Let $B^\sim \to B$ be a normal covering with translation group π. Denote by $\mathcal{D}$ the category of bounded chain complexes of finitely generated free left $\mathbb{Z}\pi$–modules. The standard SW product in $\mathcal{D}$ is $C \odot D := (C^t \otimes_{\mathbb{Z}\pi} D)^\sharp$ (compare 1.4 and 1.A.1), where C^t is C with a right action of $\mathbb{Z}\pi$. The right action and the left action are related via the involution $\sum n_g \cdot g \mapsto \sum n_g \cdot g^{-1}$ on $\mathbb{Z}\pi$.

We will use $B^\sim \to B$ to create a pair consisting of an exact functor F and a natural transformation ϕ,

$$F : \mathcal{C} \to \mathcal{D}(\mathbb{Z}\pi),$$
$$\phi : X \odot Y \to F(X) \odot F(Y)\,.$$

The functor F takes $(X \rightleftarrows B)$ to the reduced cellular chain complex of $X^\sim/B^\sim$, where $X^\sim \to X$ is the covering pulled back from $B^\sim \to B$ using $r : X \to B$. Any m–simplex z in $X \odot Y$ determines a filtration–preserving stable composite map

$$\Delta^m_+ \longrightarrow X \curlywedge Y \longrightarrow (X^\sim/B^\sim) \wedge_\pi (Y^\sim/B^\sim)$$

which in turn determines a chain map $\bar{z}$ from the cellular chain complex of Δ^m to the tensor product $F(X) \otimes_{\mathbb{Z}\pi} F(Y)$. The geometric realization of $z \mapsto \bar{z}$ is a natural map $\phi : X \odot Y \to (F(X) \otimes F(Y))^\sharp = F(X) \odot F(Y)$. (Compare 1.4 and 1.A.1.) It respects the symmetries τ.

An obvious deficiency of the pair (F, ϕ) just constructed is that the domain $\mathcal{C}$ of F is a Waldhausen category which in general fails to satisfy the axioms of §2. However, we have the stabilization $\mathcal{C}_\infty$ from 2.A.2 and 2.A.3. We can extend F to an exact functor $\mathcal{C}_\infty \to \mathcal{D}(\mathbb{Z}\pi)$ by the recipe

$$F(m, X) := \Sigma^{-m} F(X)\,.$$

Similarly ϕ extends to a natural transformation of functors on $\mathcal{C}_\infty$. Proceed as in 12.A.1 to verify that ϕ takes nondegenerate components to nondegenerate components. □

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